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Vladislav Zheligovsky

Large-Scale Perturbations of Magnetohydrodynamic Regimes

Linear and Weakly Nonlinear Stability Theory

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Christian Caron
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Vladislav Zheligovsky

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Vladislav Zheligovsky
Laboratory of Geodynamics
International Institute of Earthquake Prediction Theory
and Mathematical Geophysics
Russian Academy of Sciences
84/32 Profsoyuznaya St.
117997 Moscow
Russian Federation
vlad@mitp.ru

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Foreword

From my point of view, no other problem in the study of the nature of the Earth's magnetic field is more important than development of a qualitative theory explaining the wealth of obtained results of geodynamo computations. In other words, modern science can reproduce in computer simulations many more phenomenological facts of magnetism and paleomagnetism than we can understand. This is a new very peculiar situation in science—it just could not occur a quarter of century ago, when computer methods and hardware were incomparably weaker than nowadays. Of course, this is a developmental disease. Being mostly an optimist, I believe that science will overcome this crisis, but its depth is impressive. I will give just one example, illustrating how acute it is. While looking through the present monograph, I watched a TV performance by our colleague and a poet A. M. Gorodnitsky, who gave a brief, clear and correct account of the degree of our ignorance in the nature of geomagnetic inversions. For many years I had not a chance to see on TV such a clever discussion of a scientific problem. If a well-known poet spends this way his air time, then the problem is really challenging!

The author of this book puts forward a concept, not only suggesting to perform numerical simulations in problems concerning the geomagnetic field, but also enabling one to gain some understanding of the results obtained using computers. In view of what is stated above, the importance of this work is undoubtful.

Let us consider the approaches suggested by the author.

In most general terms, the concept that is put forward consists of identification in the description of the evolving magnetic field of the level in the hierarchy of scales, which is solely responsible for a phenomenon of interest. Nothing is more natural for a scientist of my generation—the question is how this intent is realised. Note however, that this strategic goal contradicts with the present-day numerical approach to problems in natural sciences, within the framework of which, loosely speaking, one tries to load the computer with as much as possible. Precisely this collision of paradigms is the one responsible for the acuteness of the problem under consideration. I wholly support the tactics for resolution of this contradiction, employed by the author. He does not juxtapose analytical and numerical methods, but tries to derive a sensible hierarchy of problems, in which analytical results and numerical solutions of the emerging problems mutually support each other.

In pursuing this approach, the author identifies various large-scale components of the basic equations describing the evolution of geomagnetic field and, when possible, solves the derived equations numerically. Historically this tactics was first suggested in the nineteen-hundred sixties in geophysics by S. I. Braginsky, and in the context of general physics by M. Steenbeck, F. Krause and K.-H. Rädler. The further evolution of science in the nineties has by and large abandoned these correct, as I believe, objectives and changed its course towards solution of the problem by force. Probably, it is now unproductive to discuss along which alternative trails development might have advanced, but rather we should look for positive ways out of the existing situation.

Up to this point my line of reasoning about the problem completely coincides with that of the author. Subsequently they considerably diverge—we are representatives of different scientific schools.

I would be inclined to proceed straightforwardly and pragmatically—to regard convective flows in the outer core of the Earth (or elsewhere) as a peculiar medium possessing certain integral characteristics similar to electric and magnetic permeability: turbulent diffusion and (thanks to German authors, who had introduced the heavy terminology!) α-effect, and to accept for these quantities the simplest expressions compatible with symmetries and common sense. Of course, along this way one has to vary the assumed hypotheses, consider nonlinear media, introduce by hand diverse effects—i.e. to do everything that comprises the traditional theoretical physics. This tactics is successful in electrical engineering (normal people do not calculate resistivity from the first physical principles, but use ohmmeters to measure it), in investigation of magnetic fields of galaxies, it is sufficiently fruitful in the study of cycles of magnetic activity of the Sun and stars, but we fail to apply it successfully to geodynamo problems. It is difficult to say now, whether this is linked with the scarceness of our efforts, or it is a consequence of an elusive significant physical reason.

V. Zheligovsky follows an alternative approach, whereby one derives equations for the newly introduced large-scale variables and determines the structure of coefficients in these equations systematically and mathematically rigorously. He has managed to obtain spectacular results along these lines. Just one factor makes their assessment difficult: his competitors lack entirely any results of a comparable level. Let me try, however, to describe a possible area where a comparison seems possible. We have to turn our attention to the fundamental idea of A. N. Kolmogorov, which, as it is well-known, has pointed out the direction of investigations in the theory of turbulence for decades. He postulated, that instead of examining how and why turbulent flows with apparently random properties emerge, one should assume that they are indeed random and calculate their mean characteristics employing symmetry considerations and common sense (cf. with the previous paragraph). Indeed, this approach is being developed to encompass the area of our interest, and it turns out to be possible to generalise the simple universally accepted mean-free equations (for instance, obtaining integral equations instead of differential ones), but they involve constructions so distant from the methods of astronomical or geophysical observations that we will have to return to their simplest forms if we try to apply them. Future will show, how far the theory can be advanced at this difficult stage owing to the approach developed by the author.

So, in this book the reader is offered a coherent concept explaining the world of magnetic dynamo theory. It would be beneficial for a reader making acquaintance with this concept to take into account an important idea pointed out by Heraclitus: many different convincing judgements can be presented concerning any subject. In particular, dynamo theory can be regarded at different angles. Only one point of view is exposed in the monograph. This represents its strength and weakness. One must read many books and draw comparisons in order to perceive the world of dynamo theory in full colour—the monographs of Parker, Moffatt, Mestel and others. Let this book be an important step on the ladder of knowledge—the first one for some readers, and yet another one for others.

Moscow, December 2009 Dmitriy D. Sokoloff

Preface

A challenging astrophysical problem is to explain, why many cosmic bodies possess magnetic field. Since the foundations of the theory of hydromagnetic dynamo were established in the second half of the twentieth century, magnetic field generation in cosmic bodies with molten electrically conducting cores (planets) or consisting of plasma (stars) is usually attributed to flows in their interior and are considered within the framework of this theory. However, details of these processes are not yet understood.

Geomagnetic field is an important factor influencing the biosphere. Magnetic field of the Earth protects the life on the planet, deflecting the cosmic radiation. The magnitude of the field can diminish by an order of magnitude during a magnetic field reversal, i.e. the change of polarity of magnetic field of the Earth; duration of a reversal is several thousand years. Thus, the impact of reversals on the biosphere is likely to be significant. Magnetic storms can damage or cause dysfunction of artificial satellites and supposedly can affect the general condition of humans. The influence of the variation of the solar activity on the Earth's climate has been debated. This explains, why it is important to understand the physical processes underlying the magnetic field generation and evolution.

Dramatic events experienced by magnetic field of the Sun—emergence and disappearance of sunspots—constitute a chaotic sequence. The Maunder butterfly diagram illustrates the 11-year periodicity of the number of sunspots on its visible surface. Activity of the Sun varies drastically between the periods, to near disappearance of spots during the so-called Maunder minimum throughout the second half of the seventeenth century. Magnetic field of the Earth also evolves on different time scales. Geomagnetic and paleomagnetic measurements reveal variations of magneticfield of the Earth, whose periods range from hours to millions of years. Fast variations of the geomagnetic field, the so-called geomagnetic storms, are associated with the solar activity; they cause auroras in polar regions. Magnetic field reversals have recurrence periods of several hundred thousand years. The reversals are the reason of magnetic anomalies, detected at the oceanic floor: Molten basalts extruding from mid-ocean ridges cool down and form the oceanic crust, causing the sea floor spreading; when their temperature decreases below the

Curie point, direction of magnetic field is "recorded" in the rock. These features of the Earth's and Sun's magnetic field evolution confirm that the fields have dynamic origins.

Until recently the theory could only rely on computer simulations and astrophysical observations. At present the situation has changed drastically: a possibility of magnetic field growth owing to a motion of a liquid metal has been confirmed in recent years in laboratory experiments. In first experimental dynamos flows of particular structures were arranged by pumping metal through fluid containers of special designs in order to facilitate generation. The "von Karman sodium" (VKS) experiment in Cadarache, France, was the first successful experimental dynamo,where self-organisation of the flow was not artificially restrained. Investigation of the possibility of magnetic field generation in molten metals is of prime practical importance in application to flows in pipelines of cooling systems at nuclear power plants.

Magnetic fields generated in experiments show intricate intermittent behaviour. Various spatial and temporal scales are present because an artificial flow is maintained in the facility, or because turbulence develops in molten metal. A hierarchy of scales persists in geodynamo as well. For instance, the Ekman boundary layer, emerging near the horizontal boundaries in thermal convection in a layer of rotating fluid, involves two spatial scales. Irregularities of the core-mantle boundary do not exceed 5 km, which is small compared to the radius of the fluid core, ~3500 km; they give rise to topographic core-mantle coupling, which can be responsible for the observed length of day variation. Scale separation can also take place in the entire volume of fluid. In geostrophic flows in rapidly rotating spherical or cylindrical shells the so-called Taylor columns emerge in the fluid flows, with the axes parallel to the rotation axis. Their width is significantly less than the size of the fluid container. Thin rolls emerge in thermal convection in a horizontal layer of fluid, rapidly rotating about the vertical axis, and in magnetoconvection, when the imposed magnetic field is strong. A hierarchy of spatial scales, interacting by means of processes manifesting themselves in direct and reverse energy cascades and in intermittency, is an attribute of turbulence, which plays an important role in magnetic field generation.

Because of the multiscale nature of MHD systems, direct numerical simulations in the range of parameter values of interest are problematic, if possible. Integration of equations of magnetohydrodynamics with the minimum sufficient resolution requires at least $O(R_m^{11/4})$–$O(R_m^5)$ operations to advance a solution by a (nondimensional) unit time, if finite differences or Galerkin methods, respectively, are applied (here R_m is the magnetic Reynolds number); if fast Fourier transforms can be used for simulation of space-periodic MHD systems the operation count decreases to $O(R_m^{11/4} \ln^3 R_m)$ operations. These power-law dependencies indicate that modern computers are still not powerful enough to make computations with the spatial and temporal resolution, adequate for the parameter values characterising the conditions in the interior of the Earth (R_m is estimated to be in the range 10^2–10^3) and the Sun (in various problems R_m is in the range 10^6–10^8). For instance, well-known simulations by Glatzmaier with coworkers were carried out for Taylor and Ekman numbers of the order of 10^3 and 10^{-4}–10^{-6}; these values differ by orders of magnitudes from their estimates for the liquid Earth's core, $\sim 10^9$ and $\sim 10^{-8}-10^{-15}$, respectively. (Despite this, in these simulations the predominantly dipole morphology of the magnetic field and its chaotic reversals were successfully reproduced.) Even if a single astrophysical object (e.g., the Earth) is explored, computations for ranges of parameter values must be performed, because parameters in the interior of the object and its rheology usually are not known precisely. This cannot be done just numerically due to an enormous amount of simulations involved. Thus analytical and combined analytico-computational methods are of certain value.

E. Parker suggested that formation of loops of magnetic lines by microscopic "eddies" in turbulent helical (non-parity-invariant) flows could modify the Ohm's law for the mean current. He argued that a new term, proportional to the mean magnetic field, would appear in the law. The phenomenon got the name "α-effect". This idea lies in the heart of the mean-field theory developed in the nineteen-hundred seventies and eighties and demonstrates the fruitfulness of the concept of scale separation in the theory of magnetic field generation.

A complex intermittent behaviour of magnetic field observed in experimental facilities still awaits an explanation. Equally, the study of convection in the interior of the Earth, as well as of other astrophysical problems, is far from completion. In view of the circumstances discussed above, mathematical instrumentarium, on which investigation of multiscale continuous systems relies, is indispensable for researchers conducting these studies. The monograph is dedicated to exposition of such mathematical techniques—mathematical methods for homogenisation—in application to problems of magnetic field generation and the adjacent area, MHD stability. The use of these tools for real-life problems of geo- and astrophysics is by no means straightforward, and hence we focus on model problems in regions of a simple geometry. More specifically, we consider ten problems in the order of increasing complexity: large-scale passive scalar transport; large-scale kinematic dynamos powered by steady and time-periodic flows in the entire space and by thermal convection planforms in a horizontal layer; linear and weakly nonlinear stability to large-scale perturbations of MHD regimes in entire space; weakly nonlinear stability to large-scale perturbations of nonlinear magnetic field generation by forced and free convection in a horizontal layer of fluid rotating about the vertical axis; and, finally, two kinematic dynamo problems for flows with an internal scaling and infinitely growing amplitude in axisymmetric containers.

We focus on MHD problems, for which solutions can be derived rigorously from the first principles, without a recourse to empirical closure relations—presented here is the "mathematician's" point of view on these problems; an exposition of the "physicist's" point of view can be found in the classical monograph by F. Krause and K.-H. Rädler "Mean-field magnetohydrodynamics and dynamo theory", Academic-Verlag, Berlin (1980). Application of asymptotic methods for multiscale systems gives an opportunity to derive the mean-field (or amplitude) equations, describing the large-scale dynamics. The influence of the underlying small-scale dynamics is represented in these equations by new terms (including the α-effect terms). By analogy with hydrodynamics, such terms are usually called “eddy corrections”. Computation of their coefficients boils down to numerical solution of systems of linear partial differential equations in small-scale variables. Data in these problems have a single spatial scale, the one of the small-scale MHD state, whose stability is studied. Thus, application of homogenisation methods for multiscale systems provides an opportunity to separate the large- and small-scale dynamics (under the condition that the latter is in some sense homogeneous, for instance, if the MHD regime under consideration is space-periodic). This eliminates the necessity to carry out computations with the resolution, which would be sufficient to represent accurately the entire hierarchy of large and small scales. This is a very important advantage of the combined analytico-numerical approach, discussed in the monograph, since it does not require excessive numerical resources for implementation.

The present monograph is an extended and revised translation of the book “Mathematical theory of stability of magnetohydrodynamic regimes to large-scale perturbations” (URSS, Moscow, 2009) by the same author. It is intended for a wide range of readers, including researchers working in the areas of geo- and astrophysics, fluid dynamics and magnetohydrodynamics, numerical analysts involved in direct numerical simulations in these areas, and postgraduate students. The prerequisite knowledge is confined to the university courses of Analysis III and Complex Analysis. Reading of the first eight chapters requires only an acquaintance with the basic notions of these mathematical disciplines, such as the spectrum and resolvent of a linear operator, compact and adjoint operators, the Fredholm alternative theorem and the residue theorem. Chaps. 9 and 10, where the asymptotic nature of the series that we construct is proved, demand a basic knowledge of Sobolev functional spaces. We are freely referring to notions of the magnetic dynamo theory, and it would be an advantage to be familiar with its foundations presented, e.g., in the monograph by H. K. Moffatt “Magnetic field generation in electrically conducting fluids” (Cambridge University Press, 1978).

I am most grateful to U. Frisch (Académie des Sciences) for his support of my research, collaboration and many discussions extending over a 20-year period. Most of these discussions took place in the inspiring environment of the premises of the Observatoire de Nice (a part of the Observatoire de la Côte d’Azur), which I would like to thank for hospitality. It is my pleasure to express my gratitude to numerous friends and colleagues: I am indebted to V. I. Keilis-Borok and A. A. Soloviev (Russian Academy of Sciences) for their permanent support of my work in the area of mathematical geophysics and magnetic dynamo theory; to A. M. Soward (FRS) for the long-term scientific communication and collaboration; to M. R. E. Proctor(FRS), Professors A. D. Gilbert, C. A. Jones, M. M. Vishik and K. Zhang for many discussions; to Doctor S. M. A. Gama for the lasting cooperation; to my postgraduate students at the University of Porto M. Baptista and R. Chertovskih for their enthusiasm and right questions; to my colleagues at the International institute of earthquake prediction theory and mathematical geophysics of the Russian Ac. Sci., Moscow, for the creative atmosphere and ceaseless support; and, last but not least, to my wife Doctor O. M. Podvigina for her continuous help. I am grateful to the Ministry of Education (France) for the financial support of my numerous research visits to the Observatoire de la Côte d'Azur, and to the Royal Society (UK) for financing my visits to the University of Exeter. My research was partially supported by the grants 04-05-64699, 11-05-00167-a and 07-01-341 92217-CNRSL_a from the Russian foundation for basic research, and the grant ANR-07-BLAN-0235 OTARIE from Agence nationale de la recherche (France). Computations have been carried out on the computer facilities made available in the framework of the Programme "Simulations Interactives et Visualisation en Astronomie et Mécanique (SIVAM)" and the next phase of this Programme "Mésocentre SIGAMM (Simulations Interactives et Visualisation en Géophysique, Astronomie, Mathématique et Mécanique)" hosted by Observatoire de la Côte d'Azur.

Nice – El'niki – Nice, November 2010 Vladislav Zheligovsky

Contents

Chapter 1
Introduction

Many cosmic objects—planets, stars, galaxies—possess magnetic fields. A fundamental question of modern astrophysics is: What are their sources?

Perhaps, the first attempt to answer this question was made by Gilbert in 1600 [120], who suggested that the main Earth's magnetic field can be explained by magnetisation of substance in its interior. However, nowadays it is well-known, that already at a depth as small as 30 km the temperature inside the Earth is higher than the Curie point for the constituting rocks [174]. (A Curie point of a ferromagnetic is the temperature at which it becomes paramagnetic, i.e., it fails to remain magnetised when heated above the Curie point.) Consequently, such an explanation would imply that magnetisation of the rocks is unrealistically high (see, however, [184]). According to paleomagnetic measurements, the dipolar main magnetic field of the Earth exists at least for 3×10^9 years [44] (this amounts to about of 2/3 of its entire history). Hence, it is not a relict magnetic field captured during the process of accretion of the Earth from the interplanetary substance, because the characteristic decay time of magnetic field in the core is about 25×10^3 years [48]. Other hypotheses about the source of the Earth's magnetic field were suggested, such as induction during magnetic storms [54], currents due to the presence of electric charges at fixed locations on the rotating planet, or certain atomic processes. However, a careful examination of these hypotheses has revealed their inadequacy [216].

The Earth's magnetic field experiences temporal variations on different time scales: secular variations (of periods $\sim 10^1 - 10^3$ years), the westward drift (with the speed about 0.2° per year), reversals (with periods of $\sim 10^5 - 10^6$ years) [44, 63, 69]. Magnetic reversals are responsible for the so-called magnetic anomalies of the ocean floor emerging due to thermoremanent magnetisation of molten iron-rich basalts, which are lifted from the planetary interior at mid-oceanic ridges and cool down in the course of the seafloor spreading. Discovery of magnetic anomalies and determination of apparent trajectories of migration of the Earth's magnetic poles for different continents, using paleomagnetic data, have stimulated development of the theory of plate tectonics [64, 70, 300]. Temporal

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variations are also featured by the solar magnetic field. Diagrams known as "Maunder butterflies" [180, 181] illustrate the variation, with an 11-year periodicity, of the number and latitudinal distribution of sunspots (whose emergence is related to the uplift of flux tubes of the toroidal magnetic field due to magnetic buoyancy [214]). This reveals the 22-year periodicity of the toroidal magnetic field of the Sun under the photosphere (the duplication of the period is due to the change of polarity of the field at every "flap of butterfly's wings").

These features of the main magnetic field of the Earth and of the solar magnetic field indicate that they are of a dynamic origin. That the field can be excited by magnetohydrodynamic (MHD) processes was discussed already in the beginning of the XX century [162, 163, 265]. After the foundations of the theory of hydromagnetic dynamo were developed, it became customary to answer the question about the nature of magnetic fields of various astrophysical objects—planets [33–39, 44–48, 132, 135, 257], stars [138, 185, 213, 241, 289–291] and galaxies [259]—within the framework of this theory [56, 66, 68, 156, 166, 186, 214, 215, 218, 301, 303, 311, 316, 317] (the lists of references do not pretend to be complete), although some alternative hypotheses are also put forward (e.g., [77]). Modern views on mechanisms of generation of cosmic magnetic fields are exposed in collections of review lectures [80, 242, 280] by leading researchers in this area.

Evolution of a magnetic field in a volume of a conducting fluid is described by the magnetic induction equation—a linear (in magnetic field) parabolic partial differential equation of the second order. Magnetic field affects the fluid flow by means of the Lorentz force, which is quadratic in magnetic field. Hence, while magnetic field is weak, its influence on the flow can be neglected. Consequently, if the flow is known, one can study the evolution of an initially small magnetic field (at least on short time intervals, if the field is growing) just by solving the magnetic induction equation alone. From the mathematical point of view, this is identical to the study of linear stability of the amagnetic state (in which magnetic field is zero) of the MHD system under consideration, with respect to (exclusively) magnetic perturbations. This problem is called the *kinematic dynamo problem* [186]. If magnetic field does not decay at large times, then we say that for the employed molecular magnetic diffusivity η the flow under consideration acts as a *magnetic dynamo*. If the flow is steady (i.e., the flow velocity $\mathbf{v}(\mathbf{x})$ does not depend on time), it is natural to formulate the definition of a dynamo in the terms of the spectrum of the magnetic induction operator. Suppose the volume occupied by the conducting fluid is finite and some regular conditions for the magnetic field are assumed on its boundary. Let λ denote the *dominant* (i.e. having the maximal real part) eigenvalue of the magnetic induction operator. Then, for the employed η, the flow $\mathbf{v}(\mathbf{x})$ is a kinematic dynamo, if and only if $\mathrm{Re}\,\lambda \geq 0$.

At the first stage of development of the kinematic dynamo theory, the so-called antidynamo theorems were proven, i.e. conditions were found guaranteeing that flows satisfying them cannot act as dynamos (whichever the diffusivity η is). In particular, by the well-known Cowling theorem [66–68] an axisymmetric flow cannot generate an axisymmetric magnetic field with the same axis of symmetry. No flows with a toroidal velocity [45, 86], and no plane-parallel flows (such that, at any point inside the volume of the fluid, the flow is orthogonal to a certain fixed direction) [314, 315] cannot be dynamos. The latter two statements are often referred to as the Elsasser and Zeldovich antidynamo theorems.

A further progress in development of the dynamo theory owes to the Parker's idea [213, 215] that magnetic field can be generated by a turbulent flow of a conducting fluid with the velocity, lacking reflection symmetry and having a non-zero helicity. This is the central idea of the mean-field magnetohydrodynamics [156, 283], in which a linear dependence of the mean (after averaging of the fluctuating components of the flow velocity and magnetic field over small scales) electromotive force on the mean large-scale magnetic field (as well as on its spatial derivatives, in a more general framework) is postulated. The physics of Parker's mechanism of magnetic field generation can be described as follows.

By the Alfvén's theorem [2], in the absence of magnetic diffusion magnetic field is frozen into the conducting medium, i.e. magnetic force lines are transported by the flow. If the conductivity of the flow is high, then an eddy in the turbulent flow deforms the frozen force line into a loop. This is accompanied by emergence of an electric current, parallel to the mean (not perturbed by the flow) magnetic field. If the effect does not disappear after averaging over a large ensemble of eddies (which is possible, e.g., in the case of homogeneous, isotropic, but not mirror-symmetric turbulence [156, 303]), then the Ohm's law for mean fields takes the modified form $\mathbf{j} = \sigma(\mathbf{E} + \alpha\mathbf{B})$ (here $\mathbf{j}$ is the mean electric current density, σ electric conductivity of the fluid, $\mathbf{B}$ mean magnetic field, $\mathbf{E}$ mean electric field). The new term in the right-hand side describes the so-called magnetic α-effect. The mean-field theory based on similar empirical relations is further developed in [244–246, 249, 264, 303]. The so-called test-field procedure was applied in [40, 41, 150, 151, 293] for determination of the turbulent transport coefficients, the α-tensor and turbulent diffusivity in convection and convective dynamos.

If a scale separation is present in the flow, emergence of the α-effect can be rigorously justified without making additional assumptions about statistical properties of the fluctuating components of the flow and magnetic field (i.e., components depending on the variables describing small spatial and/or temporal scales, which are traditionally called "fast"). The perturbation theory [90, 152] and the theory of averaging of elliptic operators [15, 62, 140, 155, 209, 262, 337] were applied in [306, 307] for construction of a complete asymptotic expansion of eigenfunctions of the magnetic induction operator and the associated eigenvalues for steady space-periodic flows with two characteristic spatial scales. The α-effect determines the evolution of the mean magnetic field either solely, or in interaction with molecular magnetic diffusion, depending on what relation between the scale ratio and the amplitude of the fluctuating component of the flow is assumed. (It was proposed in [307] to call it γ-effect in the former case.) Convergence of the expansion [306] to the leading-order approximation was studied numerically in [308] for the case of the α-effect co-acting with molecular diffusion. The magnetic α-effect tensors were derived in [321–323] for the kinematic dynamo problems in an axisymmetric volume of fluid whose velocity depends on the azimuthal fast variable, and for a fluid confined in a sphere with the flow depending on the three spherical fast variables. A mathematical analysis [115, 116] of weakly nonlinear MHD stability of ABC flows did not reveal any new effects due to nonlinearity. (ABC flows are space-periodic eigenfunctions of the operator curl, $\nabla\times$. They have a simple algebraic structure: each of the three components are a sum of a sine and a cosine of the Cartesian variables, with the coefficients originally denoted by A, B and C, which gave rise to their name. The flows were suggested by Arnold [8, 9] as possible examples of fast dynamos, i.e. dynamos in which the growth rate does not decay in the limit of large magnetic Reynolds numbers, $R_m \to \infty$, owing to the fact that particle trajectories in these flows exhibit stochastic behaviour in a part of the phase space [7, 78].)

With the advent of the era of computers it became possible to solve numerically the kinematic dynamo problems for model laminar flows in fluid volumes of different geometry, e.g., in a sphere [44, 46, 117, 118, 131, 160, 169, 218, 266, 324], in a flat layer [334], in spherical shells [136, 266], between two rotating coaxial cylinders [129, 260, 274–277], space-periodic [10, 104, 105, 107–110] (the reference lists are not exhaustive). When the resolution of computations is increased, dominant eigenvalues of the magnetic induction operator usually converge faster than the associated magnetic modes [46]. Nevertheless, computational complexity grows fast with the magnetic Reynolds number R_m; this represents the main difficulty of the "numerical brute force" approach. In simulations of the magnetic field evolution for a large R_m, the spatial resolution at least of the order of $R_m^{-3/4}$ is required, which can be provided by Galerkin's approximation of the field involving $\mathrm{O}(R_m^{9/4})$ basic functions. Finite-difference time-stepping is usually done in such simulations; advancement by a unit time (in the turn-over time units) then requires to make $R_m^{1/2}$ steps. (These estimations are obtained for a hydrodynamic problem under the assumption that details of flows at the Kolmogorov scale are sufficiently resolved [50]; however, the similarity of the structure of the Navier–Stokes and magnetic induction equations implies that they remain true for MHD problems.) Thus, due to quadratic nonlinearity, a straightforward integration of the magnetic induction equation over a unit time requires, in general, $\mathrm{O}(R_m^5)$ floating point operations. If the geometry of the volume of fluid and boundary conditions do not prohibit the use of the three-dimensional Fast Fourier Transform (e.g., if the basic functions are trigonometric functions or Chebyshev polynomials), then the number of operations can be reduced to $\mathrm{O}(R_m^{11/4}\ln R_m)$. In the most favourable case, when the dynamo action for an ABC or similar flow is simulated, the number of operations grows as $R_m^{11/4}$. In the spherical geometry, computational complexity of the problem can be reduced by application of the Fast Fourier Transform in the angular variables. Whichever the case, computations for a large R_m require immense computational resources. Insufficiently resolved computations can produce wrong results, including a reduced critical value of R_m for the onset of magnetic field generation. Despite being obvious, this statement has been proven in many simulations. For instance, the region of convergence of the numerical method was not attained in computations [44, 46, 169] due to the lack of resolution

[117, 218]; examples of spurious magnetic field generation obtained in computations with an insufficient resolution were also pointed out in [177, 178].

Modern parallel supercomputers give an opportunity to carry out computations of three-dimensional kinematic dynamos for $R_m \sim 10^3 - 10^4$. This is significantly smaller than what is required by the practical needs of astrophysics, which is characterised by high values of magnetic Reynolds numbers. The estimate for the Earth, $R_m = 150$ [186], is obtained under the assumption that its magnetic field is frozen into the medium of the outer core (which gives an opportunity to estimate the magnitude of the flow velocities in the core). Its actual value is supposed to be not more than a factor 3 higher or lower than this estimate, resulting from a low accuracy of determination of the electric conductivity of substance in the outer core [45]. Different problems of magnetohydrodynamics of the Sun involve much higher values: $R_m = 10^4$ [186], 10^6 [156], 3×10^7 [241].

In principle, magnetic field generation in the molten outer core can be studied by numerical solution of the system of equations, describing hydromagnetic convection: the Navier–Stokes equation (with the buoyancy, Coriolis and Lorentz forces taken into account), the magnetic induction equation and the heat equation. This approach, for instance, was followed in a well-known series of studies by Glatzmaier with coauthors [60, 122–128, 210, 254], who were computing evolutionary solutions to this system in a spherical shell representing the Earth's outer core, and successfully reproduced many features of the geomagnetic field, such as the predominantly dipolar morphology of the main field and its chaotic reversals. However, even modern computers do not give an opportunity to perform computations with the spatial and temporal resolution, which is sufficient for the parameter values characterising convection in the outer core of the Earth. The computations by Glatzmaier et al., mentioned above, were carried out for the Taylor and Ekman numbers of the order of 10^3 and $10^{-4} - 10^{-6}$, which differ by many orders of magnitude from their values in the Earth's core, $\sim 10^9$ and $\sim 10^{-8} - 10^{-15}$, respectively (the two latter estimates are obtained for the molecular and turbulent kinematic viscosity [142]; see also [11, 133]). In order to compensate for the lack of spatial resolution (for the considered parameter values), some computations were performed with the use of numerical hyperviscosity. However, this method of smoothing can significantly distort results [263, 318, 319]. For instance, when it is applied, non-axisymmetric components of the flow and magnetic field are underestimated compared to the axisymmetric ones and consequently the preferred configuration of the fields is dipolar, while it is quadrupole in the absence of hyperviscosity [49]. (Moreover, the Fourier–Galerkin system of ordinary differential equations, representing the spatial discretisation of the original system of partial differential equations, has higher stiffness, if hyperviscosity is applied.) Because of these reasons, a good qualitative agreement between results of numerical simulations in these studies and the natural geodynamo can be regarded as "surprising" [141].

Even if convection in the presence of magnetic field is studied in order to explore physical processes in a specific astrophysical body—e.g., the Earth—it must be examined in a certain region in the space of parameters, because the rheology relations [59] and parameter values [219] in the equations governing the convective dynamo in its interior are known only approximately. For instance, the available estimations of heat diffusivity in the Earth's core differ by several orders of magnitude (see, e.g., [184]). In such a study, it is desirable to identify the typical regimes and to localise the points of bifurcations, where the pattern of behaviour of the system changes. To do this exclusively by numerical means is impossible, since this would require an enormous amount of computational resources. Consequently, analytical and hybrid analytico-computational methods are also potentially valuable. An analytical approach was followed, e.g., in the well-known studies of the geodynamo by Braginsky [33–35] and Soward [278, 279], who constructed asymptotic expansions of solutions to the magnetic induction equation in order to quantify the α-effect in the MHD systems that they considered.

That magnetic field growth can be sustained by a motion of a molten metal has been proven in experiments [87]. This question is important not only for theoretical astrophysics, but also as a practical one in application to flows of molten metal in the reactor-cooling systems at nuclear power plants [3, 20, 223, 224]. No magnetic field generation by the maintained flows was observed in the first experiments [95, 165, 172, 173], since only relatively small kinematic and magnetic Reynolds numbers were obtained in the laboratory facilities. In experiments with the so-called "α-box" [284], emergence of the magnetic α-effect was observed. Generation of magnetic field was detected in the Karlsruhe experiment [192, 248, 287, 288, 298]. In it, a spiral flow of molten sodium was arranged in pipes of a diameter much smaller, than the radius of the cylindrical volume, where magnetic field was generated; metal was moving in the opposite direction in adjacent pipes (i.e., directions of flows in pipes alternated to fit the chess-board order). Such a flow, first considered by Roberts [252] and further investigated in [225, 226, 258, 297], possesses a strong α-effect, which is beneficial for generation. The "Riga" experiment at the Institute of Physics (University of Latvia, Salaspils) is an attempt to realise the Ponomarenko dynamo [233] involving a flow velocity jump on a cylindrical surface: molten sodium is drawn by a propeller into a spiral flow in the internal part of a cylindrical volume, and moves in the opposite direction in the outer part of it, the two parts being separated by a rigid pipe. Short instances of magnetic field generation were registered in this experiment [96–103]. Regimes of the evolution of magnetic field on large time intervals were studied in the experiments in Cadarache (Commissariat à l'énergie atomique, France) in facilities with molten sodium and gallium (the von Kármán sodium, VKS, and gallium, VKG, experiments) accommodated in containers of cylindrical shape [17–19, 28–30, 106, 176, 189, 190, 220, 250, 305, 309, 310]. In these experiments the possibility of magnetic field generation by a turbulent flow, which is not externally artificially constrained, has been clearly demonstrated for the first time. Experiments with flows in spherical regions were carried out at the Universities of Maryland [164, 268, 270] and Wisconsin [205]. Experimental investigation of the unsteady MHD-dynamo effect in a toroidal channel was conducted at the

Institute of continuous media mechanics (Russian Ac. Sci., Perm) [74, 89, 206]. Experimental studies of magnetoconvection aimed at geophysical applications are reviewed in [194].

In each of the experiments various spatial scales were present in the flow, either because its geometry was artificially constrained (in the Salaspils and Karlsruhe experiments), or because of emergence of developed turbulence in the volume of the molten metal (in all the rest experiments mentioned above).

Geodynamo is also characterised by the presence of structures with a hierarchy of spatial scales. The Ekman boundary layer, emerging in convective flows of a rotating fluid satisfying the no-slip condition on horizontal boundaries, represents an example of a high-contrast structure; its instability can result in magnetic field generation [237–239, 256]. The Ekman–Hartmann boundary layer can be unstable near the core–mantle boundary [75]. The core–mantle interaction, which is supposed to be responsible for the decade length-of-day variation, is another example demonstrating the importance of small scales in the geophysical context. For instance, the topographic coupling results from surface irregularities of the core–mantle boundary, whose size does not exceed 5 km (see, e.g., [31, 184]); this is small compared to the radius of the liquid core, 3,486 km [320] (the influence of the irregularities of the outer boundary of the Earth's core is considered in [4–6, 139, 253]). In these examples small-scale structures are located near the boundaries of the liquid core; scale separation can also take place in the entire volume, where convection occurs. For instance, near the onset of convection in rapidly rotating spherical shells the dominant modes are highly non-axisymmetric and have a columnar shape [81], and the so-called Taylor columns emerge in geostrophic flows in a rapidly rotating spherical or cylindrical shell; the columns are parallel to the axis of rotation, and their width is much smaller than the size of the shell (see [11, 121]). Thin rolls emerge in heat convection in a horizontal layer of fluid, rapidly rotating about the vertical axis (the width of rolls in the horizontal direction is of the order of $Ta^{-1/6}$, where Ta is the Taylor number) [14, 143–146] and in magnetoconvection with a strong imposed magnetic field (the width of cells can be of the order of $Q^{-1/6}$ or $Q^{-1/4}$, where Q is the Chandrasekhar number) [147–149, 179]. The presence of a hierarchy of interacting spatial scales, exhibiting the phenomena of direct and inverse energy cascades and intermittency, is a feature of turbulence, which plays an important rôle in the processes of generation [91, 191].

Equations for the temporal evolution of MHD perturbations (in particular, of a weak magnetic field, when the fluid flow is specified) have an important property: they are defined by a linear operator, which has for all parameter values and any geometry of the volume of fluid a non-empty kernel, unless this is forbidden by the boundary conditions. These modes have the same spatial (or spatio-temporal) scale, as the perturbed state, and non-zero spatial mean (or spatio-temporal, if the perturbed MHD state is unsteady). When the size of the fluid container is increased, the minimal in absolute value eigenvalue of the Laplace operator, describing energy dissipation due to diffusive processes (in particular, resulting from viscosity and electric resistance) tends to zero; energy dissipation in fields, slowly varying in space, is small. Therefore, physically, it is natural to pose the following question: Do growing in time perturbation modes exist, which are large-scale perturbations of the short-scale neutral modes, if the size of the MHD system is sufficiently large? This question is one of the main topics of the present monograph.

The presence of a variety of scales in MHD states suggests to apply methods of the theory of homogenisation of partial differential equations (see the monographs [15, 62, 140, 209, 217, 262]) for the mathematical analysis of problems of hydromagnetic convection, in particular, the ones in astrophysics. It is common to consider the case, where the temporal and spatial scales of a perturbation is significantly larger than the respective scales of the MHD state, whose stability is investigated. The perturbation is supposed to depend not only on the so-called *fast* spatial, $\mathbf{x}$, and temporal, t, variable, but also on *slow* variables $\mathbf{X} = \varepsilon\mathbf{x}$ and $T = \varepsilon t$ or $T = \varepsilon^2 t$. Here the spatial scale ratio $\varepsilon > 0$ is a small parameter of the problem, which can be used to construct asymptotic expansions. It is convenient to examine a weakly nonlinear perturbation (which is small in amplitude, but nonlinear effects already become prominent) under an additional assumption that the amplitude is of the order of ε, analysing exact nonlinear equations for the perturbation. We will call henceforth *small-scale* the fields which depend only on fast variables, and *large-scale* the fields depending on slow variables as well. In this terminology, we consider in this monograph stability of small-scale MHD states to large-scale perturbations.

A solution for a perturbation is constructed in the form of a power series in ε. A closed system of partial differential equations in slow variables is rigorously derived (without recourse to any empirical relations for closure) for the coefficients of the series averaged over fast variables (the mean coefficients provide a description of the large-scale structures of the perturbation). The differential equations are referred to as the system of *mean-field* or, more generally, *amplitude equations for perturbations*. The influence of small-scale structures is represented in this system by newly emerging terms (usually called eddy corrections by analogy with hydrodynamic problems) with constant coefficients. Numerical solution of the so-called *auxiliary problems*, which are linear elliptic partial differential equations in fast variables, is required for computation of the coefficients in eddy corrections. The data in the auxiliary problems involves a single scale—the one of the MHD state, whose stability is examined. Thus, application of asymptotic methods gives an opportunity to split the large-scale and small-scale dynamics (provided the latter is in some sense homogeneous) of perturbations. This eliminates the need to perform computations, resolving accurately the entire hierarchy of large and small scales in the problem, which is an important advantage of the hybrid analytico-computational method that we consider. Derivation of the amplitude equations can be regarded as an analytical justification of the numerical method, called *large eddy simulations* (LES) [114, 261], when it is applied to hydrodynamic or MHD systems with scale separation. The exact asymptotic results that we derive here for such systems—eddy tensors with the coefficients expressed in the terms of solutions to the auxiliary problems—can be used within the framework of LES for approximation of the influence of small-scale structures on large-scale ones, instead of empirical formulae and traditional methods for closure of the mean-field equations for the “large eddies”.

In hydrodynamic systems such expansions reveal the presence of the so-called AKA-effect (anisotropic kinematic α-effect) [82, 93, 292]. In the two-scale MHD stability problems, generically the mean-field equations are obtained, as in the purely hydrodynamic case, on averaging equations at order ε; in addition to the derivative in the slow time, they involve a single term representing the α-effect (independently of whether linear or weakly nonlinear perturbation is considered). However, the α-effect can turn out to be *insignificant in the leading order* i.e., the respective operator may be absent in the averaged $O(\varepsilon)$ equations. This happens, for instance, if the perturbed MHD system is parity-invariant or is symmetric about an axis (see the definitions of the symmetries in Sect. 8.4), although the presence of a symmetry is unnecessary for insignificance of the α-effect—e.g., ABC flows do not possess the AKA-effect [313]. If the α-effect is insignificant in the leading order, non-trivial mean-field equations are obtained as the *solvability conditions* for the next, ε^2-order equations in fast variables, and then new eddy effects can be encountered. In this case, the mean leading term of expansion of a linear stability mode of a steady short-scale MHD state and its growth rate are, respectively, an eigenfunction and the real part of the associated eigenvalue of the so-called *operator of combined*[1] *MHD eddy* (*turbulent*) *diffusion*. This is a linear partial differential operator of the second order in slow spatial variables; it is, in general, anisotropic and not necessarily sign-definite. If it has eigenvalues with positive real parts, one speaks of the phenomenon of *negative eddy diffusivity* [282].

This approach was applied to determine eddy diffusivity of two- [168, 207, 208, 271, 272] and three-dimensional flows [82,167, 312, 313], and it was shown that in certain parameter regions they do possess *negative eddy viscosity*. Its presence was demonstrated directly (i.e., without performing a preliminary asymptotic analysis) in a numerical study of stability of planar flows to three-dimensional large-scale perturbations [193]. In passive scalar admixture transport, eddy correction of diffusion can only enhance molecular diffusion [21, 175, 304]. Similar asymptotic expansions predict the possibility of generation of short-scale magnetic field by parity-invariant flows by means of the mechanism of magnetic negative eddy diffusivity [161, 327, 335, 336], and linear instability to large-scale perturbations of some three-dimensional parity-invariant space-periodic steady MHD states [326]. Linear stability of non-rotating convective hydromagnetic steady states, symmetric about a vertical axis, in a horizontal plane layer was considered in [13] and the equations for large-scale modes were derived.

[1] We use the modifier “combined” in definitions of eddy effects to emphasize that there is interaction of magnetic and flow velocity perturbations, and as a result each of the two perturbations is affected by them both. This mutual influence is described by operators of a similar structure, which identifies the kind of a combined eddy effect.

Near the bifurcation of the change of sign of eddy viscosity in the Kolmogorov flow, the weakly nonlinear regime satisfies an equation of the Cahn–Hilliard type with a cubic nonlinearity [195, 267]. The mean "effective" equations were applied in [84] to study the inverse energy cascade in the Kolmogorov flow. Mean-field equations for weakly nonlinear perturbations of two-dimensional steady flows represented in the terms of the stream function were explored in [92, 113]. A new additive term was introduced into the Navier–Stokes equation in [92] to accommodate the β-effect (Rossby waves), arising due to the assumed rotation of fluid, and the case of a small negative eddy viscosity was studied. Equations for perturbations of steady convective flows in a layer of fluid were considered in [72, 196]; they were derived by an analogous method, however, the derivation relied on model equations, approximately describing convective motion in the form of distorted rolls. A complete system of equations for the Boussinesq convection in a layer of fluid with rigid boundaries, formulated in the variables amplitude-phase, was considered in [197–200, 236]; a weakly nonlinear dynamics of a system of convective rolls and the emerging defects were investigated. In addition to the slow time and a slow horizontal variable, a slow phase was employed, for which a mean-field equation was derived. Magnetic field was not considered in the papers cited in this paragraph.

In this monograph we solve, in the order of increasing difficulty, several problems concerning stability of various MHD systems to large-scale perturbations, in which the α-effect is insignificant in the leading order and eddy diffusion is present: the kinematic dynamo problem for space-periodic parity-invariant steady (Chap. 3) and periodic in time (Chap. 4) three-dimensional flows, as well as for convective plan forms in a horizontal layer (Chap. 5); the linear (Chap. 6) and weakly nonlinear (Chap. 7) stability problems for MHD systems in the three-dimensional space; the weakly nonlinear stability problem for a magnetic dynamo driven by forced (Chap. 8) and free (Chap. 9) thermal convection in a plane layer of fluid rotating about the vertical axis. In the last two chapters we revisit (on a more formal mathematical basis) the kinematic dynamo problem, where in the leading order the magnetic α-effect acts simultaneously with molecular magnetic diffusion; we consider generation by a flow in an axisymmetric volume with the velocity depending on the azimuthal fast variable (Chap. 10) and by a flow in a sphere depending on the three spherical fast variables (Chap. 11). The presentation starts with a discussion of the passive scalar admixture transport problem (Chap. 2), which is not directly in the scope of our main interest, the MHD stability; it is included for pedagogical reasons, since it provides a convenient opportunity to expose the general ideas behind the multiscale formalism, which we use throughout in this monograph.

It was assumed in the study of linear stability of three-dimensional MHD states to large-scale perturbations [326], that the states are steady, space-periodic and parity-invariant; in the study of weakly nonlinear stability of three-dimensional parity-invariant MHD states [328] the assumptions that they are steady and space-periodic were abandoned. In the study of linear stability of three-dimensional convective hydromagnetic states [13], they were assumed to be steady, periodic in horizontal directions and symmetric about a vertical axis. If stability of unsteady MHD states is examined, averaging over the entire spatio-temporal domain of fast variables must be performed when considering solvability conditions in the course of construction of mean-field equations. Thus, the eddy tensors have constant coefficients (which are, in particular, independent of fast time; hence, on the mathematical grounds, the study of the dependence of eddy viscosity on time [112] seems insufficiently motivated). In the study of weakly nonlinear stability of three-dimensional convective hydromagnetic states in a layer [329–331], the assumptions that they are steady, periodic and symmetric were abandoned in favour of more general ones: it was supposed that the spatio-temporal averaging is possible (i.e., the perturbed states are homogeneous) and that the α-effect is insignificant in the leading order. Although the conditions of spatial periodicity and steadiness are convenient for computations, they had to be weakened for the mean-field equations to be applicable for the study of stability of such regimes as, for instance, chaotic spiral defects developing in the course of evolution of regular structures of thermal convection (see Fig. 5b, d, f in [26]).

Mathematically speaking, mean-field equations for large-scale weakly nonlinear perturbations of MHD states, lacking the α-effect significant in the leading order, are generalisations of the standard equations of magnetohydrodynamics. They involve some additional terms: a linear operator of combined eddy correction of magnetic diffusion and kinematic viscosity, and a quadratic operator of combined eddy correction of advection [328–331] (also, the method [327] for efficient computation of coefficients of the eddy diffusion and advection tensors, employing solutions to *auxiliary problems for the adjoint operator*, is generalised in [328–330]). Unsteadiness of the perturbed MHD state in the absence of symmetries, guaranteeing insignificance of the α-effect in the leading order, in general results in emergence of pseudodifferential operators in the mean-field equations.

Weakly nonlinear stability of magnetic field generation by forced Rayleigh–Bénard convection in a horizontal layer of fluid, rotating about a vertical axis, was considered in [329, 330]. (Convection is said to be forced, if some specified external forces and/or sources of magnetic field or heat are present inside the layer or on its boundary, which breaks the spatio-temporal invariance of the convective hydromagnetic system.) This problem, more natural for geo- or astrophysical applications than those discussed above, is algebraically more difficult. The standard method for derivation of mean-field equations for perturbations relies on the fact that there are constant vector fields in the kernel of the operator, adjoint to the linearisation of the equations for the evolution of the MHD system around the state, whose stability is considered. Mean-field equations represent the solvability condition for the equations in fast variables: by the Fredholm alternative theorem the condition consists of orthogonality of the non-homogeneous parts of the equations to the kernel of the adjoint operator, which in this case is equivalent to vanishing of their mean. (Vector fields from the kernel of the adjoint operator must satisfy the boundary conditions for fields in the domain of this operator; thus, the mean-field equations and, in particular, their number depend on the imposed boundary conditions. We consider convection in a layer with stress-free electrically conducting boundaries held at constant temperatures; then the kernel of the adjoint operator has the maximum subspace of constant vector fields, and the mean-field equations that we derive have the richest possible structure.) Rotation of fluid is represented by the Coriolis force in the Navier–Stokes equation; consequently, constant horizontal vector fields are present in the kernel of the adjoint operator, if an unbounded linear growth of pressure in horizontal directions is acceptable (so far this is just the question of choice of domains of the operator of linearisation and its adjoint). However, when the horizontal components of the Navier–Stokes equation are averaged, this results in emergence of surface integrals, which cannot be interpreted as a differential (or pseudodifferential) operator in slow variables, acting on the averaged perturbation. To overcome this difficulty, it is convenient to consider the curl of the Navier–Stokes equation (which we call the Navier–Stokes equation for vorticity). Then the equation for the mean flow perturbation is obtained as the solvability condition for order ε^3 equations, and not ε^2, as usual.

If convective hydromagnetic states are asymptotically close to a symmetric one, e.g., if a bifurcation with a loss of symmetry occurs in the MHD system and the difference between the bifurcation parameter value and the critical one is of the order of ε^2, then a term describing the combined MHD α-effect emerges in the mean-field equations despite the α-effect is insignificant in the leading order [330]. The mean-field equations are then supplemented by equations, in general involving a cubic nonlinearity, for amplitudes of short-scale zero-mean neutral (steady and oscillatory) eigenmodes of the linearisation of thermal hydromagnetic convection, which is responsible for occurrence of the bifurcation.

Equations for perturbations of magnetic field generation by free Rayleigh–Bénard convection in a horizontal layer of fluid, rotating about a vertical axis, were derived in [331]. Because of the absence of external forces and of sources of magnetic field and heat, this MHD system possesses the spatial and temporal invariance, which implies a modification of the structure of the kernel of the operator adjoint to the linearisation, in particular, its dimension increases. While the mean magnetic field perturbation is $\mathrm{O}(\varepsilon)$, the mean flow perturbation is at most $\mathrm{O}(\varepsilon^2)$, and when stability of a steady convective hydromagnetic state symmetric about a vertical axis is considered, it is $\mathrm{O}(\varepsilon^3)$. If the α-effect is insignificant in the leading order, the evolution of the perturbation is governed by a closed system comprised of 4 (when the perturbed state is steady) or 5 (when it is evolutionary) equations for amplitudes of neutral short-scale stability modes. The system is mixed: while equations for the mean magnetic field perturbation are evolutionary, the remaining ones involve neither derivatives in the slow time, nor the operators of molecular diffusion. If the convective hydromagnetic state, whose stability is investigated, possesses the symmetry (which guarantees insignificance of the α-effect in the leading order), then one of the amplitude equations is the solenoidality (in the slow horizontal variables) condition for the leading term of the mean flow, which is a partial differential equation of the third order with a cubic nonlinearity.

All chapters can be read independently, except for Chaps. 9 and 11, which rely on the material presented in Chaps. 8 and 10, respectively.

1.1 Notation

In Chaps. 2–9, we denote by $\mathbf{x}$ and $\mathbf{X} = \varepsilon\mathbf{x}$ the fast and slow spatial variables, respectively, and by t and $T = \varepsilon^s t$ the fast and slow time. In Chaps. 10–11, it proves more natural to denote by $\mathbf{x}$ the slow variables. Indices $\mathbf{x}$ and $\mathbf{X}$ in differential operators imply differentiation in the respective, fast or slow, spatial variables. $\langle\cdot\rangle$ denotes the spatial mean of a vector or a scalar field (over fast spatial variables), $\langle\!\langle\cdot\rangle\!\rangle$ spatio-temporal mean (over fast spatial variables and fast time), $\{\cdot\}$ and $\{\{\cdot\}\}$ the respective fluctuating parts of the fields.

As usual, vector quantities are printed in bold. Scalar components of vectors are enumerated by the last superscript, e.g., $(S_k(\eta))_m$ denotes the mth component of the vector field $\mathbf{S}_k(\eta)$; $\langle\mathbf{h}_0\rangle_k$ and $\langle\!\langle\mathbf{h}_0\rangle\!\rangle_k$ denote the kth components of the vector fields $\langle\mathbf{h}_0\rangle$ and $\langle\!\langle\mathbf{h}_0\rangle\!\rangle$, respectively. δ_k^j denotes the Kronecker symbol.

∇^2 denotes the Laplace operator and ∇^{-2} its inverse. $\mathscr{C}$ stands for the operator of the inverse curl, which maps a solenoidal zero-mean (in space) vector field to its solenoidal zero-mean vector potential, satisfying boundary conditions independently detailed in each chapter. $\mathscr{I}$ is the identity operator. In each chapter, $\mathscr{A}$ and $\mathscr{E}$ denote the α-effect and eddy diffusion operators, respectively, emerging in the large-scale stability problem under consideration in this chapter (as a consequence, the operators $\mathscr{E}$ introduced in different chapters are different, as are the operators $\mathscr{A}$).

We employ the following standard spaces of functions defined in region Ω: the space $\mathbb{C}^m(\Omega)$ of m times continuously differentiable functions; the Sobolev space $\mathbb{W}_2^m(\Omega)$ of m times differentiable functions; the Lebesgue space $\mathbb{L}_2(\Omega) = \mathbb{W}_2^0(\Omega)$. For $\Omega = [0, \mathbf{L}]$, the respective spaces of space-periodic functions are meant, $\mathbf{L} = (L_1, L_2, L_3)$ denoting the vector of periods in Cartesian spatial variables x_i and $[0, \mathbf{L}]$ the elementary parallelepiped of periodicity. The norm in the Sobolev space $\mathbb{W}_2^s(\Omega)$ is denoted by $|\cdot|_s$ (or by $|\cdot|_{s,\Omega}$, when Sobolev spaces of functions in different domains Ω are considered simultaneously). $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the n-dimensional real and complex spaces, respectively.

By $[\cdot]_{\partial\Omega}$ we denote the jump of a function or a vector field across the boundary $\partial\Omega$ of region Ω.

Chapter 2
Large-Scale Passive Scalar Transport

In this chapter we consider the so-called problem of passive scalar admixture transport. Instead of thinking of an abstract "admixture", the reader can imagine any physical quantity embedded into the volume of fluid and transported by fluid particles—for instance, its temperature or salinity. This problem, strictly speaking, is outside the area of the MHD stability theory, which is the central topic of the present book. The reason of its discussion is mainly pedagogical: involving a scalar quantity (as opposed to vector ones to be considered in the following chapters), this problem probably represents the simplest "real-life" example, which can be used for the introduction of the notions and asymptotic techniques employed in the analysis of MHD multiscale systems. The material of this chapter is inspired by the papers [21, 304]

2.1 The Linear Stability Problem

Let us briefly remind the concept of linear stability. Let a process, described by a scalar or a vector variable $\theta(\mathbf{x},t)$, be governed by the evolutionary equation

$$\frac{\partial\theta}{\partial t}=\mathscr{F}(\theta,\mathbf{x},t), \tag{2.1}$$

where $\mathbf{x}$ is the position in space, t time, and $\mathscr{F}$ can be a function, or an operator involving partial derivatives in $\mathbf{x}$. The theory of linear stability concerns the following question: suppose, we start monitoring the system at $t=0$ from an initial state $\theta(\mathbf{x},0)$, and this yields an evolution $\theta(\mathbf{x},t)$, or we start from a perturbed state $\theta(\mathbf{x},0)+\delta\theta(\mathbf{x},0)$ resulting in the evolution $\theta(\mathbf{x},t)+\delta\theta(\mathbf{x},t)$. Let initially the two states be infinitesimally close: $|\delta\theta(\mathbf{x},0)|\ll|\theta(\mathbf{x},0)|$. Will they remain close to each other, or will they diverge as the time is passing by?

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Both solutions, $\theta(\mathbf{x}, t)$ and $\theta(\mathbf{x}, t) + \delta\theta(\mathbf{x}, t)$, satisfy (2.1). Hence, subtracting the two respective instances of Eq. 2.1 we obtain an equation for the perturbation $\delta\theta(\mathbf{x}, t)$ of the state θ:

$$\frac{\partial}{\partial t}\delta\theta = \mathscr{F}(\theta + \delta\theta, \mathbf{x}, t) - \mathscr{F}(\theta, \mathbf{x}, t). \tag{2.2}$$

We now recall that the difference $\delta\theta$ between the two solutions is small; therefore, we can linearise (hence we are speaking of *linear* stability) the r.h.s. of (2.2) (using Taylor's expansion) and consider a simplified equation for $\delta\theta$,

$$\frac{\partial}{\partial t}\delta\theta = \mathscr{M}\delta\theta, \tag{2.3}$$

where the operator $\mathscr{M}$ is *a linearisation* of $\mathscr{F}$ around the solution $\theta(\mathbf{x}, t)$ (also called the Frechet derivative of $\mathscr{F}$, if $\mathscr{F}$ is an operator). Evidently, in general (2.3) is not any more a precise equation, unless $\mathscr{F}$ is linear in θ.

Suppose now we consider stability of a steady state $\theta(\mathbf{x})$ and hence the linear operator $\mathscr{M}$ does not explicitly involve time. Then (2.3) has solutions of the form $\delta\theta = q(\mathbf{x})\exp(\lambda t)$, substituting which into (2.3) we find

$$\lambda q(\mathbf{x}) = \mathscr{M} q.$$

In other words, λ is an eigenvalue and $q(\mathbf{x})$ the associated eigenfunction of the linearisation $\mathscr{M}$. If $\mathscr{M}$ is an elliptic operator and all its eigenvalues have strictly negative real parts, then any small perturbation $\delta\theta(\mathbf{x}, t)$ eventually exponentially decays in time (after, perhaps, an initial phase during which the transient behaviour settles down); therefore, the state $\theta(\mathbf{x})$ is *stable*. If there exists an eigenvalue of the operator $\mathscr{M}$ with a strictly positive real part, then "almost" any perturbation $\delta\theta(\mathbf{x}, t)$ exponentially grows in time, i.e. $\theta(\mathbf{x})$ is *unstable*.

2.1.1 The Governing Equations

Let us denote the concentration of a scalar admixture at time t in the point $\mathbf{x}$ by $\theta(\mathbf{x}, t)$. The evolution of the concentration in a volume of fluid moving with the velocity $\mathbf{V}(\mathbf{x}, t)$ is described by the parabolic equation of passive scalar transport

$$\frac{\partial\theta}{\partial t} + (\mathbf{V} \cdot \nabla)\theta = \mu\nabla^2\theta + f(\mathbf{x}, t) \tag{2.4}$$

In this equation, the term $\mu\nabla^2\theta$ describes how concentration θ changes in time due to diffusion (for instance, due to the Brownian motion of molecules of the liquid solvent and admixture; consistently with this, $\mu > 0$ is called *molecular diffusivity of a passive scalar*). The term $(\mathbf{V} \cdot \nabla)\theta$ describes, how concentration changes due to advection of the admixture by the flow $\mathbf{V}$. The admixture is called a *passive* scalar, since its presence does not affect the flow—$\mathbf{V}$ is not supposed to depend on θ. Finally, f represents sources ($f > 0$) or sinks ($f < 0$) of the admixture.

To study linear stability of a solution $\theta(\mathbf{x}, t)$ to Eq. 2.4, we consider Eq. 2.2 for an infinitesimal perturbation $\delta\theta$ of θ. Due to linearity of (2.2), the equation for a perturbation takes the form

$$\frac{\partial}{\partial t}\delta\theta = \mu\nabla^2\delta\theta - (\mathbf{V}\cdot\nabla)\delta\theta. \tag{2.5}$$

Its r.h.s. defines the so-called *operator of passive scalar transport*, which we denote in this chapter $\mathscr{M}$. Note that it does not involve the perturbed state θ.

In this chapter, we will consider transport by a steady zero-mean flow $\mathbf{V}(\mathbf{x})$ of incompressible fluid:

$$\nabla\cdot\mathbf{V} = 0, \tag{2.6}$$

$$\langle\mathbf{V}\rangle = 0. \tag{2.7}$$

The flow is supposed to have periods L_i in Cartesian variables x_i, i.e. to be "**L**-periodic" (where $\mathbf{L} = (L_1, L_2, L_3)$). We have employed in (2.7) the notation $\langle\cdot\rangle$ for the spatial mean:

$$\langle f\rangle = \frac{1}{L_1 L_2 L_3}\int\limits_{[0,\mathbf{L}]} f(\mathbf{x})\mathrm{d}\mathbf{x}.$$

Note that if $\langle\mathbf{V}\rangle \neq 0$, by Galilean invariance of the problem condition (2.7) can be satisfied by considering the problem in the coordinate system, co-moving with the velocity $\langle\mathbf{V}\rangle$.

As explained above, for steady flows the problem (2.5) reduces to the eigenvalue problem for the operator of passive scalar transport:

$$\mathscr{M}q = \lambda q. \tag{2.8}$$

The scalar eigenfunction $q(\mathbf{x})$ is called a *stability mode*.

2.1.2 Asymptotic Expansion of a Large-Scale Stability Mode

We consider in this chapter *large-scale stability modes* $q(\mathbf{X}, \mathbf{x})$, depending on the so-called *fast*, $\mathbf{x}$, and *slow* spatial variables, $\mathbf{X} = \varepsilon\mathbf{x}$. The *scale ratio* $\varepsilon > 0$ is a small parameter. This explains the terminology: any change $\delta\mathbf{x}$ in the fast variable $\mathbf{x}$ is accompanied by the change $\delta\mathbf{X} = \varepsilon\,\delta\mathbf{x}$ in the slow variable $\mathbf{X}$; thus, a fast change of $\mathbf{x}$ results in a slow change of $\mathbf{X}$. Such splitting of the dependence on the spatial variables into separate dependencies on the fast and slow spatial variables is a cornerstone of multiscale methods. Note that the flow $\mathbf{V}$ depends only on the fast variables. Stability modes depending only on the fast variables are called *short-scale modes*.

We will use the notation $\langle\cdot\rangle$ and $\{\cdot\}$ for the mean over a periodicity cell of the fast variables (the rectangular parallelepiped $[0, \mathbf{L}]$), also called the *spatial mean*, and the fluctuating part of a vector or a scalar field, respectively:

$$\langle f(\mathbf{X}, \mathbf{x})\rangle \equiv \frac{1}{L_1 L_2 L_3} \int\limits_{[0,\mathbf{L}]} f(\mathbf{X}, \mathbf{x})\mathrm{d}\mathbf{x}, \quad \{\mathrm{f}\} \equiv \mathrm{f} - \langle \mathrm{f}\rangle.$$

We will consider the eigenvalue problem for the partial differential operator $\mathscr{M}$, and we must supplement appropriate boundary conditions. For simplicity, we assume that the eigenfunction $q(\mathbf{X}, \mathbf{x})$ is $\mathbf{L}$-periodic in the fast variables (i.e., has the same periodicity in $\mathbf{x}$ as the flow $\mathbf{V}(\mathbf{x})$).

Since q depends on the fast and slow variables, by the chain rule differentiation in the operator of passive scalar transport is modified:

$$\nabla \to \nabla_{\mathbf{x}} + \varepsilon \nabla_{\mathbf{X}} \tag{2.9}$$

(the indices $\mathbf{x}$ and $\mathbf{X}$ denote differentiation in the respective variables). This results in emergence of new terms in the eigenvalue equation (2.8) with prefactors ε and ε^2. Consequently, a solution to the eigenvalue problem (2.8) is sought as formal asymptotic power series in the scale ratio,

$$q = \sum_{n=0}^{\infty} q_n(\mathbf{X}, \mathbf{x})\varepsilon^n, \tag{2.10}$$

$$\lambda = \sum_{n=0}^{\infty} \lambda_n \varepsilon^n. \tag{2.11}$$

2.2 The Kernel of the Passive Scalar Transport Operator and Solvability of Auxiliary Problems

Let us denote by $\mathscr{L}'$ the restriction of the passive scalar transport operator $\mathscr{M}$ to the subspace of scalar fields, $\mathbf{L}$-periodic in the fast spatial variables:

$$\mathscr{L}' q \equiv \mu \nabla_{\mathbf{x}}^2 q - (\mathbf{V} \cdot \nabla_{\mathbf{x}}) q.$$

Evidently, the kernel of $\mathscr{L}'$ is at least one-dimensional, since $\mathscr{L}' C = 0$ for any constant C. Modes from the kernel of the operator of linearisation are called *neutral short-scale modes*.

We define $\mathscr{L}$ as the restriction of the operator $\mathscr{L}'$ to the subspace of $\mathbf{L}$-periodic zero-mean scalar fields. Due to solenoidality of the flow (2.6), $\mathscr{L}' q$ is zero-mean for any $\mathbf{L}$-periodic scalar field q. Thus, $\mathscr{L}$ maps fields from this subspace into the same subspace.

2.2.1 Stability of Passive Scalar Admixture Transport

The operator $\mathscr{L}$ has a trivial ("empty") kernel. To show this, we multiply the eigenvalue equation

$$\mathscr{L}q = \lambda q$$

by the scalar field $\bar{q}(\mathbf{x})$ complex conjugate to the eigenfunction associated with an eigenvalue λ, and integrate over the parallelepiped of periodicity $\mathbf{L}$. Due to solenoidality of the flow, the real part of the resultant equation reduces to

$$-\mu \int\limits_{[0,\mathbf{L}]} |\nabla_{\mathbf{x}} q|^2 d\mathbf{x} = \mathrm{Re}\,\lambda \int\limits_{[0,\mathbf{L}]} |\mathrm{q}|^2 d\mathbf{x}. \tag{2.12}$$

Thus, the real part of any eigenvalue of the operator $\mathscr{L}$ is non-positive. Furthermore, if eigenvalue λ is imaginary, integral identity (2.11) implies that q is a constant, and hence it is not in the domain of $\mathscr{L}$.

Relation (2.12) implies that passive scalar admixture transport is linearly stable to short-scale perturbations (i.e. to perturbations, having the same parallelepiped of periodicity $\mathbf{L}$ as the velocity). Assuming now that the eigenfunction q has the vector of periods $n\mathbf{L} = (nL_1, nL_2, nL_3)$, where $n > 0$ is an arbitrary integer, and performing integration of the premultiplied eigenvalue equation over the parallelepiped $[0, n\mathbf{L}]$, we obtain a relation similar to (2.12). It implies linear stability of scalar transport to large-scale perturbations with periods commensurate with periods of the flow velocity.

Since $\mathscr{L}$ is an elliptic operator, the triviality of the kernel implies that the linearisation is invertible in the subspace of $\mathbf{L}$-periodic zero-mean fields, and the inverse operator is bounded. In other words, a problem $\mathscr{M}q = f$ is solvable and the solution is unique, provided q and f are $\mathbf{L}$-periodic and zero-mean.

2.2.2 The Hierarchy of Equations for a Large-Scale Mode

The fact, that constants and zero-mean vectors are invariant subspaces for $\mathscr{L}'$, suggests to split the terms of the series (2.10) for a stability mode scalar transport into the mean and fluctuating parts:

$$q = \sum_{n=0}^{\infty} (\langle q_n \rangle + \{q_n\})\varepsilon^n. \tag{2.13}$$

Substitution of the modified gradient (2.9) and the series (2.13) and (2.11) into the eigenvalue equation (2.8) yields

$$\sum_{n=0}^{\infty} \varepsilon^n \left(\mathscr{L}\{q_n\} + \mu(2(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\{q_{n-1}\} + \nabla_{\mathbf{X}}^2 q_{n-2}) - (\mathbf{V} \cdot \nabla_{\mathbf{X}}) q_{n-1} - \sum_{m=0}^{n} \lambda_{n-m} q_m \right) = 0\,. \tag{2.14}$$

Equating to zero each term of the series in the l.h.s. of (2.14), we obtain a hierarchy of equations. We will show in subsequent sections that they can be successively solved in $\{q_n\}$. This will uniquely define all terms in the asymptotic series (2.10) and (2.11).

The equations in the hierarchy will be considered at successive orders ε^n in two steps:

1°. Satisfy the solvability condition for the equation under consideration;
2°. Solve in q_n the resultant partial differential equation in the fast spatial variables.This simple two-step plan serves as a prototype plan for solution of all stability problems in subsequent chapters. For reader's convenience, at each stage we will indicate, which step for which n is implemented.

2.2.3 *Solvability of the Order ε^0 Equation*

In this section we begin the calculations outlined above. In the leading order ε^0, we obtain from (2.14) the first equation in the hierarchy:

$$\mathscr{L}\{q_0\} = \lambda_0 q_0. \tag{2.15}$$

Step 1° for $n = 0$. We must verify the solvability condition for (2.15): the r.h.s. must be zero-mean. This condition reduces to $\lambda_0 \langle q_0 \rangle = 0$; hence either $\lambda_0 = 0$, or $\langle q_0 \rangle = 0$. It is possible to construct the complete asymptotic expansions (2.10) and (2.11) in both cases. However, such a construction would be of a limited interest only, unless Re $\lambda_0 = 0$: For $\langle q_0 \rangle = 0$, (2.15) is an eigenvalue equation for a short-scale stability mode $q_0(\mathbf{x})$. Stability with respect to both modes, the large-scale one and the small-scale $q_0(\mathbf{x})$, is then determined by the sign of the real part of λ_0. Thus, if Re $\lambda_0 \neq 0$, construction of the large-scale mode hardly provides any new information on linear stability of passive scalar transport. Since the operator of passive scalar transport, $\mathscr{L}$, does not possess imaginary eigenvalues (see Sect. 2.1), the only case of interest is $\lambda_0 = 0$.

2.3 Perturbation of the Neutral Mode

In this section we will consider the problem for $\lambda_0 = 0$, which guarantees solvability of Eq. 2.15.

2.3.1 Solution of the Order ε^0 and ε^1 Equations

Step 2° for $n = 0$. For $\lambda_0 = 0$, equation (2.15) takes the form $\mathscr{L}\{q_0\} = 0$, i.e. the zero-mean field $\{q_0\}$ belongs to the kernel of $\mathscr{L}$, which is trivial (as shown in Sect. 2.1); hence $\{q_0\} = 0$. The leading term in the expansion (2.10) of the large-scale stability mode is supposed to be $\mathrm{O}(\varepsilon^0)$ (since modes are defined up to an arbitrary factor, this can be enforced by the appropriate normalisation of the mode). Thus, $\{q_0\} = 0$ implies $\langle q_0\rangle \neq 0$.

For $\lambda_0 = 0$ and $\{q_0\} = 0$, the equation obtained from (2.14) at order ε^1 reduces to

$$\mathscr{L}\{q_1\} = (\mathbf{V}\cdot\nabla_{\mathbf{X}})\langle q_0\rangle + \lambda_1\langle q_0\rangle. \tag{2.16}$$

Step 1° for $n = 1$. By virtue of (2.7), the first term in the r.h.s. of (2.16) is zero-mean. Since $\langle q_0\rangle \neq 0$, the spatial mean of the r.h.s. of (2.16) vanishes as long as $\lambda_1 = 0$.

Step 2° for $n = 1$. $\mathscr{L}$ is a linear partial differential operator in the fast spatial variables; $\langle q_0\rangle$ does not depend on them. Hence, by linearity, the structure of the r.h.s. of (2.16) implies

$$\{q_1\} = \sum_{k=1}^{3}\frac{\partial\langle q_0\rangle}{\partial X_k}S_k(\mathbf{x}), \tag{2.17}$$

where scalar fields $S_k(\mathbf{x})$ are zero-mean solutions to the so-called auxiliary problems:

$$\mathscr{L}S_k = V_k. \tag{2.18}$$

(By subscripts we will denote individual Cartesian components of vector quantities.) The solutions are uniquely defined, since the spatial mean of the r.h.s. of (2.18) is zero (see 2.7).

2.3.2 Eddy Diffusion

The operator of eddy diffusion emerges as the solvability condition of the next, ε^2, order equation

$$\mathscr{L}\{q_2\} = -\mu\big(2(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\{q_1\} + \nabla^2_{\mathbf{X}}\langle q_0\rangle\big) + (\mathbf{V}\cdot\nabla_{\mathbf{X}})q_1 + \lambda_2\langle q_0\rangle. \tag{2.19}$$

Step 1° for $n = 2$. Since the flow is zero-mean (2.7) and the spatial mean of a derivative of any $\mathbf{L}$-periodic field is zero, the solvability condition for equation (2.19) reduces to

$$-\mu\nabla^2_{\mathbf{X}}\langle q_0\rangle + \langle(\mathbf{V}\cdot\nabla_{\mathbf{X}})\{q_1\}\rangle + \lambda_2\langle q_0\rangle = 0.$$

Upon substitution of the solution (2.17) into the order ε^1 equation, this becomes

$$\mu\nabla^2_{\mathbf{X}}\langle q_0\rangle - \sum_{k=1}^{3}\sum_{m=1}^{3}\langle V_m S_k\rangle\frac{\partial^2\langle q_0\rangle}{\partial X_k\partial X_m} = \lambda_2\langle q_0\rangle. \tag{2.20}$$

One determines from this eigenvalue equation the leading terms of the expansions (2.10) and (2.11)—the averaged passive scalar transport stability mode, $\langle q_0\rangle$, and the associated eigenvalue, λ_2 [the fluctuating part of the leading term in the series (2.10) is then determined by expression (2.17)]. Equations for mean profiles of perturbations similar to (2.20) we call *mean-field* equations. In the leading order the mean large-scale mode $\langle q_0\rangle$ is an eigenfunction of the operator defined by the l.h.s. of (2.20). Since it involves only second-order derivatives in spatial variables, it can be interpreted as the *operator of eddy diffusion*, generically anisotropic. It is a sum of the molecular diffusion operator $\mu\nabla^2_{\mathbf{X}}$ and an operator called *eddy correction of diffusion of passive scalar*.

2.3.3 Enhancement of Molecular Diffusion

To solve the eigenvalue problem (2.20), we must specify boundary conditions for the mean profile $\langle q_0\rangle$—e.g., periodicity in the slow variables $\mathbf{X}$ (note that the assumed periodicity of $q_0(\mathbf{X},\mathbf{x})$ in the fast variables $\mathbf{x}$ together with the periodicity in $\mathbf{X}$ does not imply space-periodicity of q_0); alternatively, the problem can be considered in a finite region of the slow variables and homogeneous Dirichlet or Neumann conditions for $\langle q_0\rangle$ at the boundary can be prescribed. Interestingly, for such boundary conditions the operator of eddy diffusion is sign-definite.

To check this, note

$$\sum_{k=1}^{3}\sum_{m=1}^{3}\langle V_m S_k\rangle\frac{\partial^2\langle q_0\rangle}{\partial X_k\partial X_m} = \frac{1}{2}\sum_{k=1}^{3}\sum_{m=1}^{3}\langle V_m S_k + V_k S_m\rangle\frac{\partial^2\langle q_0\rangle}{\partial X_k\partial X_m}.$$

Using the statements of the auxiliary problems (2.18), we find

$$\begin{aligned}\frac{1}{2}\langle V_m S_k + V_k S_m\rangle &= \frac{1}{2}\langle S_k\mathscr{L}S_m + S_m\mathscr{L}S_k\rangle\\ &= -\mu\langle\nabla_{\mathbf{x}}S_k\cdot\nabla_{\mathbf{x}}S_m\rangle - \frac{1}{2}\langle S_k(\mathbf{V}\cdot\nabla_{\mathbf{x}})S_m + S_m(\mathbf{V}\cdot\nabla_{\mathbf{x}})S_k\rangle\\ &= -\mu\langle\nabla_{\mathbf{x}}S_k\cdot\nabla_{\mathbf{x}}S_m\rangle.\end{aligned}$$

Hence, multiplying (2.20) by $\langle\overline{q_0}\rangle$ (where the bar means complex conjugation) and integrating over the considered region in the slow variables (or over a periodicity cell, if periodicity in the slow variables is assumed) we find

$$\lambda_2\int|\langle q_0\rangle|^2\mathrm{d}\mathbf{X} = -\mu\int\left(|\nabla_{\mathbf{X}}\langle q_0\rangle|^2 + \left\langle\left|\nabla_{\mathbf{x}}\sum_{k=1}^{3}S_k\frac{\partial\langle q_0\rangle}{\partial X_k}\right|^2\right\rangle\right)\mathrm{d}\mathbf{X}.$$

This identity establishes that all eigenvalues of the operator of eddy diffusion are real and negative. The contribution of the eddy diffusion correction is non-positive, i.e., in passive scalar transport eddy correction of diffusion arising due to small-scale motion of fluid enhances molecular diffusion.

We can continue to implement the two-step plan and recover successively all terms in the asymptotic series (2.10) and (2.11) However, we have already reached our main goal—to derive the limit eddy diffusion operator,—and we will not perform the complete construction of the asymptotic series for the problem under consideration. The reader can do this by analogy with a similar construction carried out, for instance, for the magnetic kinematic dynamo problem considered in the next chapter.

2.4 Conclusions

1. This chapter presents a preliminary demonstration of the asymptotic techniques for multiscale problems, which will be applied in the following chapters to various MHD stability problems. We have considered the large-scale linear stability problem for passive scalar transport. Perturbations of neutral short-scale modes have been examined. We have derived equations for the leading terms in asymptotic expansions of eigenvalues (2.11) and the associated large-scale stability modes (2.10).
2. Large-scale perturbation of a constant, which is the short-scale neutral stability mode in the context of passive scalar transport, gives rise to operator (2.19) determining the leading terms in the expansions in the scale ratio of a large-scale stability mode and of the associated eigenvalue. This is a second-order partial differential operator in the slow variables, which we interpret as an eddy diffusion operator. It turns out that in the present problem eddy correction of diffusion always enhances molecular diffusion—the contribution from the eddy correction to eigenvalues associated with large-scale stability modes is always non-positive. We will see in subsequent chapters that this property is not typical for eddy diffusion operators in kinematic dynamo and MHD stability problems.

Chapter 3
Steady-Flow Dynamos

In this chapter we study large-scale magnetic modes generated by short-scale steady space-periodic parity-invariant flows of conducting fluid. We construct complete asymptotic expansions of the modes and their growth rates in power series in the scale ratio, and derive a closed set of equations for their leading terms. Computations show that a significant part of such flows can generate magnetic field by the mechanism of negative magnetic eddy diffusivity for magnetic Reynolds numbers below the critical value R_m^c for the onset of generation of the short-scale magnetic field. We present examples of flows exhibiting an anomalously large (in absolute value) negative magnetic eddy diffusivity and demonstrate that this phenomenon occurs when the magnetic Reynolds number is close to R_m^c. Finally, we show by direct simulations that even a modest scale separation, with the large scale twice larger than the small one, is beneficial for magnetic field generation (the concept of eddy diffusion is not yet applicable in this case—such a weak scale separation is outside the region of asymptotic behaviour of magnetic modes). The presentation in this chapter is based on the paper [336].

Let us briefly consider implications of our results to experimental dynamos. The estimations [87, 336] illustrate, why it is important to maintain high scale separation. Since in experiments flows of molten metal are turbulent, the rate of energy dissipation in the flow, P, behaves as $P \sim \rho(V^3/\ell)L^3$ [91]; here ρ is the metal density, V the typical flow velocity, L the size of the container of the metal, $\ell = \varepsilon L$ the typical spatial scale of the flow, and ε the scale ratio. The global magnetic Reynolds number is $R_m = LV/\eta$, where $\eta > 0$ is magnetic molecular diffusivity, implying

$$P \sim \rho\eta^3 R_m^3/(\varepsilon L). \tag{3.1}$$

Generation is possible, if R_m exceeds the critical value (usually of the order of several hundred). Consequently, for the given metal and the degree of turbulisation of the flow in the experimental facility at hand (the quantities η, ε and L thus being prescribed), the pump power P sustaining the given value of R_m is proportional to R_m^3. Thus, it is desirable to lower the critical Reynolds number as much as possible.

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Our results suggest that usually it decreases together with the scale ratio ε, so scale separation can significantly help. However, the scale ratio is also present in the denominator in (3.1), acting oppositely; thus, the onset of magnetic field generation requires the minimum of the pump power needed to maintain the generating flow for a certain optimal scale separation [222].

We show that in the absence of the α-effect scale separation implies the presence of magnetic eddy diffusion. In particular, this happens, if the flow is parity-invariant. Although this condition significantly restricts the class of considered flows, such flows can be regarded as a realistic model of time-averaged turbulent ones: parity invariance is compatible with the Navier–Stokes equation, and although, as any symmetry, it can be broken by developing instabilities, the resultant hydrodynamic attractor can be expected to have it "on average" [91]. In the next chapter we will consider generation by space-periodic parity-invariant flows, which are time-periodic; they can be regarded as a more natural model of turbulent flows.

The evolutionary magnetic mean-field equation was derived in [161] and the first example of negative magnetic eddy diffusivity was obtained ibid. for the so-called modified Taylor–Green flow. In this chapter, we construct a concise asymptotic theory for the eigenvalue problem for the magnetic induction operator, and, unlike in [82, 161] for parity-invariant flows we determine all terms of the formal asymptotic expansions of large-scale magnetic modes and their growth rates.

3.1 Kinematic Magnetic Dynamos: The Statement of the Problem

3.1.1 The Eigenvalue Problem for the Magnetic Induction Operator

Temporal evolution of a magnetic field $\mathbf{h}(\mathbf{x},t)$ in a volume of conducting fluid with the velocity $\mathbf{V}(\mathbf{x},t)$ is described by the magnetic induction equation

$$\frac{\partial \mathbf{h}}{\partial t} = \eta \nabla^2 \mathbf{h} + \nabla \times (\mathbf{V} \times \mathbf{h}). \tag{3.2}$$

The r.h.s. of this equation defines the magnetic induction operator, $\mathscr{M}$. In this chapter, we consider magnetic field generation by a steady flow $\mathbf{V}(\mathbf{x})$. For such flows, the kinematic dynamo problem reduces to the eigenvalue problem for the magnetic induction operator,

$$\mathscr{M}\mathbf{h} = \lambda \mathbf{h}. \tag{3.3}$$

Here the vector-valued eigenfunction $\mathbf{h}(\mathbf{x})$ is called a *magnetic mode*, λ is the associated eigenvalue (whose real part is the *growth rate* of this mode; a mode is decaying, if $\mathrm{Re}\,\lambda < 0$). Magnetic modes are solenoidal,

$$\nabla \cdot \mathbf{h} = 0. \tag{3.4}$$

Flows of incompressible fluid are considered,

$$\nabla \cdot \mathbf{V} = 0, \tag{3.5}$$

they are supposed to be $\mathbf{L}$-periodic (i.e. have periods L_i in Cartesian variables $x_i, \mathbf{L} = (L_1, L_2, L_3)$) and, from Sect. 3.4 onwards (except for the first half of Sect. 3.8), parity-invariant (or, in other terminology, symmetric about the centre):

$$\mathbf{V}(\mathbf{x}) = -\mathbf{V}(-\mathbf{x}). \tag{3.6}$$

3.1.2 Asymptotic Expansion of Large-Scale Magnetic Modes

A *large-scale magnetic mode* $\mathbf{h}(\mathbf{X}, \mathbf{x})$ is assumed to depend on the *fast* spatial variables $\mathbf{x}$ and on the *slow* ones $\mathbf{X} = \varepsilon \mathbf{x}$, and to be $\mathbf{L}$-periodic in $\mathbf{x}$. (The flow velocity $\mathbf{V}$ is independent of the slow variables.) It is defined in the entire three-dimensional space, or in a bounded region of the slow variables. In the former case, it is supposed to be globally bounded; in the latter, the boundary conditions in the slow variables are discussed in Sect. 3.4.2), and they are inconsequential for constructions of the preceding sections. Differentiation in (3.3) and (3.4) obeys the chain rule:

$$\nabla \to \nabla_{\mathbf{x}} + \varepsilon \nabla_{\mathbf{X}} \tag{3.7}$$

(the indices $\mathbf{x}$ and $\mathbf{X}$ denote differentiation in the respective variables). The scale ratio ε of the slow and fast variables is a small parameter.

A solution to the eigenvalue problem (3.3)–(3.4) can be sought as a power series in the scale ratio,

$$\mathbf{h} = \sum_{n=0}^{\infty} \mathbf{h}_n(\mathbf{X}, \mathbf{x}) \varepsilon^n, \tag{3.8}$$

$$\lambda = \sum_{n=0}^{\infty} \lambda_n \varepsilon^n. \tag{3.9}$$

3.1.3 The Hierarchy of Equations for the Large-Scale Modes

Let $\langle \cdot \rangle$ and $\{\cdot\}$ denote, respectively, the mean over the periodicity cell of the fast variables (we will call it the "spatial mean") and the fluctuating part of a scalar or vector field:

$$\langle \mathbf{f} \rangle = \frac{1}{L_1 L_2 L_3} \int\limits_{[0,\mathbf{L}]} \mathbf{f}(\mathbf{X}, \mathbf{x})\, d\mathbf{x}, \quad \{\mathbf{f}\} = \mathbf{f} - \langle \mathbf{f} \rangle. \tag{3.10}$$

Substituting the power series (3.8) into the solenoidality condition (3.4) and taking into account (3.7), we represent the l.h.s. of the resultant equation as a power series in ε. Considering independently the mean and the fluctuating part of its coefficients, we obtain solenoidality conditions for a large-scale magnetic mode,

$$\nabla_{\mathbf{X}} \cdot \langle \mathbf{h}_n \rangle = 0, \tag{3.11}$$

$$\nabla_{\mathbf{x}} \cdot \{\mathbf{h}_n\} + \nabla_{\mathbf{X}} \cdot \{\mathbf{h}_{n-1}\} = 0 \tag{3.12}$$

holding for all $n \geq 0$. (By definition, $\mathbf{h}_n \equiv 0$ for $n < 0$.)

Let $\mathscr{L}$ denote the restriction of the magnetic induction operator $\mathscr{M}$ to the subspace of vector fields, which are $\mathbf{L}$-periodic in space and have a zero spatial mean. (Differentiation in the fast variables only is performed in the operator $\mathscr{L}$.) Substituting the series (3.8) and (3.9) into the eigenvalue equation (3.3) and applying (3.7), we obtain

$$\sum_{n=0}^{\infty} \varepsilon^n \Bigg(\mathscr{L}\{\mathbf{h}_n\} + \eta\big(2(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\{\mathbf{h}_{n-1}\} + \nabla^2_{\mathbf{X}} \mathbf{h}_{n-2}\big)$$
$$+ \nabla_{\mathbf{x}} \times (\mathbf{V} \times \langle \mathbf{h}_n \rangle) + \nabla_{\mathbf{X}} \times (\mathbf{V} \times \mathbf{h}_{n-1}) - \sum_{m=0}^{n} \lambda_{n-m} \mathbf{h}_m \Bigg) = 0. \tag{3.13}$$

Equating to zero the coefficients of the series in the l.h.s. of (3.13), we obtain a hierarchy of equations. We will show in subsequent sections that the mean parts of these equations play the rôle of their solvability conditions, and the fluctuating parts are elliptic partial differential equations in the fast spatial variables for $\{\mathbf{h}_n\}$; solving successively the equations together with the solenoidality conditions (3.12) one can find all terms of the series (3.8) and (3.9).

3.2 The Kernel of the Magnetic Induction Operator and Solvability of Auxiliary Problems

Define $\mathscr{L}'$ as a restriction of the magnetic induction operator $\mathscr{M}$ (where differentiation is in the fast variables only) to the subspace of vector fields (of any mean, unlike in the definition of $\mathscr{L}$), $\mathbf{L}$-periodic in the fast spatial variables. The operators, adjoint to $\mathscr{L}'$ and $\mathscr{L}$ are

$$\mathscr{L}'^{*}\mathbf{h} = \eta \nabla^2_{\mathbf{x}} \mathbf{h} + (\nabla_{\mathbf{x}} \times \mathbf{h}) \times \mathbf{V}, \quad \mathscr{L}^{*}\mathbf{h} = \{\mathscr{L}'^{*}\mathbf{h}\};$$

their domains coincide with the domains of $\mathscr{L}'$ and $\mathscr{L}$, respectively. Evidently, $\mathscr{L}'^*\mathbf{e}_k = 0$, where $\mathbf{e}_k$ denote unit vectors of the Cartesian coordinate system. Hence, the kernel of $\mathscr{L}'^*$ is at least three-dimensional, and $\dim\ker\mathscr{L}' = \dim\ker\mathscr{L}'^* \geq 3$; except for a countable set of values of $\eta > 0$, the dimension is three.

We will assume henceforth in this chapter, that the spectrum of $\mathscr{L}$ does not include zero, or, in other words, neutral short-scale zero-mean magnetic modes do not exist (a *neutral magnetic mode* is an eigenfunction from the kernel of the magnetic induction operator). This assumption is not strongly restrictive, since for a given flow $\mathbf{V}$ it is not satisfied only for a countable set of magnetic molecular diffusivities η.

Let us demonstrate that under this assumption the condition

$$\langle \mathbf{f} \rangle = 0 \tag{3.14}$$

is necessary and sufficient for existence and uniqueness of a solution to the problem $\mathscr{L}\mathbf{g} = \mathbf{f}$. Averaging of this equation over the fast spatial variables shows that the condition is necessary (or, alternatively, this is implied by the facts that $\mathbf{e}_k \in \ker\mathscr{L}'^*$ and that calculation of the scalar product of a vector field with $\mathbf{e}_k$ in the Lebesgue space $\mathbb{L}_2([0,\mathbf{L}])$ amounts to calculation of the mean of the kth component of this field). Suppose now $\langle \mathbf{f} \rangle = 0$ holds true. Then the original equation is equivalent to

$$(1/\eta)\nabla^{-2}\mathscr{L}\mathbf{g} = \nabla^{-2}\mathbf{f}/\eta \tag{3.15}$$

(here ∇^{-2} denotes the inverse operator to the Laplacian in the fast variables, acting in the subspace of $\mathbf{L}$-periodic zero-mean vector fields). The operator in the l.h.s. is a sum of the identity and a compact operator [294], and hence the Fredholm alternative theorem [154, 171] is applicable. The operator $(1/\eta)\mathscr{L}^*\nabla^{-2}$, whose domain coincides with the domain of ∇^{-2}, is adjoint to the operator in the l.h.s. of (3.15). Our assumption implies that its kernel is trivial; consequently, (3.15) has a unique solution in the subspace of $\mathbf{L}$-periodic zero-mean vector fields, as required.

In what follows, we will consider equations obtained from the series (3.13) at successive orders n in two steps:

1°. Consider the spatial mean of the equation and satisfy the solvability condition.
2°. Solve the resultant partial differential equation in the fast spatial variables.

3.3 Magnetic α-Effect

The equation, obtained from (3.13) at the leading order, after simple algebraic transformations (noting that the flow is solenoidal (3.5)) becomes

$$\mathscr{L}\{\mathbf{h}_0\} + (\langle \mathbf{h}_0 \rangle \cdot \nabla_{\mathbf{x}})\mathbf{V} = \lambda_0 \mathbf{h}_0. \tag{3.16}$$

3.3.1 Solution of the Order ε^0 Equations

Step 1° for $n = 0$. Averaging of this equation yields $0 = \lambda_0 \langle \mathbf{h}_0 \rangle$, and two possibilities arise: $\lambda_0 = 0$, and/or $\langle \mathbf{h}_0 \rangle = 0$. As discussed in Sect. 2.2.3, for $\lambda_0 \neq 0$ perturbed short-scale modes are not of a particular interest unless $\operatorname{Re} \lambda_0 = 0$ (if $\langle \mathbf{h}_0 \rangle = 0$, the large-scale magnetic mode is just a perturbation of the respective short-scale mode; for $\operatorname{Re} \lambda_0 \neq 0$, the short-scale mode and the companion large-scale mode grow or decay simultaneously, i.e. the presence of large scales does not change the ability or disability of the flow to generate magnetic field). In what follows, we consider the case $\lambda = 0$. For a given flow, imaginary eigenvalues of $\mathscr{L}$ are non-generic (they occur for a countable set of molecular diffusivities only); the case of an oscillatory (i.e., associated with an imaginary eigenvalue) short-scale magnetic mode will be discussed in Sect. 3.8.

Step 2° for $n = 0$. Since $\mathscr{L}$ is a linear partial differential operator in the fast spatial variables and $\langle \mathbf{h}_0 \rangle$ is independent of them, a solution to (3.16) is

$$\{\mathbf{h}_0\} = \sum_{k=1}^{3} \langle \mathbf{h}_0 \rangle_k \mathbf{S}_k. \tag{3.17}$$

Here we have denoted $\langle \mathbf{h}_0 \rangle_k \equiv \langle \mathbf{h}_0 \rangle \cdot \mathbf{e}_k$ (i.e., the kth component of the vector field $\langle \mathbf{h}_0 \rangle$). Vector fields $\mathbf{S}_k(\mathbf{x})$ are zero-mean solutions to the so-called auxiliary problems of type I:

$$\mathscr{L}\mathbf{S}_k = -\frac{\partial \mathbf{V}}{\partial x_k}. \tag{3.18}$$

The solutions exist, because the spatial mean of the r.h.s. of (3.18) is, evidently, zero (see Sect. 3.2).

Taking the divergence of (3.18) we find (using solenoidality of the flow $\mathbf{V}$) that

$$\nabla^2_{\mathbf{x}}(\nabla_{\mathbf{x}} \cdot \mathbf{S}_k) = 0. \tag{3.19}$$

Since $\nabla_{\mathbf{x}} \cdot \mathbf{S}_k$ is a space-periodic function, this implies $\nabla_{\mathbf{x}} \cdot \mathbf{S}_k = \text{constant}$ (this can be easily shown either applying the maximum principle to the Laplace equation [296], or multiplying (3.19) by $\nabla_{\mathbf{x}} \cdot \mathbf{S}_k$ and integrating the result over a periodicity cell). Integration of this equality over a periodicity cell demonstrates that the constant in the r.h.s. is zero, i.e., $\mathbf{S}_k$ are solenoidal.

Equation 3.18 is equivalent to

$$\mathscr{L}'(\mathbf{S}_k + \mathbf{e}_k) = 0, \tag{3.20}$$

and therefore $\ker \mathscr{L}'$ contains eigenfunctions with any prescribed spatial mean. Evidently, $\dim \ker \mathscr{L} \geq \dim \ker \mathscr{L}' - 3$ and hence the kernel of $\mathscr{L}$ is non-trivial provided $\dim \ker \mathscr{L}' > 3$. Thus, the kernel of $\mathscr{L}$ is trivial if and only if $\ker \mathscr{L}'$ is three-dimensional.

Note that all non-zero-mean eigenfunctions, ordinary or generalised, of the operator $\mathscr{L}'$ are associated with the eigenvalue 0 (independently of our assumption, made in Sect. 3.2, that ker $\mathscr{L}$ is trivial): opening the brackets and averaging the eigenvalue equation $(\mathscr{L}' - \zeta_p \mathscr{I})^d \mathbf{f} = 0$ over the fast spatial variables, we obtain $\zeta^d \langle \mathbf{f} \rangle = 0$ implying $\langle \mathbf{f} \rangle = 0$. Here $\mathbf{f}$ is an ordinary (for $d = 1$) or generalised (for $d > 1$) eigenfunction, associated with an eigenvalue $\zeta \neq 0$; $\mathscr{I}$ is the identity operator.

Solvability of (3.20) does not require ker $\mathscr{L}$ to be trivial. We revoke this assumption till the end of this paragraph and assume instead that all Jordan normal form cells of $\mathscr{L}'$, that are associated with the eigenvalue zero, are of size one (which is generically the case). This holds if the kernel of $\mathscr{L}$ is trivial: then zero is a three-fold simple eigenvalue of $\mathscr{L}'$ (averaging over the fast spatial variables shows that the problem $\mathscr{L}'\mathbf{g} = \sum_{k=1}^{3} c_k(\mathbf{S}_k + \mathbf{e}_k)$ does not have a solution unless all $c_k = 0$, ruling out Jordan cells of size two or larger). Let us show that Eq. 3.18 has a solution under the relaxed assumption. If a vector field $\mathbf{s}^*$ belongs to the kernel of $\mathscr{L}^*$, then $\mathscr{L}^*\mathbf{s}^* =$ constant (as follows from the formula for $\mathscr{L}^*$ in Sect. 3.2); hence it belongs to the kernel of $\mathscr{L}'^*$; hence under the new assumption $\mathscr{L}^*\mathbf{s}^* = 0$ (and since Jordan normal forms of an operator and its adjoint have the same structure, all Jordan cells of $\mathscr{L}'^*$ associated with the eigenvalue 0 are also of size one). Consider short-scale zero-mean vector fields $\mathbf{s}_k^*(\mathbf{x})$ comprising a basis in the kernel of $\mathscr{L}^*$. By the Fredholm alternative theorem, the remaining solvability condition for the equation

$$(1/\eta)\nabla^{-2}\mathscr{L}\mathbf{S}_k = -\nabla^{-2}\mathscr{L}'\mathbf{e}_k/\eta$$

(cf. (3.15)) consists of the orthogonality of the r.h.s. to the kernel of the operator $(1/\eta)\mathscr{L}^*\nabla^{-2}$, i.e. to vector fields $\nabla^2\mathbf{s}_k^*$. The orthogonality is established now by a simple calculation:

$$\langle \nabla^{-2}\mathscr{L}'\mathbf{e}_k \cdot \nabla^2\mathbf{s}_k^* \rangle = \langle \mathscr{L}'\mathbf{e}_k \cdot \mathbf{s}_k^* \rangle = \langle \mathbf{e}_k \cdot \mathscr{L}'^*\mathbf{s}_k^* \rangle = 0$$

(we use the fact that the Laplace operator ∇^2 is self-adjoint for the condition of space periodicity at hand), concluding the demonstration.

If $\mathscr{L}'$ has one or more Jordan normal form cells[1] of size two or more, associated with the eigenvalue 0, it can be easily established by the arguments similar to the ones presented in the previous paragraph, that there exist generalised eigenfunctions of $\mathscr{L}'$, associated with the eigenvalue 0, with any prescribed spatial mean. We present now an alternative demonstration of this statement, which relies on a

[1] In the large-scale dynamo problem studied in this chapter, generically $\mathscr{L}'$ does not possess Jordan cells which are of size greater than one and are associated with the zero eigenvalue. However, we pay attention to such cases, because the mathematical constructions of this subsection can be employed, with just minor modifications, for the treatment of large-scale stability problems, where non-trivial Jordan cells are always present. We will encounter such a case in Chap. 9.

formula expressing generalised eigenfunctions as contour integrals. The operator $\mathscr{L}'$ is elliptic; its spectrum is discrete, and the number of points of the spectrum belonging to any finite region of a complex plane is finite. The operator $\mathscr{R}(\zeta) \equiv (\zeta\mathscr{I} - \mathscr{L}')^{-1} : \mathbb{L}_2([0,\mathbf{L}]) \to \mathbb{L}_2([0,\mathbf{L}])$ is called the *resolvent* of $\mathscr{L}'$ (here ζ is an arbitrary complex number not belonging to the spectrum of $\mathscr{L}'$). Denote

$$\mathscr{P} \equiv \frac{1}{2\pi \mathrm{i}} \oint\limits_{|\zeta|=a} \mathscr{R}(\zeta)\,\mathrm{d}\zeta.$$

For almost all a, the integrand is a bounded operator at any point of the contour of integration. For such an a, $\mathscr{P}$ is a projection onto the invariant subspace of $\mathscr{L}'$ associated with the eigenvalues not exceeding a in absolute value [306] (below we prove this for a small a). Suppose $\mathbf{f} \in \mathbb{L}_2([0,\mathbf{L}])$ is an arbitrary vector field from the domain of $\mathscr{L}'$, $\mathbf{g} = \mathscr{R}(\zeta)\mathbf{f}$. Averaging of the equation $\zeta\mathbf{g} - \mathscr{L}'\mathbf{g} = \mathbf{f}$ over the fast spatial variables yields $\langle\mathbf{g}\rangle = \langle\mathbf{f}\rangle/\zeta$, whereby

$$\langle\mathscr{P}\mathbf{f}\rangle = \frac{1}{2\pi \mathrm{i}} \oint\limits_{|\zeta|=a} \langle\mathbf{f}\rangle/\zeta\,\mathrm{d}\zeta = \langle\mathbf{f}\rangle. \tag{3.21}$$

Let $\mathbb{K}$ denote the $\mathscr{L}'$-invariant subspace, comprised of eigenfunctions from the kernel of $\mathscr{L}'$ and generalised eigenfunctions associated with the eigenvalue 0. Denote the maximum size of a Jordan cell of the restriction of $\mathscr{L}'$ to $\mathbb{K}$ by $d \le \dim\mathbb{K}$; there is a bound $\|\mathscr{R}(\zeta)\| \le C|\zeta|^{-d}$ in the vicinity of $\zeta = 0$. The identity

$$(\mathscr{L}')^m\mathscr{R}(\zeta) = (\mathscr{L}')^{m-1}\frac{1}{2\pi \mathrm{i}} \oint\limits_{|\zeta|=a} \zeta\mathscr{R}(\zeta)\,\mathrm{d}\zeta$$

implies

$$(\mathscr{L}')^m\mathscr{R}(\zeta) = \frac{1}{2\pi \mathrm{i}} \oint\limits_{|\zeta|=a} \zeta^m\mathscr{R}(\zeta)\,\mathrm{d}\zeta,$$

whereby

$$|(\mathscr{L}')^d\mathscr{P}\mathbf{f}| = \left|\frac{1}{2\pi \mathrm{i}} \oint\limits_{|\zeta|=a} \zeta^d\mathscr{R}(\zeta)\mathbf{f}\,\mathrm{d}\zeta\right| \le a^{d+1}\max_{|\zeta|=a}\|\mathscr{R}(\zeta)\||\mathbf{f}| \le Ca|\mathbf{f}|. \tag{3.22}$$

By the residue theorem of the theory of functions of a complex variable, the vector $\mathscr{P}\mathbf{f}$ is independent of the radius of the circle a over which integration is performed, provided $a \le a'$ and the only point of the spectrum of $\mathscr{L}'$ inside the circle of radius a' is zero. Hence, taking the limit $a \to 0$ in (3.22) we find $(\mathscr{L}')^d\mathscr{P}\mathbf{f} = 0$ for all $a \le a'$. Recall that $\mathbf{f}$ is an arbitrary vector field; by virtue of (3.21) we conclude that for any constant vector we can find a field from $\mathbb{K}$ with the spatial mean equal to this vector. In particular, if the restriction of $\mathscr{L}'$ to $\mathbb{K}$ does not possess Jordan cells of size exceeding $d = 1$, we have obtained a contour integral expression for non-zero-mean eigenfunctions from the kernel of $\mathscr{L}'$.

The inverse problem of determination of a flow generating a given neutral magnetic mode is considered in [332].

3.3.2 The Solvability Condition for the Order ε^1 Equations

Step 1° for $n = 1$. The equation obtained from (3.13) at order ε^1 is

$$\mathscr{L}\{\mathbf{h}_1\} + 2\eta(\nabla_\mathbf{x} \cdot \nabla_\mathbf{X})\{\mathbf{h}_0\} + (\langle\mathbf{h}_1\rangle \cdot \nabla_\mathbf{x})\mathbf{V} + \nabla_\mathbf{X} \times (\mathbf{V} \times \mathbf{h}_0) = \lambda_1 \mathbf{h}_0. \quad (3.23)$$

Substituting here (3.17) and averaging the result, we obtain

$$\nabla_\mathbf{X} \times \sum_{k=1}^{3} \langle \mathbf{V} \times (\mathbf{S}_k + \mathbf{e}_k)\rangle \langle \mathbf{h}_0\rangle_k = \lambda_1 \langle \mathbf{h}_0\rangle. \quad (3.24)$$

The 3×3 matrix

$$\alpha = \langle \mathbf{V} \times (\mathbf{S}_k + \mathbf{e}_k)\rangle \quad (3.25)$$

is called the magnetic *α-tensor*, and the operator in the l.h.s. of (3.24) the *operator of magnetic α-effect*. An α-effect operator is typically a linear combination of first-order derivatives in the slow spatial variables. This has important consequences. Suppose the problem is solved under the condition of periodicity in the slow spatial variables, or, alternatively, in a finite region of the slow variables invariant under reflection about the centre, $\mathbf{X} \to -\mathbf{X}$, and homogeneous Dirichlet or Neumann boundary conditions are assumed. Then the spectrum of an operator of such a structure is symmetric about the imaginary axis: if $\mathbf{g}(\mathbf{X}) = \langle \mathbf{h}_0 \rangle$ is an eigenfunction associated with an eigenvalue λ_1, then $\mathbf{g}(-\mathbf{X})$ is an eigenfunction associated with the eigenvalue $-\lambda_1$. Thus, either all eigenvalues are imaginary (a non-generic case), or there exists an eigenvalue λ_1 with a positive real part, implying magnetic field generation. Moreover, magnetic field, space-periodic in the slow variables, exhibits a superexponential growth: if $\mathbf{g}(\mathbf{X})$ is an eigenfunction associated with an eigenvalue λ_1 and n is integer, then $\mathbf{g}(n\mathbf{X})$ is an eigenfunction (possessing the same periodicity) associated with the eigenvalue $n\lambda_1$; consequently, seed magnetic fields, whose all components in the basis of eigenfunctions of the operator of the α-effect are initially non-zero, exhibit a superexponential growth. Generically, a dynamo persists, this effectively concluding the investigation of the ability of the flow $\mathbf{V}$ to generate a large-scale magnetic field. (A complete mathematical analysis under an additional condition of ellipticity of the limit operator of magnetic α-effect in the space of solenoidal vector fields was performed in [307].)

The hypothesis that the magnitude of the α-effect is proportional to the flow helicity $\int_{[0,\mathbf{L}]} \mathbf{V} \cdot (\nabla \times \mathbf{V})\, d\mathbf{x}$ was put forward by the creators of the mean-field theory [283] and critically assessed in [65, 119, 243]. This kind of dependence was derived from an empirical description of homogeneous gyrotropic turbulence in [187, 188]. The hypothesis persists in various modifications—e.g., transformed into the hypothesis of its proportionality to the difference of the flow and magnetic field helicities [23–25, 88]. Although a certain similarity between the flow helicity and the precise expression for the α-tensor, that we have derived above, is implied (under the natural assumption $\langle \mathbf{V} \rangle = 0$) by the "proportionality" of $\mathbf{S}_k$ to the r.h.s. of Eq. 3.18, $\partial \mathbf{V}/\partial x_k$, the formula (3.25) (as the expressions obtained in [306, 307, 321, 322] for other kinematic dynamo problems, see Chaps. 10 and 11) does not suggest any direct relation of the magnetic α-effect to the flow helicity (or to the difference of the flow and magnetic field helicities). As well, the formula (3.25) cannot be reduced to other heuristic expressions suggested by many authors (see, e.g., [51, 52, 76, 130, 302]) in the framework of investigation of the α-quenching.

Suppose $\langle \mathbf{V} \rangle = 0$ (this condition is satisfied in the coordinate system co-moving with the velocity $\langle \mathbf{V} \rangle$). A short calculation employing (3.18) yields an alternative expression for elements of the magnetic α-tensor:

$$\begin{aligned}\alpha_{kn} &\equiv \langle \mathbf{V} \times \mathbf{S}_k \rangle_n = \langle (\mathbf{e}_n \times \mathbf{V}) \cdot \mathbf{S}_k \rangle = q \int\limits_{[0,\mathbf{L}]} \nabla \times (\mathbf{e}_n \times \mathbf{V}) \cdot \mathscr{C}\mathbf{S}_k\, d\mathbf{x} \\ &= -q \int\limits_{[0,\mathbf{L}]} \frac{\partial \mathbf{V}}{\partial x_n} \cdot \mathscr{C}\mathbf{S}_k\, d\mathbf{x} = q \int\limits_{[0,\mathbf{L}]} \mathscr{L}\mathbf{S}_n \cdot \mathscr{C}\mathbf{S}_k\, d\mathbf{x} \\ &= -\eta \langle (\nabla \times \mathbf{S}_n) \cdot \mathbf{S}_k \rangle - \langle (\mathbf{V} \times \mathbf{S}_n) \cdot \mathbf{S}_k \rangle,\end{aligned}$$

where $q \equiv (L_1 L_2 L_3)^{-1}$, and $\mathscr{C}$ denotes the inverse curl, i.e. the operator mapping a solenoidal zero-mean space-periodic vector field to its solenoidal zero-mean space-periodic vector potential. This expression is compatible with the calculation of the α-effect in [153, 247] . It implies that in the problem at hand diagonal entries of the α-tensor,

$$\alpha_{nn} = -\eta \langle (\nabla \times (\mathbf{S}_n + \mathbf{e}_n)) \cdot (\mathbf{S}_n + \mathbf{e}_n) \rangle, \tag{3.26}$$

are proportional to the helicity of the neutral modes, and not of the flow. The formula illustrates the danger of a direct physical interpretation of mathematical results. For $\eta = 0$ the diagonal elements formally vanish; based on this one might conjecture that the magnetic α-effect has a diffusive nature and it "quenches" in the limit $R_m \to \infty$ (since the size of the periodicity cell and the velocity of the flow are of the order of unity, the magnetic Reynolds number can be defined as $1/\eta$). Evidently, this interpretation is wrong, because for $\eta = 0$ the type of the equation defining neutral magnetic modes changes from the elliptic second-order one to a first-order partial differential equation, whose solvability is not guaranteed. The conclusion about the decay of the α-effect is illegitimate, since the arguments on which it is based ignore the dependence of the neutral modes on magnetic molecular diffusivity η, and the modes may (and are likely to) develop singularities as η approaches 0. Inequalities of energy type yield bounds of the order of $1/\eta$ for the norm of the neutral mode in the Sobolev space $\mathbb{W}_2^1([0,\mathbf{L}])$. This growth is not annihilated by the factor η in (3.26). A possible growth of the magnitude of the α-effect for $\eta \to 0$ is suggested by another expression obtained by vector multiplication of (3.18) by $\nabla^{-2}\mathbf{V}/\eta$:

$$\langle \mathbf{V} \times \mathbf{S}_k \rangle = -\frac{1}{\eta}\left\langle \nabla^{-2}\mathbf{V} \times \frac{\partial \mathbf{V}}{\partial x_k} + \nabla^{-2}\mathbf{V} \times (\nabla \times (\mathbf{V} \times \mathbf{S}_k)) \right\rangle.$$

3.4 Magnetic Eddy Diffusion in Steady Parity-Invariant Flows

So far, our constructions followed [307]; here they start to diverge. We assume henceforth in this chapter that the flow velocity $\mathbf{V}(\mathbf{x})$ is parity-invariant (3.6).

3.4.1 Solution of the Order ε^1 Equations in the Absence of α-Effect

For a parity-invariant flow, the domain of the operator $\mathscr{L}$ splits into a direct sum of two subspaces: parity-invariant vector fields (such that $\mathbf{f}(\mathbf{x}) = -\mathbf{f}(-\mathbf{x})$) and parity-anti invariant ($\mathbf{f}(\mathbf{x}) = \mathbf{f}(-\mathbf{x})$). The r.h.s. of (3.18) is parity-anti invariant, and hence the solutions $\mathbf{S}_k$ are also parity-anti invariant:

$$\mathbf{S}_k(\mathbf{x}) = \mathbf{S}_k(-\mathbf{x}). \tag{3.27}$$

By virtue of this symmetry, in the case under consideration the α-tensor is a zero matrix, i.e. the α-effect is absent. Consequently, (3.24) implies $\lambda_1 = 0$.

Step 2° for $n = 1$. Substituting solutions (3.17) to the order ε^0 equation into the fluctuating part of the order ε^1 equation 3.23, and employing the solenoidality condition in the slow variables (3.11) for $n = 1$ and vector analysis identities, we find

$$\mathscr{L}\{\mathbf{h}_1\} = -(\langle \mathbf{h}_1 \rangle \cdot \nabla_{\mathbf{x}})\mathbf{V} + \sum_{k=1}^{3}\sum_{m=1}^{3}\left(-2\eta\frac{\partial \mathbf{S}_k}{\partial x_m} + V_m(\mathbf{S}_k + \mathbf{e}_k) - \mathbf{V}(S_k)_m\right)\frac{\partial \langle \mathbf{h}_0 \rangle_k}{\partial X_m} \tag{3.28}$$

(subscripts enumerate Cartesian components of vector fields; in particular, $(S_k)_m$ denotes the mth component of $\mathbf{S}_k$). By linearity of this equation,

$$\{\mathbf{h}_1\} = \sum_{k=1}^{3}\mathbf{S}_k\langle \mathbf{h}_1 \rangle_k + \sum_{k=1}^{3}\sum_{m=1}^{3}\mathbf{G}_{mk}\frac{\partial \langle \mathbf{h}_0 \rangle_k}{\partial X_m}, \tag{3.29}$$

where the nine vector fields $\mathbf{G}_{mk}(\mathbf{x})$ are solutions to auxiliary problems of type II:

$$\mathcal{L}\mathbf{G}_{mk} = -2\eta \frac{\partial \mathbf{S}_k}{\partial x_m} + V_m(\mathbf{S}_k + \mathbf{e}_k) - \mathbf{V}(S_k)_m. \tag{3.30}$$

As discussed above, for a parity-invariant flow $\mathbf{V}(\mathbf{x})$ the spatial mean of the r.h.s. (equal to the respective entry of the magnetic α-tensor) vanishes, and hence the problem does have solutions. Since the r.h.s. of equation (3.30) is parity-invariant, the same holds true for the solution: $\mathbf{G}_{mk}(\mathbf{x}) = -\mathbf{G}_{mk}(-\mathbf{x})$.

Taking the divergence of (3.30), we find using (3.20):

$$\begin{aligned}\eta\nabla^2_{\mathbf{x}}(\nabla_{\mathbf{x}} \cdot \mathbf{G}_{mk}) &= -\nabla_{\mathbf{x}} \cdot (\mathbf{e}_m \times (\mathbf{V} \times (\mathbf{S}_k + \mathbf{e}_k))) \\ &= \mathbf{e}_m \cdot (\nabla_{\mathbf{x}} \times (\mathbf{V} \times (\mathbf{S}_k + \mathbf{e}_k))) \\ &= -\eta\nabla^2_{\mathbf{x}}(S_k)_m,\end{aligned}$$

implying $\nabla_{\mathbf{x}} \cdot \mathbf{G}_{mk} + (S_k)_m =$ constant. Averaging this equation over the fast spatial variables and recalling that $\mathbf{S}_k$ are zero-mean and $\mathbf{G}_{mk}$ are space-periodic, we obtain

$$\nabla_{\mathbf{x}} \cdot \mathbf{G}_{mk} + (S_k)_m = 0. \tag{3.31}$$

This identity and expressions (3.17) and (3.29) for solutions to order ε^0 and ε^1 problems imply that the solenoidality relation (3.12) is satisfied for $n = 1$, whichever are the means $\langle \mathbf{h}_0 \rangle$ and $\langle \mathbf{h}_1 \rangle$.

3.4.2 *The Solvability Condition for the Order ε^2 Equations: The Operator of Magnetic Eddy Diffusion*

Step 1° for $n = 2$. At order ε^2 we find from (3.13) the equation

$$\begin{aligned}\mathcal{L}\{\mathbf{h}_2\} + \eta\big(2(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\{\mathbf{h}_1\} + \nabla^2_{\mathbf{X}}\mathbf{h}_0\big) + (\langle \mathbf{h}_2 \rangle \cdot \nabla_{\mathbf{x}})\mathbf{V} \\ + \nabla_{\mathbf{X}} \times (\mathbf{V} \times \{\mathbf{h}_1\}) - (\mathbf{V} \cdot \nabla_{\mathbf{X}})\langle \mathbf{h}_1 \rangle = \lambda_2 \mathbf{h}_0.\end{aligned} \tag{3.32}$$

Substituting the solution (3.29) to the order ε^1 equation and averaging the result over the fast spatial variables, we find

$$\mathcal{E}\langle \mathbf{h}_0 \rangle \equiv \eta\nabla^2_{\mathbf{X}}\langle \mathbf{h}_0 \rangle + \nabla_{\mathbf{X}} \times \sum_{k=1}^{3}\sum_{m=1}^{3} \mathbf{D}_{mk} \frac{\partial \langle \mathbf{h}_0 \rangle_k}{\partial X_m} = \lambda_2 \langle \mathbf{h}_0 \rangle, \tag{3.33}$$

where

$$\mathbf{D}_{mk} = \langle \mathbf{V} \times \mathbf{G}_{mk} \rangle. \tag{3.34}$$

Therefore, we have derived a closed eigenvalue equation (3.33), defining together with the solenoidality condition (3.11) for $n = 0$ the leading terms in the expansions of the magnetic mode (3.8) and the associated eigenvalue (3.9), $\mathbf{h}_0$ and λ_2, respectively (the fluctuating part of the eigenmode is now completely determined by relation (3.17)). We call (3.33) *mean-field* equations for mean profiles of large-scale magnetic modes. The mean magnetic field $\langle\mathbf{h}_0\rangle$ is an eigenfunction of the operator $\mathscr{E}$, which is a sum of second-order derivatives; this enables us to interpret it as the *operator of magnetic eddy diffusion*.The difference between the magnetic eddy and molecular diffusion operators—i.e., the operator with the coefficients (3.34)—is called *magnetic eddy correction*. Generically, magnetic eddy diffusion is anisotropic, and, unlike the operator of molecular diffusion, the operator of magnetic eddy diffusion is not necessarily sign-definite. Solenoidal fields constitute an invariant subspace of $\mathscr{E}$, and in view of solenoidality of $\langle\mathbf{h}_0\rangle$ (condition (3.11) for $n = 0$) we restrict $\mathscr{E}$ to solenoidal vector fields. If the operator has an eigenvalue with a positive real part associated with a solenoidal eigenfunction, one speaks of the phenomenon of *negative magnetic eddy diffusivity*.

The operator $\mathscr{E}$ has constant coefficients. Consequently, if the magnetic mode is defined in the entire three-dimensional space of the slow variables, eigenfunctions of $\mathscr{E}$ are Fourier harmonics

$$\langle\mathbf{h}_0\rangle = \widetilde{\mathbf{h}}e^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}},$$

where $\widetilde{\mathbf{h}}$ and $\mathbf{q}$ are constant three-dimensional vectors satisfying the equations

$$-\eta|\mathbf{q}|^2\widetilde{\mathbf{h}} - \mathbf{q} \times \sum_{k=1}^{3}\sum_{m=1}^{3}\mathbf{D}_{mk}\widetilde{h}_k q_m = \lambda_2\widetilde{\mathbf{h}}, \qquad (3.35)$$

(this follows from the magnetic mean-field equation (3.33)) and

$$\widetilde{\mathbf{h}} \cdot \mathbf{q} = 0 \qquad (3.36)$$

(this is a consequence of the solenoidality condition (3.11) for $n = 0$). Global boundedness of the mean field $\mathbf{h}_0$ in space implies that $\mathbf{q}$ is a real vector (it determines the orientation of the front of the magnetic mode, regarded as an amplitude-modulated plane wave). Hence, the large-scale magnetic mode, averaged over small spatial scales, turns out to be space-periodic. (The fluctuating part of the mode is periodic only if the wave vector $\mathbf{q}$ and the scale ratio ε satisfy certain rationality conditions.) Since ε is an arbitrary small number, it suffices to consider wave vectors $\mathbf{q}$ of unit length.

The mode can be also defined in a bounded region of the slow variables, with the mean field satisfying some physically meaningful boundary conditions (e.g., for a dielectric medium or for a perfect conductor surrounding the conducting fluid). However, the fluctuating part of the mode generically will not satisfy them, and therefore the condition of periodicity in the slow variables for the mean magnetic field, arising for a mode defined in the entire space, seems to be the most natural one. Nevertheless, for the sake of generality, here and in what follows we consider mean-field problems for arbitrary boundary conditions, separately discussing the particular case of a mode defined in the entire space of the slow variables.

Step 2° for $n = 2$. The fluctuating part of the order ε^2 equation (3.32) takes the form

$$\mathscr{L}\{\mathbf{h}_2\} = -\eta\left(2\sum_{k=1}^{3}\sum_{m=1}^{3}\left(\frac{\partial \mathbf{S}_k}{\partial x_m}\frac{\partial\langle\mathbf{h}_1\rangle_k}{\partial X_m} + \sum_{l=1}^{3}\frac{\partial \mathbf{G}_{mk}}{\partial x_l}\frac{\partial^2\langle\mathbf{h}_0\rangle_k}{\partial X_m \partial X_l}\right) + \nabla^2_{\mathbf{X}}\{\mathbf{h}_0\}\right)$$
$$- (\langle\mathbf{h}_2\rangle\cdot\nabla_{\mathbf{x}})\mathbf{V} - \nabla_{\mathbf{X}}\times\left(\sum_{k=1}^{3}(\mathbf{V}\times(\mathbf{S}_k+\mathbf{e}_k))\langle\mathbf{h}_1\rangle_k\right.$$
$$\left. + \sum_{k=1}^{3}\sum_{m=1}^{3}\{\mathbf{V}\times\mathbf{G}_{mk}\}\frac{\partial\langle\mathbf{h}_0\rangle_k}{\partial X_m}\right) + \lambda_2\{\mathbf{h}_0\}. \tag{3.37}$$

At this stage $\langle\mathbf{h}_2\rangle$ and $\partial\langle\mathbf{h}_1\rangle_k/\partial X_m$ remain unknown here. However, the x-dependent factors in the terms involving these unknown functions are the same in (3.37), as in the terms in (3.28) involving $\langle\mathbf{h}_1\rangle$ and $\partial\langle\mathbf{h}_0\rangle_k/\partial X_m$, respectively; hence

$$\{\mathbf{h}_2\} = \sum_{k=1}^{3}\mathbf{S}_k\langle\mathbf{h}_2\rangle_k + \sum_{k=1}^{3}\sum_{m=1}^{3}\mathbf{G}_{mk}\frac{\partial\langle\mathbf{h}_1\rangle_k}{\partial X_m} + \mathbf{h}'_2,$$

where the zero-mean vector field $\mathbf{h}'_2(\mathbf{X},\mathbf{x})$ is uniquely defined by the equation with a known zero-mean r.h.s.

$$\mathscr{L}\mathbf{h}'_2 = -\eta\left(2\sum_{k=1}^{3}\sum_{m=1}^{3}\sum_{l=1}^{3}\frac{\partial \mathbf{G}_{mk}}{\partial x_l}\frac{\partial^2\langle\mathbf{h}_0\rangle_k}{\partial X_m\partial X_l} + \nabla^2_{\mathbf{X}}\{\mathbf{h}_0\}\right)$$
$$- \nabla_{\mathbf{X}}\times\sum_{k=1}^{3}\sum_{m=1}^{3}\{\mathbf{V}\times\mathbf{G}_{mk}\}\frac{\partial\langle\mathbf{h}_0\rangle_k}{\partial X_m} + \lambda_2\{\mathbf{h}_0\}. \tag{3.38}$$

3.5 Complete Asymptotic Expansion of Large-Scale Magnetic Modes

We discuss in this section, how to solve successively the equations emerging from the series (3.13) at orders ε^N, $N > 2$. The operator of magnetic eddy diffusion $\mathscr{E}$ acts in the space of solenoidal vector fields depending on the slow variables and, unless magnetic field is defined in the entire space of the slow variables, satisfying some imposed boundary conditions. We assume that the operator $\mathscr{E} - \lambda_2\mathscr{I}$ has a bounded inverse in the $\mathscr{E}$-invariant subspace, complementary to the one spanned by the eigenfunction $\langle\mathbf{h}_0\rangle$; in particular, λ_2 is a simple (multiplicity one) eigenvalue of $\mathscr{E}$. (If the region in the slow variables is bounded, regular boundary conditions are considered, and the eddy diffusion operator is elliptic, this condition is satisfied for all molecular diffusivities η except for at most a countable set of values.)

Suppose while solving the equations at orders up to ε^{N-1} we have obtained the following information:

- the fields $\mathbf{h}_n(\mathbf{X},\mathbf{x})$ for all $n<N-2$;
- for $n=N-1$ and $n=N-2$,

$$\{\mathbf{h}_n\} = \sum_{k=1}^{3}\mathbf{S}_k\langle\mathbf{h}_n\rangle_k + \sum_{k=1}^{3}\sum_{m=1}^{3}\mathbf{G}_{mk}\frac{\partial\langle\mathbf{h}_{n-1}\rangle_k}{\partial X_m} + \mathbf{h}'_n, \tag{3.39}$$

  where zero-mean fields $\mathbf{h}'_n(\mathbf{X},\mathbf{x})$ are known for all $n\le N-1$;
- the quantities λ_n for all $n<N$.

Step 1° for $n=N$. Substitution of (3.39) for $n=N-1$ and averaging over the fast spatial variables reduces the equation obtained from the series (3.13) at order ε^N to

$$(\mathscr{E}-\lambda_2\mathscr{I})\langle\mathbf{h}_{N-2}\rangle - \lambda_N\langle\mathbf{h}_0\rangle = -\nabla_{\mathbf{X}}\times\langle\mathbf{V}\times\mathbf{h}'_{N-1}\rangle + \sum_{m=1}^{N-3}\lambda_{N-m}\langle\mathbf{h}_m\rangle. \tag{3.40}$$

Here the r.h.s. is a known solenoidal (in the slow variables) vector field. Projecting out Eq. (3.40) in the domain of $\mathscr{E}$ into the invariant subspace spanned by $\langle\mathbf{h}_0\rangle$, we find uniquely λ_N. Since the operator $\mathscr{E}-\lambda_2\mathscr{I}$ in the complementary $\mathscr{E}$-invariant subspace has an inverse, we solve (3.40) by finding a solenoidal field $\langle\mathbf{h}_{N-2}\rangle$ (note the solenoidality condition (3.11) for $n=N-2$) in this subspace up to an additive term proportional to $\langle\mathbf{h}_0\rangle$, which we assume to be zero (this is a normalisation condition: emergence of such terms in $\langle\mathbf{h}_n\rangle$ for $n\ge 1$ is equivalent to multiplication of the mode $\mathbf{h}(\mathbf{X},\mathbf{x})$ by a power series in ε, evidently not altering the shape of the mode).

Step 2° for $n=N$. Now $\mathbf{h}_{N-2}$ is completely determined by formula (3.39) for $n=N-2$. After $\mathbf{h}_{N-2}$ and (3.39) for $n=N-1$ are substituted, the fluctuating part of the equation, obtained from the series (3.13) at order ε^N, takes the form

$$\begin{aligned}\mathscr{L}\{\mathbf{h}_N\} = &-\eta\left(2\sum_{k=1}^{3}\sum_{m=1}^{3}\left(\frac{\partial\mathbf{S}_k}{\partial x_m}\frac{\partial\langle\mathbf{h}_{N-1}\rangle_k}{\partial X_m} + \sum_{l=1}^{3}\frac{\partial\mathbf{G}_{mk}}{\partial x_l}\frac{\partial^2\langle\mathbf{h}_{N-2}\rangle_k}{\partial X_l\partial X_m}\right)\right.\\ &\left.+\,2(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\mathbf{h}'_{N-1} + \nabla^2_{\mathbf{X}}\{\mathbf{h}_{N-2}\}\right) - (\langle\mathbf{h}_N\rangle\cdot\nabla_{\mathbf{x}})\mathbf{V}\\ &-\nabla_{\mathbf{X}}\times\left(\sum_{k=1}^{3}(\mathbf{V}\times(\mathbf{S}_k+\mathbf{e}_k))\langle\mathbf{h}_{N-1}\rangle_k + \sum_{k=1}^{3}\sum_{m=1}^{3}\{\mathbf{V}\times\mathbf{G}_{mk}\}\frac{\partial\langle\mathbf{h}_{N-2}\rangle_k}{\partial X_m}\right.\\ &\left.+\,\{\mathbf{V}\times\mathbf{h}'_{N-1}\}\right) + \sum_{m=0}^{N-2}\lambda_{N-m}\{\mathbf{h}_m\}.\end{aligned} \tag{3.41}$$

So far, only the terms involving the factors $\langle\mathbf{h}_N\rangle$ and derivatives of $\langle\mathbf{h}_{N-1}\rangle_k$ remain undetermined. As in (3.37), $\mathbf{x}$-dependent coefficients in these terms coincide with those in the terms containing $\langle\mathbf{h}_1\rangle$ and derivatives of $\langle\mathbf{h}_0\rangle_k$ in (3.28), respectively.

Thus, (3.41) yields an expression for $\{\mathbf{h}_N\}$ analogous to (3.39). An equation for $\mathbf{h}'_N$ is obtained from (3.41) by elimination of all terms proportional to $\langle\mathbf{h}_N\rangle$ and derivatives of $\langle\mathbf{h}_{N-1}\rangle_k$ (for $N = 2$ it coincides with (3.38)); in the resultant equation the only unknown function is $\mathbf{h}'_N$, and the solvability condition is evidently satisfied.

Thus, the equation obtained from the series (3.13) at order ε^N can be solved, with all required properties stated in the beginning of this section reproduced. It remains to verify, that the solenoidality conditions (3.12) are satisfied for the solutions that we have found. It is easy to do this by induction. We assume that (3.12) holds true for all $n<N$ and consider equations obtained from the series (3.13) at orders ε^{N-1} and ε^N. Adding the divergence of the former in the slow variables with the divergence of the latter in the fast variables, employing the identity

$$\nabla_{\mathbf{x}} \cdot (\nabla_{\mathbf{X}} \times \mathbf{a}(\mathbf{X},\mathbf{x})) + \nabla_{\mathbf{X}} \cdot (\nabla_{\mathbf{x}} \times \mathbf{a}(\mathbf{X},\mathbf{x})) = 0,$$

the solenoidality conditions (3.11) for $n \le N-2$ and the induction assumptions, we find

$$\eta\nabla^2_{\mathbf{x}}(\nabla_{\mathbf{x}} \cdot \{\mathbf{h}_N\} + \nabla_{\mathbf{X}} \cdot \{\mathbf{h}_{N-1}\}) = 0.$$

This implies (3.12) for $n = N$, as required.

We have constructed a formal complete asymptotic expansion of large-scale magnetic modes in the power series in the scale ratio. It can be easily shown that it possesses the following properties: all $\mathbf{h}_n$ and $\mathbf{h}'_n$ are parity-anti invariant for all even n, and parity-invariant for all odd n; in particular, $\langle\mathbf{h}_n\rangle = 0$ for all odd n. Consequently, (3.43) implies that $\lambda_n = 0$ for all odd n, i.e. the series (3.9) in which the eigenvalue is expanded is, in fact, a power series in ε^2.

Let now the magnetic mode be defined in the entire space of the slow variables. As we have established in the previous section, λ_2 is then an eigenvalue of the 3×3 matrix (3.35) in the invariant subspace orthogonal to the wave vector (see (3.36)). We denote its second eigenvalue by λ'_2 and the associated eigenvector from this subspace by $\widetilde{\mathbf{h}}'$ (assuming $|\widetilde{\mathbf{h}}| = |\widetilde{\mathbf{h}}'| = 1$), and consider the generic case $\lambda_2 \ne \lambda'_2$. The domain of the operator of magnetic eddy diffusion $\mathscr{E}$ splits into a direct sum of invariant subspaces of vector fields of the form[2] $\mathbf{h}(\mathbf{x})\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}$, categorised by constant wave vectors $\mathbf{q}$. Solution of the hierarchy of equations is now

[2] This suggests an alternative approach to investigation of large-scale dynamos. Instead of developing a *singular* expansion of large-scale magnetic modes, as we have done in this chapter, one might consider an eigenfunction of this particular form. Substituting it into the eigenvalue equation for the magnetic induction operator and cancelling out the exponent $\exp(\mathrm{i}\varepsilon\mathbf{q}\cdot\mathbf{x})$, we would reduce the problem to a short-scale eigenvalue problem in $\mathbf{h}(\mathbf{x})$ for a modified operator parameterised by the scale ratio ε and the wave vector $\mathbf{q}$. Solutions to this problem admit a *regular* expansion in the power series in ε, which it is perhaps easier to construct. We have not pursued this approach, because it is inapplicable for weakly nonlinear stability problems.

considerably simplified, because the dependence of solutions on the slow variables is via the factor $e^{i\mathbf{q}\cdot\mathbf{X}}$:

$$\mathbf{h}_n = \mathbf{g}_n(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{X}}. \tag{3.42}$$

In particular, we can relax the condition that the inverse to $\mathscr{E} - \lambda_2\mathscr{I}$ exists in the codimension 1 invariant subspace: it suffices to assume that the eigenvalue λ_2 has multiplicity one in the two-dimensional subspace of solenoidal Fourier harmonics (in the slow variables), associated with the wave vector $\mathbf{q}$ under consideration.

To consider further the equation at order ε^N, we substitute (3.42) into (3.40) and cancel out the factor $e^{i\mathbf{q}\cdot\mathbf{X}}$. Invariant projection of the resultant equation parallel to $\widetilde{\mathbf{h}}$ in $\mathbb{C}^3$ yields

$$\lambda_N = i\widetilde{\mathbf{h}}\cdot\mathscr{P}_{\widetilde{\mathbf{h}}}(\mathbf{q}\times\langle\mathbf{V}\times\mathbf{g}'_{N-1}\rangle), \tag{3.43}$$

where $\mathbf{h}'_{N-1} = \mathbf{g}'_{N-1}(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{X}}$ and $\mathscr{P}_{\mathbf{u}}$ is projection parallel to a vector $\mathbf{u}$. Invariant projection parallel to $\widetilde{\mathbf{h}}'$ yields

$$\langle\mathbf{g}_{N-2}\rangle = \frac{1}{\lambda'_2 - \lambda_2}\left(-i\mathscr{P}_{\widetilde{\mathbf{h}}'}(\mathbf{q}\times\langle\mathbf{V}\times\mathbf{g}'_{N-1}\rangle) + \sum_{m=1}^{N-3}\lambda_{N-m}\langle\mathbf{g}_m\rangle\right).$$

Hence, all $\langle\mathbf{g}_n\rangle$ for $n > 0$ are parallel to $\widetilde{\mathbf{h}}'$ (the arbitrary additive term parallel to $\widetilde{\mathbf{h}}$ is set to zero as a normalisation condition). The fluctuating part of Eq. 3.14 obtained from (3.13) at order ε^N yields a relation of the kind of (3.39); the r.h.s. of the equation for $\mathbf{h}'_N$ is now proportional to $e^{i\mathbf{q}\cdot\mathbf{X}}$, and therefore the solution admits the form $\mathbf{h}'_N = \mathbf{g}'_N(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{X}}$.

3.6 How Rare is Negative Magnetic Eddy Diffusivity?

The approach, employed by [113] to study how common the phenomenon of negative eddy viscosity is among two-dimensional flows, was applied in [336] to explore how often negative magnetic eddy diffusivity is encountered among "turbulent" flows. For a given flow, the auxiliary problems (3.18) and (3.30) were solved numerically[3] in order to evaluate coefficients (3.34) of the magnetic eddy diffusion operator and find, using Eqs. 3.35 and 3.36, the maximum of the real part

[3] It is more efficient to proceed by solving a smaller number of auxiliary problems for the adjoint operator (considered in Sect. 4.4) than auxiliary problems of type II, as discussed in the next chapter.

of eigenvalues of this operator over wave vectors $\mathbf{q}$ of unit length. This procedure was repeated for an ensemble of flows, comprised of Fourier harmonics with random amplitudes and a prescribed energy spectrum, and the fraction of flows was determined, which exhibit magnetic negative eddy diffusivity, but do not act as short-scale dynamos.

Flows in a cube of periodicity of size $L = 2\pi$ are expanded in Fourier series:

$$\mathbf{V} = \sum_{|\mathbf{k}|=1}^{N} \mathbf{V_k} e^{\mathbf{ik\cdot x}}. \tag{3.44}$$

The field $\mathbf{V}$ is real as long as $\mathbf{V_k} = \overline{\mathbf{V_{-k}}}$. The conditions of parity invariance (3.6) and solenoidality (3.5) amount to $\mathrm{Re}\,\mathbf{V_k} = 0$ and $\mathbf{k} \cdot \mathbf{V_k} = 0$, respectively.

Two ensembles of flows were examined in [336], different in the shape of the energy spectrum

$$E_K = \sum_{K-1<|\mathbf{k}|\leq K} |\mathbf{V_k}|^2$$

(here $K > 0$ is an integer): an exponential spectrum with $E_K \sim 10^{-\xi K}$, and a hyperbolic spectrum with $E_K \sim 1/K$ (modelling flows with a fast and slow energy spectrum fall off, respectively). The series (3.44) cut off was at $N = 10$ for the exponential spectrum ($\xi = 2/3$ was set), and at $N = 7$ for the hyperbolic one. The flows were generated by the following algorithm (applied for a half of wave vectors): (i) For each $\mathbf{k}$ in the ball $1 \leq |\mathbf{k}| \leq N$, three-dimensional real vectors $\mathbf{V}'_\mathbf{k}$ are pseudo randomly generated with each component uniformly distributed at the interval $[-0.5, 0.5]$. (ii) $\mathbf{V}'_\mathbf{k}$ is projected on the plane normal to $\mathbf{k}$. (iii) The resultant vectors are normalised in spherical shells of width 1, so that when they are interpreted as imaginary parts of a half of Fourier coefficients of the flow (3.44), and the remaining half is obtained by complex conjugation so that the flow is real, the latter has a desirable energy spectrum fall off and the r.m.s. velocity 1.

Each ensemble is comprised of 100 sample flows. If magnetic molecular diffusivity η is too large, eddy diffusivity is close to the molecular one (Eqs. 3.18, 3.30 and 3.34 imply that $\mathbf{D}_{mk} = \mathrm{O}(\eta^{-2})$ for $\eta \to \infty$). If η is too small, then a short-scale dynamo operates, concealing the large-scale generation (since magnetic field growth rates in the two processes are $\mathrm{O}(\varepsilon^0)$ and $\mathrm{O}(\varepsilon^2)$, respectively). For each sample, computations were performed for three intermediate values, $\eta = 0.3$, 0.2 and 0.1, chosen following preliminary numerical experiments. For each combination of $\mathbf{V}$ and η, a dominant (i.e., associated with an eigenvalue with the largest real part) short-scale zero-mean magnetic mode was computed by an adapted version of the code[4] [325] and it was verified that the growth rate is negative[5] (see

Fig. 3.1). (Recall that magnetic modes with a non-zero spatial mean are associated with a zero eigenvalue—see Sect. 3.2.)

Solutions to the auxiliary problems are also sought in the form of Fourier series. Iterative methods can be used for numerical solution of the discretised auxiliary problems, e.g. the generalised conjugate gradients method [12] for linear systems of equations, or the method [231] for general nonlinear systems of equations employing an extremal property of roots of Chebyshev polynomials and its modification [336]. They are designed for systems, where the linear operator (in our case, $\mathscr{L}$, or, more generally, the linearisation around the MHD regime under examination) is self-adjoint and positive-definite. Since this is not so for the auxiliary problems—the magnetic induction operator $\mathscr{L}$ is not self-adjoint—in principle,convergence can stall. However, for the large values of η that were used in this numerical experiment, we did not encounter such a nasty behaviour.

All computations were made with the resolution of 64^3 Fourier harmonics. The Fast Fourier Transform [240] and dealiasing [32, 50, 221] were employed for computation of advective terms. Energy spectra of solutions to the auxiliary problems fall off fast and monotonically for $K \geq 2$: for $\eta = 0.1$ the spectra usually decrease by 7–8 orders of magnitude for flows with the hyperbolically falling off spectrum, and by 14–15 orders of magnitude for flows with the exponentially falling off one. The code was tested against the computations [161] for the modified Green–Taylor flow.

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[4] Contrary to [178] we cannot recommend integration of the magnetic induction equation (3.2) in time as an *efficient* method for the study of magnetic field generation in the kinematic regime, or for computation of dominant magnetic modes (more generally, of MHD instability modes). This "method" is advantageous compared to the power method in that an efficient time-difference scheme can be applied for integration, whereas application of the power method is algebraically equivalent to integration of this equation by the Euler's scheme. However, on the one hand, as explained in [325], much larger "steps of integration" can be used in the power method, since the accuracy of integration is not an issue—the step size is only constrained by the region of convergence of the method, i.e. by the largest in magnitude subdominant eigenvalue; on the other, the method [325], built up on the power method, involves optimisation of computation of the dominant mode by transition to"trajectories" (in the terms of the analogy with integration of the evolutionary equation), which are better approximations of exponential ones. Simulation of an evolution of a magnetic field for solution of the short-scale kinematic dynamo problem is practical only if (1) an efficient specialised time-stepping scheme such as [71 201–203] is used, where the part of the linearisation of space-discretised differential equations,which is responsible for the stiffness of the system, is integrated precisely; (2) convergence to the dominant magnetic mode is optimised by enabling transitions to new trajectories with a sequence of initial conditions converging to the mode (this optimisation can be performed applying the methods [325]); (3) the discrepancy $|\mathscr{L}\mathbf{h} - \zeta\mathbf{h}|$ is computed to test the termination condition for the computation, rather than a loosely defined "exponential pattern of the evolution of the mode" is examined.

[5] Every sample flow can be regarded as a solution to the Navier–Stokes equation with an appropriate steady forcing. Another condition for physical soundness of the kinematic dynamo problem is hydrodynamic stability of the flow, but this holds true if viscosity is large enough.

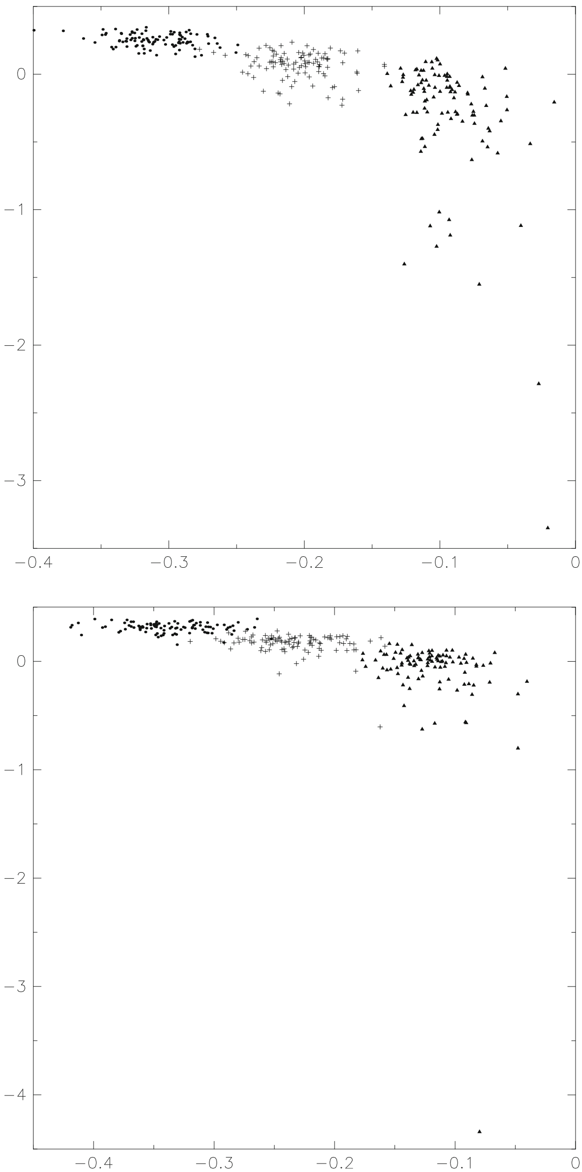

Fig. 3.1 Minimum magnetic eddy diffusivity η_{eddy} (*the vertical axis*) versus the growth rate of the dominant short-scale magnetic mode (*the horizontal axis*) for $\eta = 0.3$ (*dots*), 0.2 (*pluses*) and 0.1 (*triangles*) in the two ensembles of 100 flows with the exponential (*the upper panel*) and hyperbolic (*the lower panel*) energy spectrum fall off

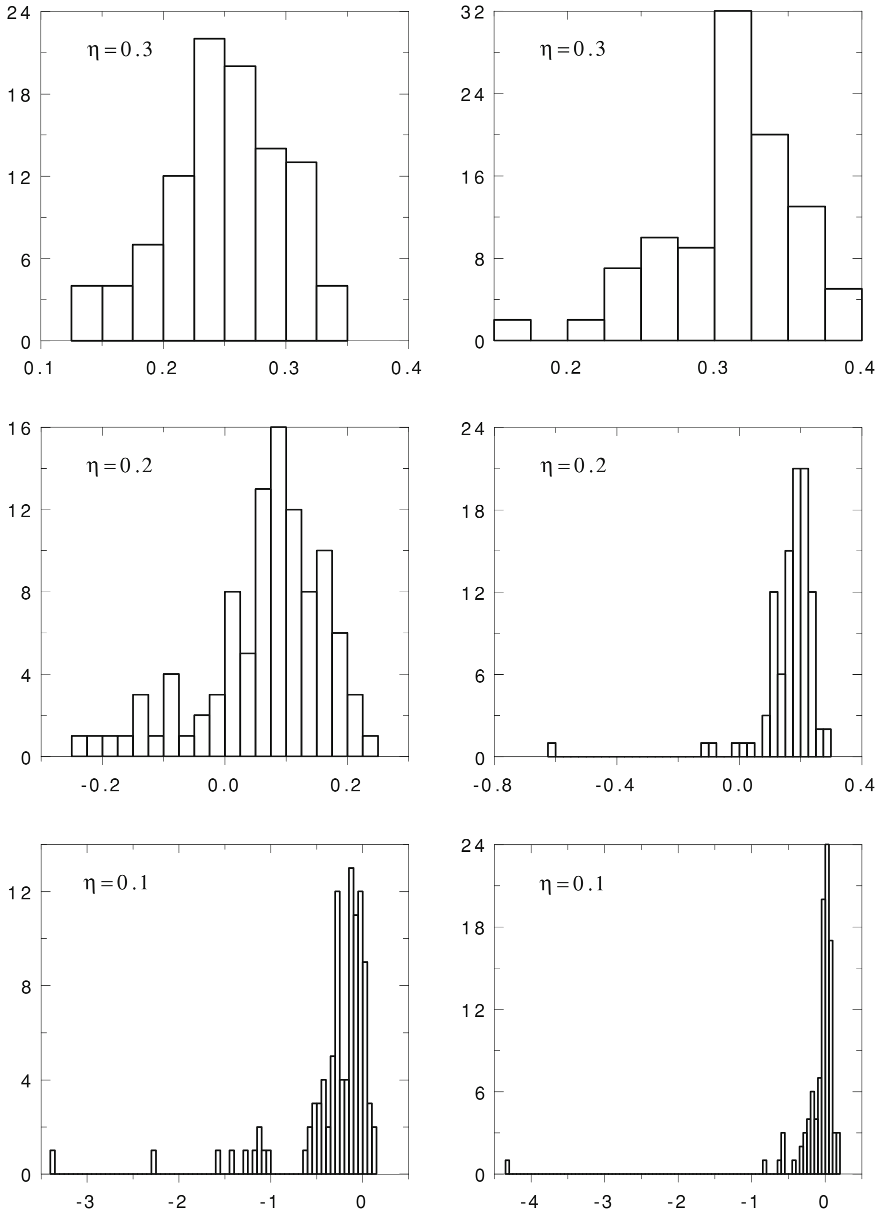

Fig. 3.2 Histograms of magnetic eddy diffusivity η_{eddy} (*the horizontal axis*) in the two ensembles of parity-invariant flows with the exponential (*the left column*) and hyperbolic (*the right column*) energy spectrum fall off

Table 3.1 Statistics of magnetic eddy diffusivity in the two ensembles of flows

	Exponential spectrum (%)			Hyperbolic spectrum (%)		
	$\eta = 0.3$	$\eta = 0.2$	$\eta = 0.1$	$\eta = 0.3$	$\eta = 0.2$	$\eta = 0.1$
$\mathrm{Re}\,\eta_{\mathrm{eddy}} < 0$	0	18	86	0	4	53
$\mathrm{Re}\,\eta_{\mathrm{eddy}} < \eta$	83	96	98	30	63	94
$\mathrm{Im}\,\lambda_2 \neq 0$	7	2	2	8	5	3

After the coefficients (3.34) were determined, the minimum magnetic eddy diffusivity

$$\eta_{\mathrm{eddy}} = \min_{|\mathbf{q}|=1}(-\mathrm{Re}\,\lambda_2(\mathbf{q})) \tag{3.45}$$

was evaluated. For a given $\mathbf{q}$, $\lambda_2(\mathbf{q})$, defined by the eigenvalue problem (3.35)–(3.36), is a root of a quadratic equation, and thus magnetic eddy diffusivity can be complex (like eddy viscosity of a three-dimensional flow, see [312, 313]). To accommodate such cases (which are infrequent—see Table 3.1) real parts of roots are considered in (3.45).

The statistics of the obtained values of magnetic eddy diffusivity (Table 3.1 and Fig. 3.2) shows that for smaller molecular diffusivities magnetic eddy correction is more likely to be negative, and the fraction of flows with negative eddy diffusivity is larger. Eddy diffusivity is generally lower in flows with the exponentially falling off energy spectrum than in flows with the hyperbolic one. This indicates that small wave number harmonics in flows are more important for eddy correction to be negative.

The fluctuating part of a dominant large-scale magnetic mode (more precisely, $\{\mathbf{h}_0\}$) for a sample flow with the exponentially falling off energy spectrum is shown in Fig. 3.3 in a short-scale periodicity cell (the mean field $\langle\mathbf{h}_0\rangle$ is of the order of the fluctuating part). The pattern can be continued periodically in space with the harmonic modulation prescribed by the factor $e^{i\mathbf{q}\cdot\mathbf{X}}$. It is easy to verify that the $\mathbf{L}$-periodicity and parity invariance of the flow imply its parity invariance about each point in the regular grid $(n_1L_1/2, n_2L_2/2, n_3L_3/2)$, where n_i are arbitrary integers. A cube of size $L/2 = \pi$ involving 8 points of the grid inside the cube of periodicity (whose vertices do not belong to the grid) is shown by dashed lines. Each centre of parity invariance is, evidently, a stagnation point of the flow. All stagnation points are located inside regions of small kinetic energy density shown on the upper panel. In the vicinity of stagnation points with one-dimensional unsteady manifolds magnetic flux ropes can emerge [111, 186]. Magnetic structures shown in Fig. 3.3 are apparently such flux ropes.

3.7 Strongly Negative Magnetic Eddy Diffusivity

Histograms Fig. 3.2 for $\eta = 0.1$ reveal several large in magnitude negative magnetic eddy diffusivities. In this section we discuss the mechanism of their emergence. Figure 3.1 provides a key: it shows that eddy diffusivities are

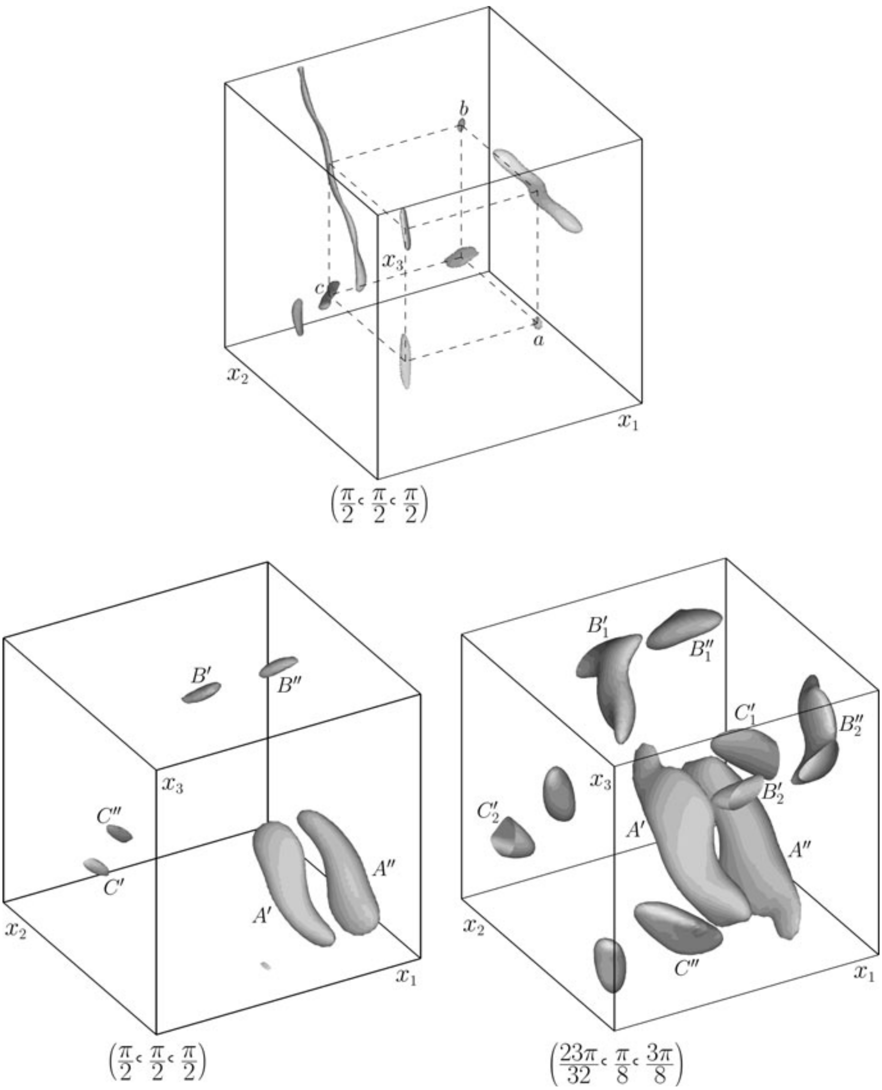

Fig. 3.3 A magnetic mode for $\eta = 0.1$ and a sample flow with the exponentially falling off energy spectrum. *The upper panel*: isosurfaces of $|\mathbf{V}|^2$ at the level of 0.002 of the maximum. *The lower panels*: isosurfaces of $|\mathbf{h}|^2$ at the levels of 0.6 (*left*) and 0.4 (*right*) of the maximum. *Dashed lines*: an elementary cube of the grid of parity invariance centres of the flow. Magnetic structures labelled A, B and C are associated with the flow stagnation points at the vertices of the cube labelled a, b and c, respectively. *Ordinary and double primes* in the labels refer to disjoint parts of the same structure. *Subscripts* enumerate fragments of structures, artificially produced by cutting them with faces of the shown periodicity cubes

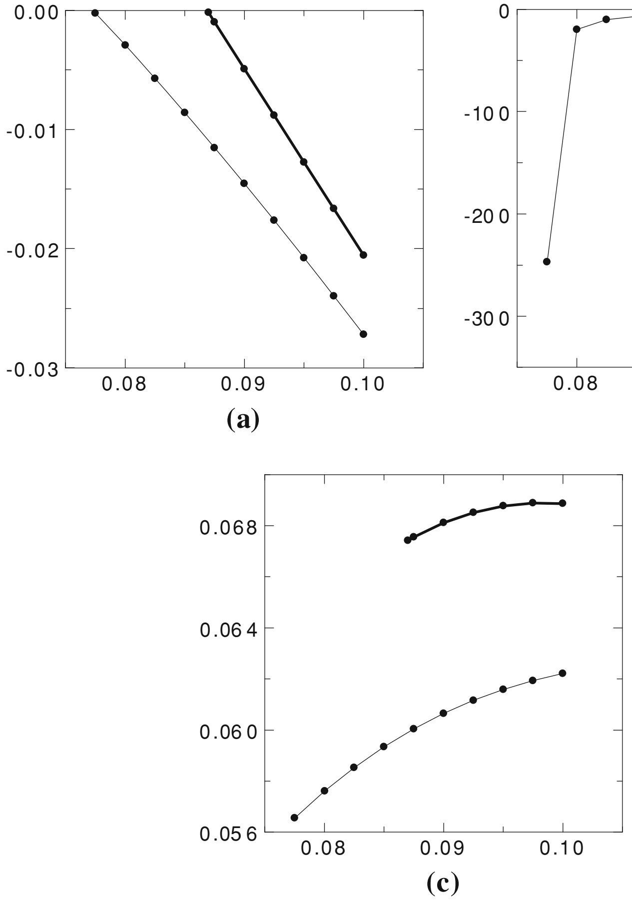

Fig. 3.4 Maximum growth rates ζ_1 of short-scale magnetic modes (**a**), minimum magnetic eddy diffusivities η_{eddy} (**b**) and the products $\zeta_1\eta_{\mathrm{eddy}}$ (**c**) as functions of molecular diffusivity η (*the horizontal axes*). *Lines* of different widths refer to two flow samples, shown by two points with the minimum η_{eddy} in the lower right part of Fig. 3.1a

strongly negative, when the growth rate of the dominant zero-mean short-scale magnetic mode is close to zero. The respective eigenvalue of the magnetic induction operator $\mathscr{L}$ is real. Solutions to the auxiliary problems (3.18) and (3.30) are obtained by application of the operator $\mathscr{L}^{-1}$ to their right-hand sides, and hence the smallness of an eigenvalue of $\mathscr{L}$ can result in an anomalously large solution and, in favourable conditions, in a strongly negative magnetic eddy diffusivity. This explanation was verified numerically by checking that large in magnitude magnetic eddy diffusivities are approximately inversely proportional to the small in absolute value dominant eigenvalue of $\mathscr{L}$ (see Fig. 3.4).

Let $\mathbf{f}_p(\eta)$ be the basis of short-scale **L**-periodic zero-mean magnetic modes:

$$\mathscr{L}\mathbf{f}_p(\eta) = \zeta_p(\eta)\mathbf{f}_p(\eta), \quad |\mathbf{f}_p(\eta)| = 1, \quad \nabla \cdot \mathbf{f}_p(\eta) = 0$$

(for the sake of simplicity we assume in this section that $\mathscr{L}$ does not involve Jordan normal form cells of size 2 or higher). Suppose for η decreasing to the critical value η_c for the onset of generation, the dominant magnetic mode becomes zero:

$$\mathrm{Re}\,\zeta_p(\eta) < 0 \quad \text{for all } p \text{ and } \eta < \eta_c; \quad \zeta_1(\eta_c) = 0.$$

Generically, the loss of magnetic stability occurs in one of the invariant subspaces—of parity-invariant, or parity-antiinvariant solenoidal vector fields. This gives rise to two versions of the mechanism that we are considering.

Suppose short-scale parity-invariant magnetic fields are the first to loose stability. For $\eta < \eta_c$ solutions $\mathbf{S}_k(\eta)$ to the auxiliary problems of type I (3.18) depend on η smoothly. The auxiliary problem of type II (3.30) can be recast as

$$\sum_p \zeta_p(\eta)\gamma_{mk,p}(\eta)\mathbf{f}_p(\eta) = \sum_p \kappa_{mk,p}(\eta)\mathbf{f}_p(\eta), \tag{3.46}$$

where

$$-2\eta\frac{\partial \mathbf{S}_k(\eta)}{\partial x_m} + V_m(\mathbf{S}_k(\eta) + \mathbf{e}_k) - \mathbf{V}(S_k(\eta))_m = \sum_p \kappa_{mk,p}(\eta)\mathbf{f}_p(\eta)$$

and

$$\mathbf{G}_{mk}(\eta) = \sum_p \gamma_{mk,p}(\eta)\mathbf{f}_p(\eta)$$

are expansions of the r.h.s. of (3.30) and $\mathbf{G}_{mk}(\eta)$ in the basis of magnetic modes, and summation is over parity-invariant modes. From (3.46), $\gamma_{mk,p}(\eta) = \kappa_{mk,p}(\eta)/\zeta_p(\eta)$. Since generically $\kappa_{mk,1}(\eta_c) \neq 0$,

$$\mathbf{G}_{mk}(\eta) \approx \frac{\kappa_{mk,1}(\eta_c)}{\zeta_1(\eta)}\mathbf{f}_1(\eta_c) \to \infty,$$

for $\eta \to \eta_c$, and the leading, in this limit, part of the eigenvalue equation (3.35) is

$$\frac{\mathbf{A}(\mathbf{q})}{\zeta_1(\eta)}\sum_{k=1}^{3}\sum_{m=1}^{3}\kappa_{mk,1}(\eta_c)\widetilde{h}_k q_m = -\lambda_2\widetilde{\mathbf{h}}, \tag{3.47}$$

where $\mathbf{A}(\mathbf{q}) \equiv \mathbf{q} \times \langle \mathbf{V} \times \mathbf{f}_1(\eta_c)\rangle$. Two eigenvalues can be found from (3.47) and (3.36). One of them, $\lambda_2' = 0$, is of no interest; the associated eigenvector $\widetilde{\mathbf{h}}'$ satisfies

$$\sum_{k=1}^{3}\sum_{m=1}^{3}\kappa_{mk,1}(\eta_c)\widetilde{h}_k' q_m = 0$$

and (3.36). The second eigenvalue and the associated eigenvector are

$$\lambda_2 = -\frac{a(\mathbf{q})}{\zeta_1(\eta)}, \quad a(\mathbf{q}) \equiv \sum_{k=1}^{3}\sum_{m=1}^{3} \kappa_{mk,1}(\eta_c) A_k(\mathbf{q}) q_m, \quad \widetilde{\mathbf{h}} = \mathbf{A}(\mathbf{q})$$

(we exclude from consideration wave vectors $\mathbf{q}$, for which $\mathbf{A}(\mathbf{q}) = 0$). Hence, if $\widehat{a} \equiv \max_{|\mathbf{q}|=1} a(\mathbf{q}) > 0$, then the minimum magnetic eddy diffusivity admits in the limit $\eta \to \eta_c$ arbitrarily large in magnitude negative values:

$$\eta_{\mathrm{eddy}} \approx \widehat{a}/\zeta_1(\eta) \to -\infty.$$

Now suppose short-scale parity-antiinvariant magnetic fields become unstable the first. Then solutions $\mathbf{S}_k(\eta)$ to the auxiliary problems of type I (3.18) tend to infinity. The problem can be expressed as

$$\sum_p \zeta_p(\eta) s_{k,p}(\eta) \mathbf{f}_p(\eta) = \sum_p \rho_{k,p}(\eta) \mathbf{f}_p(\eta),$$

where

$$-\frac{\partial \mathbf{V}}{\partial x_k} = \sum_p \rho_{k,p}(\eta) \mathbf{f}_p(\eta)$$

and

$$\mathbf{S}_k(\eta) = \sum_p s_{k,p}(\eta) \mathbf{f}_p(\eta)$$

are expansions of the r.h.s. of (3.18) and $\mathbf{S}_k(\eta)$ in parity-antiinvariant magnetic modes. Thus, $s_{k,p}(\eta) = \rho_{k,p}(\eta)/\zeta_p(\eta)$. Since generically $\rho_{k,1}(\eta_c) \neq 0$, we find in the limit $\eta \to \eta_c$

$$\mathbf{S}_k(\eta) \approx \frac{\rho_{k,1}(\eta_c)}{\zeta_1(\eta)} \mathbf{f}_1(\eta_c) \to \infty.$$

Let γ_m denote a solution to the problem

$$\mathscr{L}\gamma_m = -2\eta_c \frac{\partial \mathbf{f}_1(\eta_c)}{\partial x_m} + \mathbf{V}_m \mathbf{f}_1(\eta_c) - \mathbf{V}(\mathbf{f}_1(\eta_c))_m.$$

Then

$$\mathbf{G}_{mk}(\eta) \approx \frac{\rho_{k,1}(\eta_c)}{\zeta_1(\eta)} \gamma_m,$$

and hence the leading part of the eigenvalue equation (3.35) is

$$\frac{\mathbf{B}(\mathbf{q})}{\zeta_1(\eta)} \sum_{k=1}^{3} \rho_{k,1}(\eta_c) \widetilde{h}_k = -\lambda_2 \widetilde{\mathbf{h}}, \tag{3.48}$$

where

$$\mathbf{B}(\mathbf{q}) \equiv \sum_{m=1}^{3} \mathbf{q} \times \langle \mathbf{V} \times \gamma_m \rangle q_m.$$

Equations 3.48 and 3.36 yield two eigenvalues. One is $\lambda_2' = 0$ with the associated eigenvector $\widetilde{\mathbf{h}}'$ normal to $\mathbf{q}$ and satisfying

$$\sum_{k=1}^{3} \rho_{k,1}(\eta_h) \widetilde{h}_k' = 0.$$

The second eigenvalue and eigenvector are

$$\lambda_2 = -\frac{b(\mathbf{q})}{\zeta_1(\eta)}, \quad b(\mathbf{q}) \equiv \sum_{k=1}^{3} \rho_{k,1}(\eta_c) B_k(\mathbf{q}), \quad \widetilde{\mathbf{h}} = \mathbf{B}(\mathbf{q})$$

(we consider only such wave vectors $\mathbf{q}$, that $\mathbf{B}(\mathbf{q}) \neq 0$). Consequently, for $\eta \to \eta_c$ the minimum magnetic eddy diffusivity again admits arbitrarily large in magnitude negative values:

$$\eta_{\text{eddy}} \approx \widehat{b}/\zeta_1(\eta) \to -\infty,$$

provided $\widehat{b} \equiv \max_{|\mathbf{q}|=1} b(\mathbf{q}) > 0$.

A singular behaviour of this kind indirectly suggests that the asymptotic expansion constructed in Sects. 3.1–3.5 is inapplicable for $\eta = \eta_c$. The increase of the kernel of $\mathscr{L}^*$ for the critical value $\eta = \eta_c$ implies the respective increase of the number of solvability conditions for problems in the fast variables, obtained from the series (3.13) at various orders. Of course, a similar asymptotic expansion can be constructed for $\eta = \eta_c$. The problem for the limit operator will then involve mean fields $\langle \mathbf{h}_0 \rangle$ as before, and additional variables—amplitudes of the short-scale eigenfunctions from the kernel of $\mathscr{L}$, which will be also present in the leading term of the expansion of a large-scale magnetic mode. If the flow is parity-invariant and the kernel of $\mathscr{L}$ involves parity-invariant short-scale neutral modes, then the algebra is similar to that in the presence of the α-effect: the limit operator emerges in the analysis of solvability of the equation obtained from the series (3.13) at order ε, and it is a first-order partial differential operator in the slow variables; the associated eigenvalue is $O(\varepsilon)$. If the flow is parity-invariant and all short-scale modes in $\ker \mathscr{L}$ are parity-antiinvariant, then the limit operator is a second-order partial differential operator in the slow variables, generalising the operator of magnetic eddy diffusion, $\mathscr{E}$, derived in Sect. 3.4.2.

3.8 Perturbation of an Oscillatory Short-Scale Magnetic Mode

We consider now the second possibility for solvability of the order ε^0 equation (3.16): $\langle \mathbf{h}_0 \rangle = 0$, $\lambda_0 \neq 0$. As discussed in Sect. 3.1, the case is physically interesting when λ_0 is imaginary. If such an eigenvalue exists, generically it has multiplicity one, and we will assume that the problem is considered under this condition. In this section we assume that the flow $\mathbf{V}(\mathbf{x})$ is parity-invariant (3.6) in Sect. 3.8.3 onwards.

Step 2° for $n = 0$. By virtue of Eq. 3.16, $\mathbf{h}_0$ is now an eigenfunction of the short-scale magnetic induction operator, associated with the imaginary eigenvalue λ_0:

$$\mathbf{h}_0(\mathbf{X}, \mathbf{x}) = c_0(\mathbf{X})\mathbf{S}(\mathbf{x}), \quad \mathscr{L}\mathbf{S}(\mathbf{x}) = \lambda_0 \mathbf{S}(\mathbf{x}). \tag{3.49}$$

Note that the eigenvalue equation for the short-scale oscillatory mode $\mathbf{S}$ implies $\langle \mathbf{S}(\mathbf{x}) \rangle = 0$, and hence the condition $\langle \mathbf{h}_0 \rangle = 0$ is automatically enforced for solutions of the form (3.49). Taking the divergence of this eigenvalue equation, we obtain

$$\eta \nabla^2 (\nabla \cdot \mathbf{S}) = \lambda_0 \nabla \cdot \mathbf{S},$$

i.e. $\nabla \cdot \mathbf{S}$ is a scalar eigenfunction of the Laplacian. However, under the condition of space periodicity all eigenvalues of the Laplacian are real and negative, this implying that $\mathbf{S}$ is solenoidal.

3.8.1 Solvability Condition for Auxiliary Problems

Let $\mathbf{S}^*(\mathbf{x})$ denote the eigenfunction of the operator $\mathscr{L}'^*$ (the adjoint of $\mathscr{L}'$) associated with the eigenvalue λ_0. We normalise it by the condition

$$\int_{[0,\mathbf{L}]} \mathbf{S}(\mathbf{x}) \cdot \mathbf{S}^*(\mathbf{x})\, \mathrm{d}\mathbf{x} = 1.$$

Consider an equation

$$(\mathscr{L} - \lambda_0)\mathbf{h} = \mathbf{f}. \tag{3.50}$$

Scalar multiplying it by $\mathbf{S}^*(\mathbf{x})$ and integrating over a short-scale periodicity cell (this projects the equation into the subspace spanned by $\mathbf{S}(\mathbf{x})$), we find a necessary condition for solvability of this problem:

$$\int_{[0,\mathbf{L}]} \mathbf{f}(\mathbf{x}) \cdot \mathbf{S}^*(\mathbf{x})\, \mathrm{d}\mathbf{x} = 0. \tag{3.51}$$

As in Sect. 3.2, it can be shown, that this condition is also sufficient. To do this, we note that $\langle \mathbf{h} \rangle = -\langle \mathbf{f} \rangle / \lambda_0$, reduce Eq. 3.50 to the one with the zero-mean data, and apply to the new equation the inverse Laplacian, thus obtaining a reduced problem, which involves a linear operator of the correct structure for application of the

Fredholm alternative theorem. The solvability condition for the reduced problem then can be rendered in the form (3.51) noting that $\{\mathbf{S}^*\}$ is the eigenfunction of the operator $\mathscr{L}^*$ associated with the eigenvalue λ_0, and

$$\langle \mathbf{S}^* \rangle = -\langle \mathbf{V} \times (\nabla \times \{\mathbf{S}^*\})\rangle/\lambda_0.$$

In other words, the operator $\mathscr{L}' - \lambda_0$ is invertible in the invariant subspace, complementary to the subspace spanned by $\mathbf{S}(\mathbf{x})$.

In the present context, it is convenient to join the relations for divergencies (3.11) and (3.12) into a single one,

$$\nabla_{\mathbf{x}} \cdot \mathbf{h}_n + \nabla_{\mathbf{X}} \cdot \mathbf{h}_{n-1} = 0. \tag{3.52}$$

Solenoidality of $\mathbf{S}(\mathbf{x})$ verifies this equation for $n = 0$.

3.8.2 Order ε^1 Equation: The α-Effect

Step $1°$ for $n = 1$. For $\mathbf{h}_0$ satisfying (3.49), the equation obtained from series (3.13) at order ε^1 reduces to

$$(\mathscr{L}' - \lambda_0)\mathbf{h}_1 = -2\eta\nabla_{\mathbf{x}}\mathbf{S} \cdot \nabla_{\mathbf{X}} c_0 - \nabla_{\mathbf{X}} c_0 \times (\mathbf{V} \times \mathbf{S}) + \lambda_1 \mathbf{S} c_0. \tag{3.53}$$

The mean part of this equation can be used to determine the mean $\langle \mathbf{h}_1 \rangle$:

$$\langle \mathbf{h}_1 \rangle = \nabla_{\mathbf{X}} c_0 \times \langle \mathbf{V} \times \mathbf{S} \rangle / \lambda_0.$$

The solvability condition (3.51) for Eq. (3.53) reduces to

$$\mathscr{A} c_0 \equiv \sum_{k=1}^{3} \alpha_k \frac{\partial c_0}{\partial X_k} = \lambda_1 c_0, \tag{3.54}$$

where

$$\alpha_k \equiv \int_{[0,\mathbf{L}]} \left(2\eta \frac{\partial \mathbf{S}}{\partial x_k} + \mathbf{e}_k \times (\mathbf{V} \times \mathbf{S}) \right) \cdot \mathbf{S}^* \, d\mathbf{x}. \tag{3.55}$$

Thus, the subleading term λ_1 in the expansion (3.9) of the eigenvalue (the real part of λ_1 is the leading term in the expansion of the growth rate of the large-scale magnetic mode under consideration), and the amplitude $c_0(\mathbf{X})$ of the leading term in the expansion (3.8) of the large-scale mode are solutions to the eigenvalue problem for the operator $\mathscr{A}$ defined by (3.54). By analogy with the mean-field case, this operator is called the *operator of the α-effect* acting on perturbation of an oscillatory mode acting on large-scale perturbations of a short-scale oscillatory magnetic mode. This operator is a sum of first-order derivatives in the slow spatial variables, generically implying a superexponential growth of a seed large-scale magnetic field (see Sect. 3.3.2).

Thus, in investigation of generation of large-scale magnetic modes which are perturbations of oscillatory short-scale modes $\mathbf{S}(\mathbf{x})$, we obtain an *amplitude*[6] *equation* for the modulating amplitude $c_0(\mathbf{X})$ of the mode $\mathbf{S}(\mathbf{x})$, involving the α-effect operator. In subsections that follow we consider the case, where this α-effect operator is zero.

3.8.3 Order ε^1 Equations in the Absence of the α-Effect

We consider henceforth magnetic field generation by parity-invariant flows (3.6). The domain of the operator $\mathscr{L}'$ is a direct sum of two subspaces, of parity-invariant and parity-antiinvariant vector fields, and the oscillatory mode $\mathbf{S}$ and eigenfunction $\mathbf{S}^*$ are simultaneously either parity-invariant, or parity-antiinvariant. In both cases, expressions (3.55) imply $\alpha_k = 0$, i.e. the α-effect is absent and hence (3.54) reduces to $\lambda_1 = 0$.

Step 2° for $n = 1$. Consequently, the parity (with respect to the fast variable $\mathbf{x}$) of the r.h.s. of the order ε^1 equation (3.53) is opposite to that of $\mathbf{S}$, and in the respective invariant subspace the operator $\mathscr{L}' - \lambda_0$ is invertible. Owing to the structure of the r.h.s. of (3.53) and its linearity, solutions to this equation are

$$\mathbf{h}_1(\mathbf{X}, \mathbf{x}) = c_1(\mathbf{X})\mathbf{S}(\mathbf{x}) + \sum_{k=1}^{3} \frac{\partial c_0}{\partial X_k} \mathbf{G}_k(\mathbf{x}). \tag{3.56}$$

Here, the three vector fields $\mathbf{G}_k$ are $\mathbf{L}$ periodic solutions to auxiliary problems of type II':

$$(\mathscr{L}' - \lambda_0)\mathbf{G}_k = -2\eta \frac{\partial \mathbf{S}}{\partial x_k} - \mathbf{e}_k \times (\mathbf{V} \times \mathbf{S}). \tag{3.57}$$

The parity of the r.h.s. of this equation is opposite to that of $\mathbf{S}$, and therefore the solvability condition (3.51) is satisfied; since parity-invariant and parity-antiinvariant fields compose invariant subspaces of $\mathscr{L}$, the parity of $\mathbf{G}_k$ is opposite to that of $\mathbf{S}$.

Taking the divergence of (3.57) and adding the kth component of the eigenvalue equation (3.49) for the short-scale magnetic mode $\mathbf{S}$, we obtain

$$\eta\nabla^2(\nabla \cdot \mathbf{G}_k + S_k) = \lambda_0(\nabla \cdot \mathbf{G}_k + S_k).$$

This identity implies that $\nabla \cdot \mathbf{G}_k + S_k = 0$, and therefore the relation (3.52) for $n = 1$ holds true.

---

[6] Mean-field equations are a particular case of amplitude equations, since the mean fields can be interpreted as amplitudes, depending on the slow variables and modulating the respective steady non-zero-mean short-scale modes, see, for instance, (3.17).

The mean part of Eq. 3.57 is equivalent to

$$\langle \mathbf{G}_k \rangle = \mathbf{e}_k \times \langle \mathbf{V} \times \mathbf{S} \rangle / \lambda_0.$$

3.8.4 Order ε^2 Equations in the Absence of α-Effect: Eddy Diffusion Revisited

Step 1° for $n = 2$. At order ε^2 we obtain from (3.13) the equation

$$(\mathscr{L}' - \lambda_0)\mathbf{h}_2 = -\eta(2(\nabla_\mathbf{x} \cdot \nabla_\mathbf{X})\mathbf{h}_1 + \nabla^2_\mathbf{X}\mathbf{h}_0) - \nabla_\mathbf{X} \times (\mathbf{V} \times \mathbf{h}_1) + \lambda_2 \mathbf{h}_0. \quad (3.58)$$

Substituting here the solution (3.56) to the order ε^1 equation, scalar multiplying by $\mathbf{S}^*(\mathbf{x})$ and integrating over a short-scale periodicity cell, we find

$$\mathscr{E}_{\mathrm{osc}} c_0 \equiv \eta \nabla^2_\mathbf{X} c_0 + \sum_{k=1}^{3} \sum_{m=1}^{3} D_{mk} \frac{\partial^2 c_0}{\partial X_k \partial X_m} = \lambda_2 c_0, \quad (3.59)$$

where

$$D_{mk} \equiv \int_{[0,\mathbf{L}]} \left(2\eta \frac{\partial \mathbf{G}_k}{\partial x_m} + \mathbf{e}_m \times (\mathbf{V} \times \mathbf{G}_k) \right) \mathbf{S}^* \, \mathrm{d}\mathbf{x}.$$

Thus, we have derived an eigenvalue equation for determination of the amplitude $c_0(\mathbf{X})$ of the short-scale oscillatory magnetic mode $\mathbf{S}(\mathbf{x})$ in the leading term in the expansion of the large-scale stability mode (3.8), and the subleading term λ_2 in the expansion of the associated eigenvalue (3.9); the real part of λ_2 is the leading term in expansion of the growth rate of the large-scale mode. The operator $\mathscr{E}_{\mathrm{osc}}$ defined by (3.59) is a sum of second-order derivatives, and we interpret it as the operator of eddy diffusion, acting on large-scale perturbation of a short-scale oscillatory magnetic mode. The operator is, in general, anisotropic. The coefficients D_{mk} in $\mathscr{E}_{\mathrm{osc}}$ are constant; consequently, if the mode is defined in the entire space of the slow variables and is globally bounded, the eigenfunctions are Fourier harmonics. If the eddy diffusion operator $\mathscr{E}_{\mathrm{osc}}$ has an eigenvalue with a positive real part (which is not ruled out), one speaks of generation of a large-scale magnetic mode by the mechanism of *negative magnetic eddy diffusivity*.

As in Sects. 3.3 and 3.4, and in Sect. 3.8.2, we interrupt here solution of the problem at hand, although it is possible to solve further equations from the hierarchy arising from (3.13) and thus to construct complete asymptotic expansions (3.8) and (3.9).

3.9 Magnetic Field Generation with Moderate Scale Separation

In this section we somewhat deviate from the main topic of the present book and consider magnetic field generation for finite scale ratios. Two questions were studied numerically in [336]: (i) How do growth rates of dominant magnetic modes depend on the scale ratio for flows with negative magnetic eddy diffusivity? (ii) How does the critical magnetic Reynolds number behave when ε is modestly lowered from $\varepsilon = 1$?

Computations are simplified by the fact that for a fixed scale ratio ε magnetic modes can be sought in the form

$$\mathbf{H} = \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{q}\cdot\mathbf{x}}\mathbf{h}(\mathbf{x}; \varepsilon, \mathbf{q}), \tag{3.60}$$

(agreeing with the structure of the large-scale magnetic modes, discussed at the end of Sect. 3.5). Substituting (3.60) into the eigenvalue equation (3.3) for the magnetic induction operator and cancelling out the factor $\mathrm{e}^{\mathrm{i}\varepsilon\mathbf{q}\cdot\mathbf{x}}$, we obtain a short-scale eigenvalue problem for a modified operator parameterised by the quantities ε and $\mathbf{q}$ (cf. [94, 252]). The modified operator and the solenoidality condition are obtained from the original ones by the modification of gradients

$$\nabla \to \nabla_{\mathbf{x}} + \mathrm{i}\varepsilon\mathbf{q}.$$

To find the dominant eigenvalue for the given scale ratio ε, it remains to maximise the growth rates over directions $\mathbf{q}$. Since this requires a large amount of computations (lowering of the condition number in auxiliary problems makes them significantly less demanding numerically than computation of a dominant eigenvalue), maximisation over all $\mathbf{q}$ of unit length was not performed. Instead, only "binary" wave vectors were considered, every component of which takes the values 0 or 1; then ε can be interpreted as the scale ratio along the respective coordinate axes. For every flow, the binary wave vector was chosen as follows: For $\eta = 0.1$ the vector $\mathbf{q}$, minimising magnetic eddy diffusivity, was determined. For all four considered flows it turned out to be approximately parallel to a binary one, and the latter was used in computations.

Maximum growth rates for four samples from the ensemble of flows with a hyperbolic energy spectrum, exhibiting negative magnetic eddy diffusivity, are shown in Fig. 3.5. The spectra of the modified operators have the following properties: (i) For wave vectors $\mathbf{q}$ with integer components, for any two values of ε differing by 1, the spectra coincide, and eigenfunctions associated with the same eigenvalue coincide up to a factor $\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}$. (ii) Due to parity invariance of the flow (3.6), for any two values of ε differing in sign, the spectra coincide, and eigenfunctions associated with the same eigenvalue are transformed one into another by the mapping $\mathbf{H}(\mathbf{x}) \to \mathbf{H}(-\mathbf{x})$. Consequently, the plots in Fig. 3.5 can be continued over the entire range of ε by 1-periodicity and the symmetry about the vertical line $\varepsilon = 1/2$. Property (ii) implies, that the spectrum is complex-conjugate: eigenvalues λ and $\overline{\lambda}$ are associated with eigenfunctions $\mathbf{H}(\mathbf{x})$ and $\overline{\mathbf{H}(-\mathbf{x})}$, respectively.

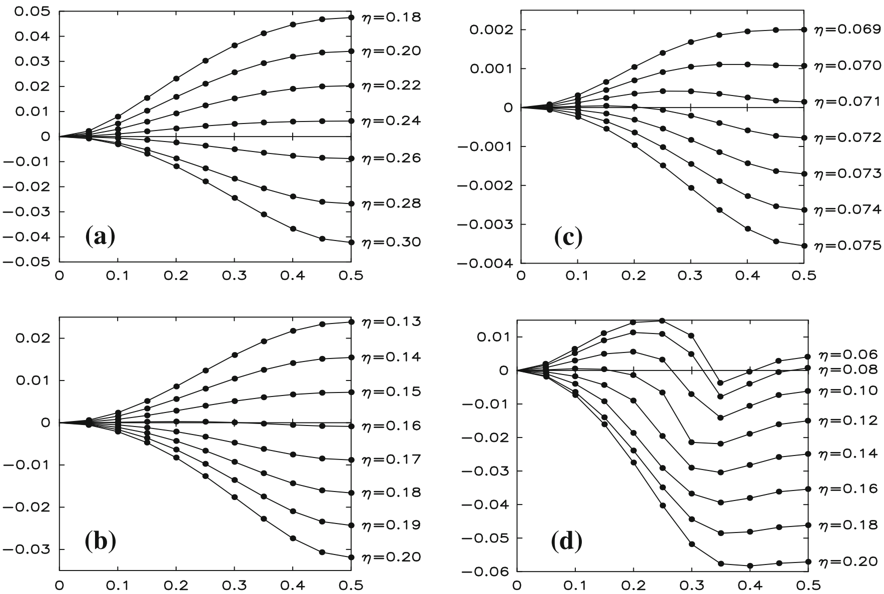

Fig. 3.5 Maximum growth rates (*the vertical axis*) of magnetic modes as a function of the scale ratio ε (*the horizontal axis*) for different molecular diffusivities η for four samples from the ensemble of flows with the hyperbolic energy spectrum

Computations presented in Fig. 3.5 were performed with the resolution of 64^3 Fourier harmonics by an adapted version of the code [325]. Energy spectra of magnetic modes decrease at least by 4 orders of magnitude for $\eta = 0.13$ and 5 orders for $\eta = 0.3$, confirming that adequate resolution was employed. For $\eta = 0.07$ it decreases by 3 orders only. Although from a conservative point of view this is the minimum acceptable fall off, the growth rates differ from those computed for test purposes with the resolution of 32^3 Fourier harmonics only in the third significant digit. This is in line with the observation [46] that eigenvalues are less sensitive to insufficient resolution than the fine structure of eigenfunctions.

In agreement with the eigenvalue series expansion (3.9) for $\lambda_0 = \lambda_1 = 0$ (see Sects. 3.3.1 and 3.4.1), near $\varepsilon = 0$ plots in Fig. 3.5 have a parabolic shape. When magnetic eddy diffusivity is negative (i.e., the growth rates are positive in the limit of $\varepsilon \to 0$), generation typically occurs for all scale ratios, or ceases only for considerably large ones (i.e., the intervals of η, for which it ceases for small ε, are very short). Switching of analytical branches of dominant magnetic modes near $\varepsilon = 0.3$ is responsible for windows of dynamo disappearance in Fig. 3.5d.

Why maximum growth rates tend to zero when $\varepsilon \to 0$, deserves a comment. The eigenfunction of the modified operator, $\mathbf{h}(\mathbf{x}; \varepsilon, \mathbf{q})$, is not required to have a zero spatial mean. Nevertheless, unless ε is integer, the spatial mean of the mode (3.60) is zero; this explains why some dominant eigenvalues are negative in

Fig. 3.5, despite there always exist non-zero-mean magnetic modes associated with a zero eigenvalue (see Sect. 3.2). However, for $\varepsilon \to 0$, coefficients of the modified operator tend to those of the non-modified magnetic induction operator, and hence eigenvalues and eigenfunctions of the former tend to the ones of the latter. Computations show that $\langle \mathbf{h}(\mathbf{x}; \varepsilon, \mathbf{q}) \rangle$ does not tend to zero in the limit under consideration. Since non-zero-mean eigenfunctions of the magnetic induction operator belong to its kernel, for $\varepsilon = 0$ the zero growth rate is obtained. (Recall that we deliberately exclude from consideration the combinations of the flow and magnetic molecular diffusivity, for which short-scale generation is possible.)

Finally, we study how a moderate scale separation (more precisely, doubling of the spatial scale of magnetic field) affects the critical magnetic Reynolds number for the onset of magnetic field generation. We define magnetic Reynolds numbers for the maximum length scale present in the problem—that of the magnetic field; the r.m.s. velocity is 1 (see an outline of the algorithm for generation of flows in the previous section), implying $R_m = 1/(\varepsilon\eta)$. Computations were done in [336] for 30 samples from the ensemble of flows with a hyperbolic energy spectrum fall off (see Fig. 3.6). For $\varepsilon = 1/2$, computations for all seven possible binary $\mathbf{q} \neq 0$ were made with the resolution of 32^3 Fourier harmonics, and afterwards the critical magnetic diffusivity was refined with the resolution of 64^3 Fourier harmonics for the wave vector $\mathbf{q}$, for which the maximum of low-resolution critical values was admitted. The secant method was used to determine the critical magnetic molecular diffusivity, for which the dominant mode is steady or time-periodic; iterations were terminated when the real part of the eigenvalue was below 10^{-5}.

In all computations for $\varepsilon = 1/2$ energy spectra decrease by at least 4 orders of magnitude; hence the 64^3 harmonics resolution is sufficient. The most demanding computations are for $\varepsilon = 1$ and $R_m > 50$, for which energy spectra decrease by 2–3 orders of magnitude; however, in agreement with the remark above, the

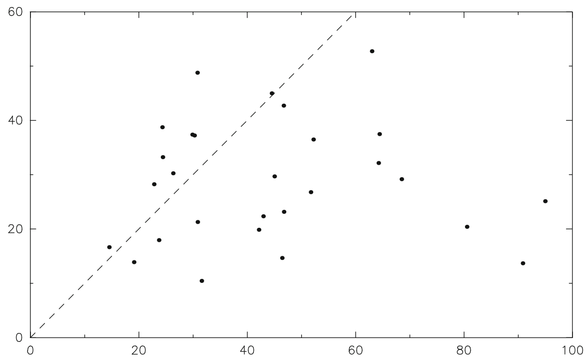

Fig. 3.6 Critical magnetic Reynolds numbers for the onset of short-scale magnetic field generation ($\varepsilon = 1$; *the horizontal axis*) versus critical magnetic Reynolds numbers for the onset of generation of magnetic field of a twice larger spatial scale ($\varepsilon = 1/2$; *the vertical axis*) for 28 parity-invariant flows with a hyperbolic energy spectrum fall off. *Dashed line*: equal Reynolds numbers

eigenvalues are nevertheless determined accurately enough (the critical magnetic Reynolds numbers computed with the resolutions of 32^3 and 64^3 harmonics differ in the third significant digit).

Only 28 out of the 30 considered flow samples are shown in Fig. 3.6. For the two remaining flows, the critical magnetic Reynolds numbers for $\varepsilon = 1/2$ are $R_m = 30.7$ and 25.3, but these flows do not generate a short-scale ($\varepsilon = 1$) magnetic field for $R_m \leq 100$. Thus, in 70% scale of magnetic field has resulted in a decrease of the critical value. The same effect was observed in [79, 105] for ABC flows and in [204] for the modified Taylor–Green flow.

3.10 Conclusions

1. In this chapter we have constructed asymptotic expansions of large-scale magnetic modes (3.8) generated by short-scale steady space-periodic modes and their growth rates in power series in the scale ratio, and derived a closed set of equations for the leading terms of the expansions. Large-scale perturbations of neutral (belonging to the kernel of the magnetic induction operator $\mathscr{L}'$) and oscillatory (associated with imaginary eigenvalues of $\mathscr{L}'$) short-scale modes have been considered. It is shown that generically the α-effect takes place. The amplitudes of the short-scale modes, comprising the leading term of the expansion of a magnetic mode, are a solution to the eigenvalue problem for the operator of α-effect, (3.24) or (3.54), respectively. An operator of α-effect is a partial differential operator of the first order of a special structure, whose spectrum is symmetric about the imaginary axis, and therefore generically the kinematic dynamo operates (under certain restrictions on the shape of the region of the slow variables, where the mode resides). Moreover, space-periodic (in the slow variables) magnetic fields experience superexponential growth. If the flow is parity-invariant, then the limit operators, (3.33) or (3.59), respectively, are partial differential operators of the second order, describing magnetic eddy diffusion; generically it is anisotropic.
2. Magnetic eddy diffusivity was computed for 600 combinations of magnetic molecular diffusivity and space-periodic parity-invariant flows composed of random-amplitude Fourier harmonics with a prescribed exponential or hyperbolic fall off of the energy spectrum. The results demonstrate that negative magnetic eddy diffusivity is a physically realisable phenomenon: for the considered magnetic molecular diffusivities $\eta = 0.1$, 0.2 and 0.3, no short-scale kinematic magnetic dynamo operates, but for $\eta = 0.1$ 86% of flows with the exponential energy spectrum and 53% of flows with the hyperbolic energy spectrum generate large-scale magnetic field by the mechanism of negative magnetic eddy diffusivity.
3. Examples of space-periodic parity-invariant flows with an anomalously strong negative magnetic eddy diffusivity are obtained. We have explained how this

happens, when the magnetic Reynolds number is close to the critical value for the onset of short-scale magnetic field generation.

4. Direct numerical simulations have shown that doubling of the spatial scale of magnetic field decreases the critical magnetic Reynolds number for the onset of magnetic field generation in 70% of instances. A detailed study of the dependence of the growth rate of dominant magnetic modes on the scale ratio ε for four sample flows indicates that flows with negative magnetic eddy diffusivity can cease to kinematically generate magnetic field at small finite scale ratios ε, but this typically happens only in a short interval of magnetic molecular diffusivities. Switching of analytical branches of dominant magnetic modes, occurring when ε is varied, can give rise to windows, where no generation takes place.

Chapter 4
Time-Periodic Large-Scale Magnetic Dynamos

We investigate in this chapter generation of large-scale magnetic fields by short-scale parity-invariant flows, which are space- and time-periodic. The presentation is based on [335].

We have demonstrated in the previous chapter that steady parity-invariant flows exhibit negative eddy diffusivity relatively often. It might be desirable to perform a similar investigation for unsteady turbulent flows, but this problem is numerically intensive and solving it consumes significant computational resources. For this reason, we consider an "intermediate" class of time-periodic flows and focus on flows of, perhaps, the simplest in this class dependence on time,

$$\mathbf{V}(\mathbf{x},t) = \mathbf{U}(\mathbf{x}) + \sqrt{\omega}(\mathbf{V}_c(\mathbf{x})\cos\omega t + \mathbf{V}_s(\mathbf{x})\sin\omega t), \tag{4.1}$$

which can still provide an insight into the effects featured by flows involving more complicated periodic time dependencies.

Like for steady flows, parity invariance implies the absence of the α-effect in the magnetic dynamo under consideration. In order to derive the tensor of eddy correction of magnetic diffusion, we consider, as in the previous chapter, expansions of magnetic modes and the associated eigenvalues in asymptotic power series in the scale ratio ε. As before, magnetic eddy diffusion is anisotropic, and the coefficients of the operator of eddy correction of magnetic diffusion can be expressed in the terms of solutions to short-scale auxiliary problems. For the flow (4.1), the derivation of the mean-field equations is simpler than in the general case, and auxiliary problems have a reduced numerical complexity.

Space- and time-periodic flows of two kinds were extensively studied within the framework of fast kinematic dynamo theory. Both flows are generalisations of ABC flows; they are comprised of a small number of trigonometric functions. In "modulated waves" considered in [57, 211] the time dependence of the form (4.1) was assumed. Generation by "circularly polarised" flows was studied analytically in [57, 234, 235]. Because of the considered time dependence, trajectories of fluid

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particles in these flows exhibit chaotic behaviour (which is a necessary condition for a dynamo to be fast), despite the flows depend only on two spatial coordinates. Independence on the third coordinate gave a possibility to separate the variables, and as a result to simulate the evolution of magnetic fields for large magnetic Reynolds numbers $R_m \sim 10^4$ [107, 110] with a sufficient resolution. Circularly polarised flows depending on three spatial coordinates were considered in [42, 43].

We investigate here the dynamo action of flows, which are more physically realistic; as in the previous chapter, computations were performed for synthetic flows, comprised of Fourier harmonics of random amplitudes with an exponential fall off of the energy spectrum. (As we have shown in Chap. 3, flows comprised of random-amplitude harmonics with a slow—hyperbolic—fall off of the energy spectrum are less efficient generators, and we do not consider them here). They can be regarded as a model of turbulent flows, which is more precise than similar steady flows considered in Chap. 3. Even for the simple time dependence (4.1), computation of the tensor of eddy correction of magnetic diffusion requires considerable computer resources. For this reason we did not collect statistics for time-dependent flows as large, as for steady flows. Because of the random choice of amplitudes of Fourier harmonics, we can nevertheless hope that our numerical results are typical for a sufficiently wide class of flows resembling turbulent ones. (By contrast, if a term with a sine or a cosine is missing in (4.1), no genericity of results can be expected: for instance, as shown in Sect. 4.6, in this case the contribution from the time-dependent part of the flow to magnetic eddy diffusion disappears in the limit of high frequencies).

Our computations confirm the general result obtained for the steady flows: scale separation is beneficial for magnetic field generation. However, time-periodic flows turn out to be, in general, less efficient dynamos than the steady ones. The case of high temporal frequencies ω is considered in Sect. 4.6. It is shown that due to the presence of the factor $\sqrt{\omega}$ in the time-dependent terms in (4.1), their contribution to magnetic eddy diffusivity is finite in the limit $\omega \to \infty$. We have numerically verified that magnetic eddy diffusivity can remain negative at large ω.

4.1 The Floquet Problem for Large-Scale Magnetic Modes

4.1.1 Statement of the Problem

We consider in this chapter the kinematic dynamo problem for a flow with a velocity $\mathbf{V}(\mathbf{x}, t)$, which has a period $\bar{T} = 2\pi/\omega$ in time t, and periods L_i in the spatial variables x_i. As in the previous chapter, we assume that the fluid is incompressible,

$$\nabla \cdot \mathbf{V} = 0,$$

and the flow is parity-invariant,

$$\mathbf{V}(\mathbf{x},t) = -\mathbf{V}(-\mathbf{x},t). \tag{4.2}$$

The temporal evolution of a magnetic field $\mathbf{h}$ is governed by the magnetic induction equation,

$$\mathscr{M}\mathbf{h} \equiv -\frac{\partial \mathbf{h}}{\partial t} + \eta\nabla^2\mathbf{h} + \nabla \times (\mathbf{V} \times \mathbf{h}) = 0. \tag{4.3}$$

Substituting here $\mathbf{h} \to \mathbf{h}(\mathbf{x},t)\mathrm{e}^{\lambda t}$, we find that a *magnetic mode* $\mathbf{h}(\mathbf{x},t)$ is a solution to the Floquet problem for the operator of magnetic induction, $\mathscr{M}$:

$$\mathscr{M}\mathbf{h} = \lambda\mathbf{h}. \tag{4.4}$$

The mode is solenoidal,

$$\nabla \cdot \mathbf{h} = 0, \tag{4.5}$$

and has the same period $\bar{T}$ in time, as the flow has. Re λ is the growth (or decay, depending on the sign) rate of the magnetic mode. Eigenfunctions of the problem (4.4), (4.5) can be determined up to a factor $\mathrm{e}^{\mathrm{i}J\omega t}$, where J is an arbitrary integer; the associated modes differ by the factor $\mathrm{i}J\omega$. Hence it is natural to impose the normalisation condition

$$|\mathrm{Im}\,\lambda| \le \omega/2. \tag{4.6}$$

4.1.2 Asymptotic Expansion of Large-Scale Magnetic Modes

We study large-scale magnetic modes $\mathbf{h}(\mathbf{X},\mathbf{x},t)$ that are globally bounded and depend on the fast, $\mathbf{x}$, and slow, $\mathbf{X} = \varepsilon\mathbf{x}$, spatial variables (the flow $\mathbf{V}$ is independent of the slow variables). Consequently, we differentiate in (4.4) and (4.5) using the chain rule:

$$\nabla \to \nabla_\mathbf{x} + \varepsilon\nabla_\mathbf{X}. \tag{4.7}$$

The spatial scale ratio ε is a small parameter. A mode is supposed to have the same vector of spatial periods $\mathbf{L}$ in the fast spatial variables, as the flow. A discussion of the boundary conditions in Sect. 3.4.2, remains applicable without any modification for the case of time-periodic flows considered here.

We will seek a solution to the Floquet problem (4.4) in the form of power series

$$\lambda = \sum_{n=0}^{\infty} \lambda_n \varepsilon^n, \tag{4.8}$$

$$\mathbf{h} = \sum_{n=0}^{\infty} \mathbf{h}_n(\mathbf{X}, \mathbf{x}, t)\varepsilon^n. \tag{4.9}$$

Substitution of the series (4.8) and (4.9) into equations (4.4) and (4.5) defining magnetic modes, and expansion of the result into a power series in ε yields a hierarchy of equations.

4.1.3 Solvability of Auxiliary Problems

In this subsection we consider the hierarchy of short-scale equations emerging for the Floquet problem (4.4) for an arbitrary time- and space-periodic flow $\mathbf{V}(\mathbf{x}, t)$. We establish conditions for solvability of auxiliary problems and will present, in the next subsection, a sketch of solution of the Floquet problem in this general case. Solution for the flow of the simple structure (4.1) is discussed in details in Sects. 4.2 and 4.3.

Let us denote the spatio-temporal mean over the fast variables $\mathbf{x}$ and t, and the fluctuating part of a field f by

$$\langle\!\langle f \rangle\!\rangle \equiv \frac{1}{\bar{T}} \int_0^{\bar{T}} \langle f(\mathbf{X}, \mathbf{x}, t)\rangle \mathrm{d}t, \qquad \{\!\{ f \}\!\} \equiv f - \langle\!\langle f \rangle\!\rangle,$$

respectively; the spatial mean $\langle f \rangle$ over the fast variables $\mathbf{x}$ was defined in Sect. 3.1.3.

We denote by $\mathscr{L}'$ the reduction of the operator of magnetic induction, $\mathscr{M}$, defined by (4.3), on the linear space of vector fields, $\mathbf{L}$-periodic in space and $\bar{T}$-periodic in time, and by $\mathscr{L}$ the reduction of $\mathscr{L}'$ on the subspace of vector fields with a vanishing spatio-temporal mean. Differentiation in the fast variables $\mathbf{x}$ and t only is performed in $\mathscr{L}'$ and $\mathscr{L}$.

The operators, adjoint to $\mathscr{L}'$ and $\mathscr{L}$, take the form

$$\mathscr{L}'^{*}\mathbf{h} = \frac{\partial \mathbf{h}}{\partial t} + \eta \nabla^2_{\mathbf{x}} \mathbf{h} + (\nabla_{\mathbf{x}} \times \mathbf{h}) \times \mathbf{V}, \qquad \mathscr{L}^{*}\mathbf{h} = \{\!\{ \mathscr{L}'^{*}\mathbf{h} \}\!\}$$

(like $\mathscr{L}$, $\mathscr{L}^*$ acts in the subspace of vector fields with a vanishing spatio-temporal mean). Since, evidently, $\mathscr{L}'^{*}\mathbf{e}_k = 0$, the kernel of $\mathscr{L}'^{*}$ is at least three-dimensional. We will consider the generic case, in which its dimension is equal to three (for a given flow $\mathbf{V}(\mathbf{x}, t)$ this does not hold only for a countable set of values of η). In this case the kernels of $\mathscr{L}^*$ and $\mathscr{L}$ are trivial. This implies that the spatio-temporal means of vector fields from $\ker \mathscr{L}'$ are non-zero.

Let us show that under this assumption

$$\langle\!\langle \mathbf{f} \rangle\!\rangle = 0 \tag{4.10}$$

is the solvability condition for the problem

$$\mathscr{L}'\mathbf{g} = \mathbf{f}, \tag{4.11}$$

where the r.h.s. is $\mathbf{L}$-periodic in space and $\bar{T}$-periodic in time and belongs to the Lebesgue space $\mathbb{L}_2([0, \mathbf{L}] \times [0, \bar{T}])$.

First, we consider the problem

$$-\frac{\partial \mathbf{g}}{\partial t} + \eta \nabla^2 \mathbf{g} = \mathbf{f}.$$

Under condition (4.10) its solution possessing the same periodicities as the r.h.s. can be constructed as a Fourier series:

$$\mathbf{g} = -\sum_{m,\mathbf{n}} \frac{\hat{\mathbf{f}}_{m,\mathbf{n}}}{\mathrm{i}m\omega + 4\pi^2 \eta \sum_{q=1}^{3} (n_q/L_q)^2} \mathrm{e}^{\mathrm{i}\left(m\omega t + 2\pi \sum_{q=1}^{3} n_q x_q / L_q\right)}, \tag{4.12}$$

where the term for $m = 0, \mathbf{n} = 0$ is absent due to (4.10), and

$$\mathbf{f} = \sum_{m,\mathbf{n}} \hat{\mathbf{f}}_{m,\mathbf{n}} \mathrm{e}^{\mathrm{i}\left(m\omega t + 2\pi \sum_{q=1}^{3} n_q x_q / L_q\right)}$$

is the Fourier series for the field $\mathbf{f}$.

Equation (4.12) defines the operator $(-\partial/\partial t + \eta \nabla^2)^{-1}$, acting in the subspace of zero-mean vector fields possessing the periodicities under consideration and belonging to the Lebesgue space $\mathbb{L}_2([0, \mathbf{L}] \times [0, \bar{T}])$. We can therefore consider now the problem

$$\mathbf{g} + \mathscr{B}\mathbf{g} = \left(-\frac{\partial}{\partial t} + \eta \nabla^2\right)^{-1} \mathbf{f}, \tag{4.13}$$

which is equivalent to the problem (4.11), (4.10) where

$$\mathscr{B}\mathbf{g} \equiv \left(-\frac{\partial}{\partial t} + \eta \nabla^2\right)^{-1} (\nabla \times (\mathbf{V} \times \mathbf{g})).$$

The operator $\mathscr{L}'^{*}\left(\frac{\partial}{\partial t} + \eta \nabla^2\right)^{-1}$ defined in the above-mentioned subspace is adjoint to the operator in the l.h.s. of (4.13). Our assumption implies that its kernel is trivial. Therefore, by the Fredholm alternative theorem [154, 171] this problem (and (4.11) simultaneously with it) has a unique solution in this subspace, since the operator $\mathscr{B}$ is compact.

The latter statement is well-known in the theory of parabolic partial differential equations. For the reader's convenience we will give now an elementary proof of compactness of the operator

$$\tilde{\mathscr{B}}\mathbf{g} \equiv \left(-\frac{\partial}{\partial t} + \eta\nabla^2\right)^{-1} \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\mathbf{a}_j \cdot \mathbf{g})$$

acting in the same subspace under the condition that vector fields $\mathbf{a}_j(\mathbf{x}, t)$ are sufficiently smooth and have the periodicities under consideration; clearly, this implies compactness of $\mathscr{B}$. It suffices to show that $\tilde{\mathscr{B}}$ is a limit of finite-dimensional operators [154]. We expand $\mathbf{a}_j$ and $\mathbf{g}$ into Fourier series:

$$\mathbf{a}_j = \sum_{m,\mathbf{n}} \hat{\mathbf{a}}_{j,m,\mathbf{n}} \mathrm{e}^{\mathrm{i}\left(m\omega t + 2\pi \sum_{q=1}^{3} n_q x_q / L_q\right)}, \quad \mathbf{g} = \sum_{m,\mathbf{n}} \hat{\mathbf{g}}_{m,\mathbf{n}} \mathrm{e}^{\mathrm{i}\left(m\omega t + 2\pi \sum_{q=1}^{3} n_q x_q / L_q\right)}.$$

Consequently,

$$\tilde{\mathscr{B}}\mathbf{g} = \sum_{M,\mathbf{N}} \sum_{m,\mathbf{n}} \sum_{j=1}^{3} \frac{-2\pi\mathrm{i}\hat{\mathbf{a}}_{j,m,\mathbf{n}} \cdot \hat{\mathbf{g}}_{M-m,\mathbf{N}-\mathbf{n}} N_j / L_j}{\mathrm{i}M\omega + 4\pi^2\eta \sum_{q=1}^{3} (N_q/L_q)^2} \mathrm{e}^{\mathrm{i}\left(M\omega t + 2\pi \sum_{q=1}^{3} N_q x_q / L_q\right)}.$$

Let us estimate the norm of the operator

$$\tilde{\mathscr{B}}_K\mathbf{g} \equiv \sum_{\substack{|M| \geq K, \\ |\mathbf{N}| \geq K}} \sum_{m,\mathbf{n}} \sum_{j=1}^{3} \frac{-2\pi\mathrm{i}\hat{\mathbf{a}}_{j,m,\mathbf{n}} \cdot \hat{\mathbf{g}}_{M-m,\mathbf{N}-\mathbf{n}} N_j / L_j}{\mathrm{i}M\omega + 4\pi^2\eta \sum_{q=1}^{3} (N_q/L_q)^2} \mathrm{e}^{\mathrm{i}\left(M\omega t + 2\pi \sum_{q=1}^{3} N_q x_q / L_q\right)}.$$

By the Cauchy–Schwarz–Buniakowski inequality and Parseval's theorem,

$$\begin{aligned}
|\tilde{\mathscr{B}}_K\mathbf{g}|^2 &\leq \sum_{\substack{|M| \geq K, \\ |\mathbf{N}| \geq K}} \left| \sum_{m,\mathbf{n}} \sum_{j=1}^{3} \frac{2\pi\hat{\mathbf{a}}_{j,m,\mathbf{n}} \cdot \hat{\mathbf{g}}_{M-m,\mathbf{N}-\mathbf{n}} N_j / L_j}{\mathrm{i}M\omega + 4\pi^2\eta \sum_{q=1}^{3} (N_q/L_q)^2} \right|^2 \\
&\leq \sum_{\substack{|M| \geq K, \\ |\mathbf{N}| \geq K}} \left(\frac{4\pi^2 \sum_{q=1}^{3} (N_q/L_q)^2}{M^2\omega^2 + \left(4\pi^2\eta \sum_{q=1}^{3} (N_q/L_q)^2\right)^2} \left(\sum_{m,\mathbf{n}} \sum_{j=1}^{3} |\hat{\mathbf{a}}_{j,m,\mathbf{n}}|^2 (|\mathbf{n}| + |m| + 1)^{\alpha} \right) \right. \\
&\quad \times \left. \left(\sum_{m,\mathbf{n}} |\hat{\mathbf{g}}_{M-m,\mathbf{N}-\mathbf{n}}|^2 (|\mathbf{n}| + |m| + 1)^{-\alpha} \right) \right) \\
&\leq \left(\frac{|\mathbf{g}| \max_q L_q}{2\pi\eta K} \right)^2 \left(\sum_{m,\mathbf{n}} \sum_{j=1}^{3} |\hat{\mathbf{a}}_{j,m,\mathbf{n}}|^2 (|\mathbf{n}| + |m| + 1)^{\alpha} \right) \sum_{m,\mathbf{n}} (|\mathbf{n}| + |m| + 1)^{-\alpha}.
\end{aligned}$$

The last sum converges for any $\alpha > 4$; the second factor is finite by the assumption on the regularity of $\mathbf{a}_j$. The inequality implies, that for the given $\mathbf{a}_j$, for any $\alpha > 4$ and $\delta > 0$ one can choose such a K that $\|\tilde{\mathscr{B}}_K\| < \delta$. Thus, we have shown that $\tilde{\mathscr{B}}$ is a limit of finite-dimensional operators $\tilde{\mathscr{B}} - \tilde{\mathscr{B}}_K$.

As a corollary, we find that

$$\mathscr{L}^{-1} = (\mathscr{I} + \mathscr{B})^{-1} \left(-\frac{\partial}{\partial t} + \eta\nabla^2\right)^{-1},$$

(where $\mathscr{I}$ is the identity) is a compact operator (being a product of a bounded operator and a compact one [154). In particular, this implies [154], that every eigenvalue of $\mathscr{L}$ is of a finite multiplicity, and the absolute values of eigenvalues of $\mathscr{L}$ tend to infinity.

4.1.4 Solution of the Floquet Problem for a Generic Time-Periodic Parity-Invariant Flow

The derivation of the mean-field equation and the magnetic eddy diffusion operator for an arbitrary parity-invariant periodic flow follows the derivation for a steady parity-invariant flow, considered in Chap. 3, with two changes:

- The spatio-temporal averaging $\langle\langle\cdot\rangle\rangle$ is performed in place of the spatial averaging $\langle\cdot\rangle$.
- The operators $\mathscr{M}$, $\mathscr{L}$ and $\mathscr{L}'$ are defined by (4.3) and have the domains defined in Sect. 4.1.3.

As a result, the magnetic mean-field equation takes the form (3.33), and the tensor of eddy correction of magnetic diffusion can be expressed in the terms of solutions to the auxiliary problems of type II, $\mathbf{G}_{mk}$, as the spatio-temporal means

$$\mathbf{D}_{mk} = \langle\langle \mathbf{V} \times \mathbf{G}_{mk} \rangle\rangle \tag{4.14}$$

(cf. 3.34).

4.2 Magnetic Eddy Diffusion in Time-Periodic Parity-Invariant Flows of the Simple Structure

In Sects. 4.2 and 4.3 we will construct complete asymptotic expansions of large-scale magnetic modes in the terms of coefficients of the Fourier series in time and of the associated eigenvalues, for flows whose dependence on time is of the simple structure (4.1). The mean-field eigenvalue equations are employed in Sect. 4.5 for a numerical investigation of generation of large-scale magnetic fields.

4.2.1 The Hierarchy of Equations for Flows of the Simple Structure

It is convenient to recast the flow (4.1) in the form

$$\mathbf{V}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x}) + \mathbf{W}(\mathbf{x})\mathrm{e}^{\mathrm{i}\omega t} + \overline{\mathbf{W}}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}, \tag{4.15}$$

where we have denoted

$$\mathbf{W} = \frac{1}{2}\sqrt{\omega}(\mathbf{V}_c - \mathrm{i}\mathbf{V}_s). \tag{4.16}$$

By our assumptions concerning the flow $\mathbf{V}$, vector fields $\mathbf{U}(\mathbf{x})$ and $\mathbf{W}(\mathbf{x})$ are $\mathbf{L}$-periodic in the fast spatial variables and independent of the fast time and of the slow variables. They are solenoidal,

$$\nabla_{\mathbf{x}} \cdot \mathbf{U} = \nabla_{\mathbf{x}} \cdot \mathbf{W} = 0, \tag{4.17}$$

and parity-invariant:

$$\mathbf{U}(\mathbf{x}) = -\mathbf{U}(-\mathbf{x}), \quad \mathbf{W}(\mathbf{x}) = -\mathbf{W}(-\mathbf{x}). \tag{4.18}$$

Let us expand a magnetic mode $\mathbf{h}$ into a Fourier series in time:

$$\mathbf{h}(\mathbf{X}, \mathbf{x}, t) = \sum_{j=-\infty}^{\infty} \hat{\mathbf{h}}_j(\mathbf{X}, \mathbf{x})\mathrm{e}^{\mathrm{i}j\omega t}. \tag{4.19}$$

For the flow (4.15), the coefficients in the series satisfy

$$\lambda\hat{\mathbf{h}}_j = -\mathrm{i}j\omega\hat{\mathbf{h}}_j + \eta\nabla^2\hat{\mathbf{h}}_j + \nabla \times (\mathbf{W} \times \hat{\mathbf{h}}_{j-1} + \mathbf{U} \times \hat{\mathbf{h}}_j + \overline{\mathbf{W}} \times \hat{\mathbf{h}}_{j+1}) \tag{4.20}$$

for all j. We seek a solution to this system of equations in the form of asymptotic series

$$\hat{\mathbf{h}}_j(\mathbf{X}, \mathbf{x}) = \sum_{n=0}^{\infty} \hat{\mathbf{h}}_{j,n}(\mathbf{X}, \mathbf{x})\varepsilon^n \tag{4.21}$$

and (4.8). By virtue of (4.21) and (4.19), terms in the asymptotic expansion (4.9) have the Fourier series

$$\mathbf{h}_n = \sum_{j=-\infty}^{\infty} \hat{\mathbf{h}}_{j,n}\mathrm{e}^{\mathrm{i}j\omega t}.$$

After the gradient is transformed according to (4.7) and the coefficients are split into the spatial mean and fluctuating parts, the solenoidality condition (4.5) applied to the large-scale magnetic mode reduces to

$$\nabla_{\mathbf{X}} \cdot \langle\hat{\mathbf{h}}_{j,n}\rangle = 0, \tag{4.22}$$

$$\nabla_{\mathbf{x}} \cdot \{\hat{\mathbf{h}}_{j,n}\} + \nabla_{\mathbf{X}} \cdot \{\hat{\mathbf{h}}_{j,n-1}\} = 0 \tag{4.23}$$

for all j and $n \geq 0$ (we define $\hat{\mathbf{h}}_{j,n} \equiv 0$ for $n < 0$).

Let is expand $\mathscr{L}\mathbf{f}$ into a Fourier series in time:

$$\mathscr{L}\mathbf{f} = \sum_{j=-\infty}^{\infty} \mathscr{L}_j\mathbf{f}\mathrm{e}^{\mathrm{i}j\omega t};$$

here the operators $\mathscr{L}_j$ are defined by the identities

$$\mathscr{L}_j\mathbf{f} \equiv \eta\nabla^2_{\mathbf{x}}\hat{\mathbf{f}}_j + \nabla_{\mathbf{x}} \times (\mathbf{W} \times \hat{\mathbf{f}}_{j-1} + \mathbf{U} \times \hat{\mathbf{f}}_j + \overline{\mathbf{W}} \times \hat{\mathbf{f}}_{j+1}) - \mathrm{i}j\omega\hat{\mathbf{f}}_j, \tag{4.24}$$

and $\hat{\mathbf{f}}_j$ are the Fourier coefficients of the field $\mathbf{f}$:

$$\mathbf{f}(\mathbf{X}, \mathbf{x}, t) = \sum_{j=-\infty}^{\infty} \hat{\mathbf{f}}_j(\mathbf{X}, \mathbf{x})\mathrm{e}^{\mathrm{i}j\omega t}.$$

Implementing the transformation (4.7) of the spatial gradients, and substituting the series (4.21) and (4.8), we find from the eigenvalue equations (4.20):

$$\begin{aligned}\sum_{n=0}^{\infty}\varepsilon^n\Big(&-\mathrm{i}j\omega\langle\hat{\mathbf{h}}_{j,n}\rangle + \mathscr{L}_j\{\mathbf{h}_n\} + \eta\big(2(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\{\hat{\mathbf{h}}_{j,n-1}\} + \nabla^2_{\mathbf{X}}\hat{\mathbf{h}}_{j,n-2}\big)\\ &+\nabla_{\mathbf{x}} \times \big(\mathbf{W}\times\langle\hat{\mathbf{h}}_{j-1,n}\rangle + \mathbf{U}\times\langle\hat{\mathbf{h}}_{j,n}\rangle + \overline{\mathbf{W}}\times\langle\hat{\mathbf{h}}_{j+1,n}\rangle\big)\\ &+\nabla_{\mathbf{X}} \times \big(\mathbf{W}\times\hat{\mathbf{h}}_{j-1,n-1} + \mathbf{U}\times\hat{\mathbf{h}}_{j,n-1} + \overline{\mathbf{W}}\times\hat{\mathbf{h}}_{j+1,n-1}\big) - \sum_{m=0}^{n}\lambda_{n-m}\hat{\mathbf{h}}_{j,m}\Big) = 0.\end{aligned} \tag{4.25}$$

In the subsections that follow, we will consider in detail the equations emerging from this series at successive orders of ε in three steps:

1°. Consider the spatial mean of the equation at order ε^n for $j = 0$. Satisfying the solvability condition, find λ_n and $\langle\hat{\mathbf{h}}_{0,n-2}\rangle$.
2°. Consider the spatial means of the equations for $j \neq 0$ and find $\langle\hat{\mathbf{h}}_{j,n}\rangle$.
3°. Solve in $\{\hat{\mathbf{h}}_{0,n}\}$ the partial differential equation in the fast spatial variables, which represents the fluctuating part of the equation at order ε^n.

4.2.2 Solution of Order ε^0 Equations

The following equations result from the leading (order ε^0) term of the series (4.25):

$$\begin{aligned}&\mathscr{L}_j\{\mathbf{h}_0\} + (\langle\hat{\mathbf{h}}_{j-1,0}\rangle\cdot\nabla_{\mathbf{x}})\mathbf{W} + (\langle\hat{\mathbf{h}}_{j,0}\rangle\cdot\nabla_{\mathbf{x}})\mathbf{U} + (\langle\hat{\mathbf{h}}_{j+1,0}\rangle\cdot\nabla_{\mathbf{x}})\overline{\mathbf{W}}\\ &\quad= \lambda_0\hat{\mathbf{h}}_{j,0} + \mathrm{i}j\omega\langle\hat{\mathbf{h}}_{j,0}\rangle.\end{aligned} \tag{4.26}$$

Steps 1° and 2° for $n = 0$. Averaging of (4.26) over the fast spatial variables yields

$$0 = (\lambda_0 + \mathrm{i}j\omega)\langle\hat{\mathbf{h}}_{j,0}\rangle.$$

In what follows, we assume $\langle\!\langle\mathbf{h}_0\rangle\!\rangle = \langle\hat{\mathbf{h}}_{0,0}\rangle \neq 0$. In this case (note also the normalisation condition (4.6)), the averaged equations imply

$$\langle \hat{\mathbf{h}}_{j,0} \rangle = 0 \quad \text{for all } j \neq 0, \quad \lambda_0 = 0; \tag{4.27}$$

thus, the mean of the leading term in the expansion of the large-scale magnetic mode over the fast spatial variables is independent of the fast time. Another physically interesting case is the one of an imaginary eigenvalue λ_0 (see a comment on this issue in Sect. 3.3.1), but since it is non-generic, we will not consider it. The reader can do this by analogy with our derivations in Sect. 3.8, for a steady flow.

Step 3° for $n = 0$. Since the operators $\mathscr{L}_j$ are linear and the slow variables are not involved in their definition (4.24), the fluctuating parts of the short-scale equations (4.26) have a solution

$$\{\mathbf{h}_{j,0}\} = \sum_{k=1}^{3} \hat{\mathbf{S}}_{j,k} \langle \hat{\mathbf{h}}_{0,0} \rangle_k, \tag{4.28}$$

where vector fields $\hat{\mathbf{S}}_{j,k}(\mathbf{x})$ are solutions to auxiliary problems of type I:

$$\mathscr{L}_j \mathbf{S}_k = -\frac{\partial}{\partial x_k}(\delta_1^j \mathbf{W} + \delta_0^j \mathbf{U} + \delta_{-1}^j \overline{\mathbf{W}}),$$
$$\mathbf{S}_k = \sum_{j=-\infty}^{\infty} \hat{\mathbf{S}}_{j,k}(\mathbf{x}) e^{\mathrm{i}j\omega t}; \tag{4.29}$$

δ_i^j denotes the Kronecker symbol. (Therefore, the field $\mathbf{S}_k$ is a solution to the problem (3.18) with the operator $\mathscr{L}$ defined in Sect. 4.1.3 for the flow (4.15): (4.29) is a restatement of (3.18) in the terms of coefficients of the Fourier series in time).

By the assumption that the kernel of $\mathscr{L}$ is trivial, the problem (4.29) has a unique solution with the zero mean $\langle\langle \mathbf{S}_k \rangle\rangle = 0$. Averaging (4.29) over the fast spatial variables for $j \neq 0$, we find $-\mathrm{i}j\omega \langle \hat{\mathbf{S}}_{j,k} \rangle = 0$, and therefore $\langle \mathbf{S}_k \rangle = 0$ at any time t. Due to uniqueness of solutions and the structure of the operators $\mathscr{L}_j$ (4.24), equations (4.29) imply

$$\hat{\mathbf{S}}_{-j,k} = \overline{\hat{\mathbf{S}}_{j,k}}.$$

Taking the divergence of (4.29), we conclude by the same arguments as were applied for solutions to auxiliary problems of type I for a steady flow (Sect. 3.3.1), that $\hat{\mathbf{S}}_{j,k}$ are solenoidal for all j and k. Due to parity invariance of the flow (4.18), parity-invariant and parity-antiinvariant vector fields constitute two invariant subspaces of the operator $\mathscr{L}$; consequently, parity-antiinvariance of the r.h.s. of (4.29) implies parity-antiinvariance of the solution to the auxiliary problem,

$$\hat{\mathbf{S}}_{j,k}(\mathbf{x}) = \hat{\mathbf{S}}_{j,k}(-\mathbf{x}). \tag{4.30}$$

4.2.3 Solution of Order ε^1 Equations

In view of solenoidality of the flow (4.17) and of the spatial mean of the first term in expansion of the mode (relation (4.22) for $j = n = 0$) and its independence of time (4.27), the second (order ε^1) term of the series (4.25) becomes

$$\begin{aligned}&\mathscr{L}_j\{\mathbf{h}_1\} + 2\eta(\nabla_\mathbf{x}\cdot\nabla_\mathbf{X})\{\hat{\mathbf{h}}_{j,0}\} + (\langle\hat{\mathbf{h}}_{j-1,1}\rangle\cdot\nabla_\mathbf{x})\mathbf{W} + (\langle\hat{\mathbf{h}}_{j,1}\rangle\cdot\nabla_\mathbf{x})\mathbf{U}\\ &+ (\langle\hat{\mathbf{h}}_{j+1,1}\rangle\cdot\nabla_\mathbf{x})\overline{\mathbf{W}} + \nabla_\mathbf{X}\times\left(\mathbf{W}\times\{\hat{\mathbf{h}}_{j-1,0}\} + \mathbf{U}\times\{\hat{\mathbf{h}}_{j,0}\} + \overline{\mathbf{W}}\times\{\hat{\mathbf{h}}_{j+1,0}\}\right)\\ &-\left((\delta_1^j\mathbf{W} + \delta_0^j\mathbf{U} + \delta_{-1}^j\overline{\mathbf{W}})\cdot\nabla_\mathbf{X}\right)\langle\hat{\mathbf{h}}_{0,0}\rangle = \lambda_1\delta_0^j\hat{\mathbf{h}}_{0,0} + \mathrm{i}j\omega\langle\hat{\mathbf{h}}_{j,1}\rangle. \qquad (4.31)\end{aligned}$$

Steps 1° and 2° for $n = 1$. Upon substitution of expressions (4.28) for the fluctuating parts$\{\mathbf{h}_{j,0}\}$ and averaging over the fast spatial variables, we obtain

$$\nabla_\mathbf{X}\times\sum_{k=1}^{3}\langle\mathbf{W}\times\hat{\mathbf{S}}_{j-1,k} + \mathbf{U}\times\hat{\mathbf{S}}_{j,k} + \overline{\mathbf{W}}\times\hat{\mathbf{S}}_{j+1,k}\rangle\langle\hat{\mathbf{h}}_{0,0}\rangle_k = \lambda_1\delta_0^j\langle\hat{\mathbf{h}}_{0,0}\rangle + \mathrm{i}j\omega\langle\hat{\mathbf{h}}_{j,1}\rangle.$$

Due to parity invariance of the flow (4.18) and parity-antiinvariance of the solutions to the auxiliary problems of type I (4.30), the spatial means of vector products in the l.h.s. of this equation (describing the α-effect) vanish, and hence

$$\langle\hat{\mathbf{h}}_{j,1}\rangle = 0 \quad \text{for all } j\neq 0; \quad \lambda_1 = 0. \qquad (4.32)$$

Step 3° for $n = 1$. Upon substitution of (4.28) and (4.32), the fluctuating part of (4.31) becomes

$$\begin{aligned}\mathscr{L}_j\{\mathbf{h}_1\} =& -(\langle\hat{\mathbf{h}}_{0,1}\rangle\cdot\nabla_\mathbf{x})(\delta_1^j\mathbf{W} + \delta_0^j\mathbf{U} + \delta_{-1}^j\overline{\mathbf{W}})\\ &+\sum_{k=1}^{3}\sum_{m=1}^{3}\Bigg(-2\eta\frac{\partial\hat{\mathbf{S}}_{j,k}}{\partial x_m} + (\delta_1^j W_m + \delta_0^j U_m + \delta_{-1}^j\overline{W}_m)\mathbf{e}_k\\ &-\mathbf{e}_m\times(\mathbf{W}\times\hat{\mathbf{S}}_{j-1,k} + \mathbf{U}\times\hat{\mathbf{S}}_{j,k} + \overline{\mathbf{W}}\times\hat{\mathbf{S}}_{j+1,k})\Bigg)\frac{\partial\langle\hat{\mathbf{h}}_{0,0}\rangle_k}{\partial X_m}. \qquad (4.33)\end{aligned}$$

Hence, by linearity of these equations,

$$\{\hat{\mathbf{h}}_{j,1}\} = \sum_{k=1}^{3}\hat{\mathbf{S}}_{j,k}\langle\hat{\mathbf{h}}_{0,1}\rangle_k + \sum_{k=1}^{3}\sum_{m=1}^{3}\hat{\mathbf{G}}_{j,mk}\frac{\partial\langle\hat{\mathbf{h}}_{0,0}\rangle_k}{\partial X_m}, \qquad (4.34)$$

where vector fields $\hat{\mathbf{G}}_{j,mk}(\mathbf{x})$ are solutions to auxiliary problems of type II:

$$\begin{aligned}\mathscr{L}_j\mathbf{G}_{mk} =& -2\eta\frac{\partial\hat{\mathbf{S}}_{j,k}}{\partial x_m} + W_m(\hat{\mathbf{S}}_{j-1,k} + \delta_1^j\mathbf{e}_k) - \mathbf{W}(\hat{S}_{j-1,k})_m + U_m(\hat{\mathbf{S}}_{j,k} + \delta_0^j\mathbf{e}_k)\\ &-\mathbf{U}(\hat{S}_{j,k})_m + \overline{W}_m(\hat{\mathbf{S}}_{j+1,k} + \delta_{-1}^j\mathbf{e}_k) - \overline{\mathbf{W}}(\hat{S}_{j+1,k})_m, \qquad (4.35)\\ \mathbf{G}_{mk} =& \sum_{j=-\infty}^{\infty}\hat{\mathbf{G}}_{j,mk}(\mathbf{x})\mathrm{e}^{\mathrm{i}j\omega t},\end{aligned}$$

satisfying $\langle\!\langle \mathbf{G}_{mk} \rangle\!\rangle = 0$. (Equivalently, $\mathbf{G}_{mk}$ is a solution to the problem (4.30) with the operator $\mathscr{L}$ defined in Sect. 4.1.3 for the flow (4.15).) Due to parity invariance of the flow (4.18) and parity-antiinvariance of $(\hat{S}_{j,k})_m$ for all j, k and m (4.30), the spatial means of the r.h.s. of (4.35) vanish, and hence averaging the equation for $j \neq 0$ we find $\langle \hat{\mathbf{G}}_{j,mk} \rangle = 0$; consequently, $\langle \mathbf{G}_{mk} \rangle = 0$ at any time t. The standard arguments applied to (4.35) reveal that the solutions are real parity-invariant vector fields:

$$\hat{\mathbf{G}}_{-j,mk} = \overline{\hat{\mathbf{G}}_{j,mk}}; \quad \hat{\mathbf{G}}_{j,mk}(\mathbf{x}) = -\hat{\mathbf{G}}_{j,mk}(-\mathbf{x}).$$

Subtracting equation (4.29), expressing the statement of the auxiliary problem of type I, from the divergence of (4.35), and employing the same arguments, as in the case of steady flows (Sect. 3.4.1), we find

$$\nabla_{\mathbf{x}} \cdot \hat{\mathbf{G}}_{j,mk} + (\hat{S}_{j,k})_m = 0, \tag{4.36}$$

i.e. identities (3.31) remain valid, if the flow $\mathbf{V}$ is time-periodic. By virtue of solenoidality of $\hat{\mathbf{S}}_{j,k}$ and (4.36), expressions (4.28) and (4.34) for the fluctuating parts of the first two terms in the expansion of the magnetic mode in the scale ratio imply that relation (4.23) is satisfied for $n = 1$.

4.2.4 The Solvability Condition for Order ε^2 Equations: The Operator of Magnetic Eddy Diffusion

Solenoidality (4.17) of the flow (4.15) and of the spatial mean of the second term in the expansion of the magnetic mode ((4.22) for $j = 0,\ n = 1$), its independence of time (4.32) and vanishing of the first two terms in the expansion of the eigenvalue ((4.27) and (4.32)) imply, that the order ε^2 equation arising from the series (4.25) reduces to

$$\begin{aligned}
&\mathscr{L}_j\{\mathbf{h}_2\} + \eta\big(2(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\{\hat{\mathbf{h}}_{j,1}\} + \nabla^2_{\mathbf{X}}(\delta^j_0\langle\hat{\mathbf{h}}_{0,0}\rangle + \{\hat{\mathbf{h}}_{j,0}\})\big)\\
&+ (\langle\hat{\mathbf{h}}_{j-1,2}\rangle \cdot \nabla_{\mathbf{x}})\mathbf{W} + (\langle\hat{\mathbf{h}}_{j,2}\rangle \cdot \nabla_{\mathbf{x}})\mathbf{U} + (\langle\hat{\mathbf{h}}_{j+1,2}\rangle \cdot \nabla_{\mathbf{x}})\overline{\mathbf{W}}\\
&+ \nabla_{\mathbf{X}} \times \big(\mathbf{W} \times \{\hat{\mathbf{h}}_{j-1,1}\} + \mathbf{U} \times \{\hat{\mathbf{h}}_{j,1}\} + \overline{\mathbf{W}} \times \{\hat{\mathbf{h}}_{j+1,1}\}\big)\\
&- \big((\delta^j_1\mathbf{W} + \delta^j_0\mathbf{U} + \delta^j_{-1}\overline{\mathbf{W}}) \cdot \nabla_{\mathbf{X}}\big)\langle\hat{\mathbf{h}}_{0,1}\rangle = \lambda_2(\delta^j_0\langle\hat{\mathbf{h}}_{0,0}\rangle + \{\hat{\mathbf{h}}_{j,0}\}) + \mathrm{i}j\omega\langle\hat{\mathbf{h}}_{j,2}\rangle.
\end{aligned} \tag{4.37}$$

Step 1° for $n = 2$. In view of the expression (4.34) for the second term in expansion of the magnetic mode, parity invariance (4.18) of the flow (4.15) and parity-antiinvariance (4.30) of solutions to the auxiliary problem of type I, the spatial mean of this equation is

$$\begin{aligned}
\eta\delta^j_0\nabla^2_{\mathbf{X}}\langle\hat{\mathbf{h}}_{0,0}\rangle + \nabla_{\mathbf{X}} \times \sum_{m=1}^{3}\sum_{k=1}^{3}\langle\mathbf{W} \times \hat{\mathbf{G}}_{j-1,mk} + \mathbf{U} \times \hat{\mathbf{G}}_{j,mk} + \overline{\mathbf{W}} \times \hat{\mathbf{G}}_{j+1,mk}\rangle\frac{\partial\langle\hat{\mathbf{h}}_{0,0}\rangle_k}{\partial X_m}&\\
= \lambda_2\delta^j_0\langle\hat{\mathbf{h}}_{0,0}\rangle + \mathrm{i}j\omega\langle\hat{\mathbf{h}}_{j,2}\rangle.&
\end{aligned} \tag{4.38}$$

We find from (4.38) for $j = 0$, that the spatio-temporal mean of the leading term in the expansion of the large-scale magnetic mode, $\langle \hat{\mathbf{h}}_{0,0} \rangle$, and the associated eigenvalue λ_2 are solutions to the eigenvalue problem for the magnetic eddy diffusion operator

$$\mathscr{E}\langle \hat{\mathbf{h}}_{0,0} \rangle \equiv \eta \nabla^2 \langle \hat{\mathbf{h}}_{0,0} \rangle + \nabla_{\mathbf{X}} \times \sum_{m=1}^{3} \sum_{k=1}^{3} \mathbf{D}_{mk} \frac{\partial \langle \hat{\mathbf{h}}_{0,0} \rangle_k}{\partial X_m} = \lambda_2 \langle \hat{\mathbf{h}}_{0,0} \rangle, \tag{4.39}$$

where

$$\mathbf{D}_{mk} = 2\mathrm{Re}\langle \overline{\mathbf{W}} \times \hat{\mathbf{G}}_{1,mk} \rangle + \langle \mathbf{U} \times \hat{\mathbf{G}}_{0,mk} \rangle. \tag{4.40}$$

Equation (4.39) is identical to the magnetic mean-field equation (3.33), which we have derived for a steady flow in Chap. 3. The difference is in that the tensor of eddy correction of magnetic diffusion is determined in the terms of solutions to the auxiliary problems of type II (4.35) by relations (4.14), which reduce to (4.40) for a flow of the simple structure (4.15). The eigenfunction has now the sense of the leading term in the expansion of the large-scale magnetic mode, averaged not only over the fast spatial variables, but also over the fast time. Now relations (4.27) and (4.28) completely determine the Fourier coefficients $\hat{\mathbf{h}}_{j,0}$ of the leading term in the expansion of the large-scale magnetic mode in the spatial scale ratio ε.

As in the case of steady flows, the spatio-temporal means of globally bounded large-scale magnetic modes, that are defined in the entire space of the slow variables, are Fourier harmonics $\langle \hat{\mathbf{h}}_{0,0} \rangle = \widetilde{\mathbf{h}} e^{i\mathbf{q}\cdot\mathbf{X}}$; here $\widetilde{\mathbf{h}}$ and $\mathbf{q}$ are constant three-dimensional vectors satisfying the eigenvalue equation

$$-\eta|\mathbf{q}|^2 \widetilde{\mathbf{h}} - \mathbf{q} \times \sum_{k=1}^{3} \sum_{m=1}^{3} \mathbf{D}_{mk} \widetilde{h}_k q_m = \lambda_2 \widetilde{\mathbf{h}}, \tag{4.41}$$

and the orthogonality relation

$$\widetilde{\mathbf{h}} \cdot \mathbf{q} = 0, \tag{4.42}$$

which is a consequence of solenoidality in the slow variables of the spatio-temporal mean of the leading term in the expansion of the large-scale magnetic mode in the power series in the scale ratio ε ((4.22) for $j = n = 0$).

Step $2°$ for $n = 2$. For $j \neq 0$, (4.38) implies

$$\langle \hat{\mathbf{h}}_{j,2} \rangle = \frac{1}{\mathrm{i}j\omega} \nabla_{\mathbf{X}} \times \sum_{m=1}^{3} \sum_{k=1}^{3} \langle \mathbf{W} \times \hat{\mathbf{G}}_{j-1,mk} + \mathbf{U} \times \hat{\mathbf{G}}_{j,mk} + \overline{\mathbf{W}} \times \hat{\mathbf{G}}_{j+1,mk} \rangle \frac{\partial \langle \hat{\mathbf{h}}_{0,0} \rangle_k}{\partial X_m}.$$

4.2.5 Solution of Order ε^2 Equations

Step $3°$ for $n = 2$. After the expressions (4.28) and (4.34) for the first two terms in the expansion of the Fourier coefficients of the large-scale magnetic mode are substituted, the fluctuating part of (4.38) becomes

$$\begin{aligned}\mathscr{L}_j\{\mathbf{h}_2\} = &-\eta\left(2\sum_{k=1}^{3}\sum_{m=1}^{3}\left(\frac{\partial\hat{\mathbf{S}}_{j,k}}{\partial x_m}\frac{\partial\langle\hat{\mathbf{h}}_{0,1}\rangle_k}{\partial X_m}+\sum_{l=1}^{3}\frac{\partial\hat{\mathbf{G}}_{j,mk}}{\partial x_l}\frac{\partial^2\langle\hat{\mathbf{h}}_{0,0}\rangle_k}{\partial X_m\partial X_l}\right)+\nabla^2_{\mathbf{X}}\{\hat{\mathbf{h}}_{j,0}\}\right)\\
&-(\langle\hat{\mathbf{h}}_{j-1,2}\rangle\cdot\nabla_{\mathbf{x}})\mathbf{W}-(\langle\hat{\mathbf{h}}_{j,2}\rangle\cdot\nabla_{\mathbf{x}})\mathbf{U}-(\langle\hat{\mathbf{h}}_{j+1,2}\rangle\cdot\nabla_{\mathbf{x}})\overline{\mathbf{W}}\\
&+\left((\delta_1^j\mathbf{W}+\delta_0^j\mathbf{U}+\delta_{-1}^j\overline{\mathbf{W}})\cdot\nabla_{\mathbf{X}}\right)\langle\hat{\mathbf{h}}_{0,1}\rangle\\
&-\nabla_{\mathbf{X}}\times\left(\sum_{k=1}^{3}\left(\mathbf{W}\times\hat{\mathbf{S}}_{j-1,k}+\mathbf{U}\times\hat{\mathbf{S}}_{j,k}+\overline{\mathbf{W}}\times\hat{\mathbf{S}}_{j+1,k}\right)\langle\hat{\mathbf{h}}_{0,1}\rangle_k\right.\\
&\left.+\sum_{k=1}^{3}\sum_{m=1}^{3}\{\mathbf{W}\times\hat{\mathbf{G}}_{j-1,mk}+\mathbf{U}\times\hat{\mathbf{G}}_{j,mk}+\overline{\mathbf{W}}\times\hat{\mathbf{G}}_{j+1,mk}\}\frac{\partial\langle\hat{\mathbf{h}}_{0,0}\rangle_k}{\partial X_m}\right)+\lambda_2\{\hat{\mathbf{h}}_{j,0}\}.\end{aligned} \tag{4.43}$$

In this equation, the factors depending on the fast variables in the terms involving the unknown quantities $\langle\hat{\mathbf{h}}_{0,2}\rangle_k$ and $\partial\langle\hat{\mathbf{h}}_{0,1}\rangle_k/\partial X_m$ are the same, as in the terms in (4.33) involving $\langle\hat{\mathbf{h}}_{0,1}\rangle_k$ and $\partial\langle\hat{\mathbf{h}}_{0,0}\rangle_k/\partial X_m$, respectively. By linearity, this implies

$$\{\hat{\mathbf{h}}_{j,2}\} = \sum_{k=1}^{3}\hat{\mathbf{S}}_{j,k}\langle\hat{\mathbf{h}}_{0,2}\rangle_k + \sum_{k=1}^{3}\sum_{m=1}^{3}\hat{\mathbf{G}}_{j,mk}\frac{\partial\langle\hat{\mathbf{h}}_{0,1}\rangle_k}{\partial X_m} + \hat{\mathbf{h}}'_{j,2}.$$

The vector field $\mathbf{h}'_2 = \sum_j \hat{\mathbf{h}}'_{j,2}(\mathbf{X},\mathbf{x})e^{ij\omega t}$, whose spatio-temporal mean is zero (and hence, as implied by the governing equations, the spatial mean is zero at any time) can be uniquely determined from the equations, obtained from (4.43) by replacing $\{\mathbf{h}_2\}\to\mathbf{h}'_2$ and removing all terms containing the factors $\langle\hat{\mathbf{h}}_{0,2}\rangle_k$ and derivatives of $\langle\hat{\mathbf{h}}_{0,1}\rangle_k$ (the r.h.s. of the modified equation is already known at this stage).

4.3 Complete Asymptotic Expansion of Large-Scale Modes for Flows of the Simple Structure

Systems of equations obtained from the series (4.25) at orders ε^n for $n>2$ constitute a hierarchy. In this section we discuss, how its solution can be constructed, if λ_2 is an eigenvalue of multiplicity one of the magnetic eddy diffusion operator $\mathscr{E}$, acting in the space of solenoidal vector fields satisfying the desirable boundary conditions.

Suppose the following information was extracted from orders up to ε^{N-1} systems of equations:

- Vector fields $\langle\hat{\mathbf{h}}_{j,n}\rangle$ for all $j\neq0$ and $n<N$.
- Vector fields $\langle\hat{\mathbf{h}}_{0,n}\rangle$ for all $n<N-2$.
- Vector fields $\{\hat{\mathbf{h}}_{j,n}\}$ for all j and $n<N-2$.

- Expressions for $\{\hat{\mathbf{h}}_{j,n}\}$ of the form

$$\{\hat{\mathbf{h}}_{j,n}\} = \sum_{k=1}^{3} \hat{\mathbf{S}}_{j,k} \langle \hat{\mathbf{h}}_{0,n} \rangle_k + \sum_{k=1}^{3} \sum_{m=1}^{3} \hat{\mathbf{G}}_{j,mk} \frac{\partial \langle \hat{\mathbf{h}}_{0,n-1} \rangle_k}{\partial X_m} + \hat{\mathbf{h}}'_{j,n} \tag{4.44}$$

  for $n = N - 1$ and $n = N - 2$, where the zero-mean fields $\hat{\mathbf{h}}'_{j,n}$ are known.
- The quantities λ_n for all $n < N$.

Upon substitution of the expressions (4.44) for $n = N - 1$ and averaging over the fast spatial variables, the equation obtained from the series (4.25) at order ε^N becomes

$$\begin{aligned} &\eta \nabla^2 \langle \hat{\mathbf{h}}_{j,N-2} \rangle + \nabla_{\mathbf{X}} \times \sum_{m=1}^{3} \sum_{k=1}^{3} \langle \mathbf{W} \times \hat{\mathbf{G}}_{j-1,mk} + \mathbf{U} \times \hat{\mathbf{G}}_{j,mk} \\ &+ \overline{\mathbf{W}} \times \hat{\mathbf{G}}_{j+1,mk} \rangle \frac{\partial \langle \hat{\mathbf{h}}_{j,N-2} \rangle_k}{\partial X_m} + \nabla_{\mathbf{X}} \times \langle \mathbf{W} \times \hat{\mathbf{h}}'_{j-1,N-1} + \mathbf{U} \times \hat{\mathbf{h}}'_{j,N-1} + \overline{\mathbf{W}} \times \hat{\mathbf{h}}'_{j+1,N-1} \rangle \\ &= \sum_{m=0}^{N-2} \lambda_{N-m} \langle \hat{\mathbf{h}}_{j,m} \rangle + \mathrm{i} j \omega \langle \hat{\mathbf{h}}_{j,N} \rangle. \end{aligned} \tag{4.45}$$

Step 1° for $n = N$. Let us consider (4.45) for $j = 0$:

$$\begin{aligned} &(\mathscr{E} - \lambda_2) \langle \hat{\mathbf{h}}_{0,N-2} \rangle - \lambda_N \langle \hat{\mathbf{h}}_{0,0} \rangle \\ &\quad = \sum_{m=1}^{N-3} \lambda_{N-m} \langle \hat{\mathbf{h}}_{0,m} \rangle - \nabla_{\mathbf{X}} \times \left(2\mathrm{Re} \langle \overline{\mathbf{W}} \times \hat{\mathbf{h}}'_{1,N-1} \rangle + \langle \mathbf{U} \times \hat{\mathbf{h}}'_{0,N-1} \rangle \right); \end{aligned} \tag{4.46}$$

here the r.h.s. is known. Let $\mathbb{D}$ denote the domain of the operator $\mathscr{E}$: solenoidal vector fields which are defined in a bounded region of the slow variables, satisfy the desirable boundary conditions and belong to the Sobolev space $\mathbb{W}_2^1$. By invariant projection in $\mathbb{D}$ of the equation (4.46) parallel to $\langle \hat{\mathbf{h}}_{0,0} \rangle$, we determine uniquely λ_N. Assuming that the operator $\mathscr{E} - \lambda_2$ is invertible in the complementary $\mathscr{E}$-invariant subspace of $\mathbb{D}$ (this is equivalent to the assumption that λ_2 is an eigenvalue of multiplicity one), from (4.46) we find $\langle \hat{\mathbf{h}}_{0,N-2} \rangle$ in this subspace uniquely up to an arbitrary additive term, which is a multiple of $\langle \hat{\mathbf{h}}_{0,0} \rangle$; we set it equal to zero. As in the case of a steady flow, this is a normalisation condition: the presence of such terms in $\langle \hat{\mathbf{h}}_{0,n} \rangle$ for $n \geq 1$ is equivalent to multiplication of the magnetic mode (4.19) by an insignificant scalar factor, which is a power series in ε. $\{\hat{\mathbf{h}}_{j,N-2}\}$ are now defined by the expressions (4.44); thus we have now completely determined $\hat{\mathbf{h}}_{j,N-2}$ for all j.

Step 2° for $n = N$. Afterwards, we determine $\langle \hat{\mathbf{h}}_{j,N} \rangle$ for $j \neq 0$ from (4.45).

Step 3° for $n = N$. Upon substitution of (4.44) for $n = N - 1$, the fluctuating part of the equation obtained from the series (4.25) at order ε^N reduces to

$$
\begin{aligned}
\mathscr{L}_j\{\mathbf{h}_N\} = -\eta\Bigg(& 2\sum_{k=1}^{3}\sum_{m=1}^{3}\left(\frac{\partial\hat{\mathbf{S}}_{j,k}}{\partial x_m}\frac{\partial\langle\hat{\mathbf{h}}_{0,N-1}\rangle_k}{\partial X_m}+\sum_{l=1}^{3}\frac{\partial\hat{\mathbf{G}}_{j,mk}}{\partial x_l}\frac{\partial^2\langle\hat{\mathbf{h}}_{0,N-2}\rangle_k}{\partial X_l\partial X_m}\right)\\
&+2(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\hat{\mathbf{h}}'_{j,N-1}+\nabla^2_{\mathbf{X}}\hat{\mathbf{h}}_{j,N-2}\Bigg)-(\langle\hat{\mathbf{h}}_{j-1,N}\rangle\cdot\nabla_{\mathbf{x}})\mathbf{W}+(\mathbf{W}\cdot\nabla_{\mathbf{X}})\langle\hat{\mathbf{h}}_{j-1,N-1}\rangle\\
&-(\langle\hat{\mathbf{h}}_{j,N}\rangle\cdot\nabla_{\mathbf{x}})\mathbf{U}+(\mathbf{U}\cdot\nabla_{\mathbf{X}})\langle\hat{\mathbf{h}}_{j,N-1}\rangle-(\langle\hat{\mathbf{h}}_{j+1,N}\rangle\cdot\nabla_{\mathbf{x}})\overline{\mathbf{W}}+(\overline{\mathbf{W}}\cdot\nabla_{\mathbf{X}})\langle\hat{\mathbf{h}}_{j+1,N-1}\rangle\\
&-\nabla_{\mathbf{X}}\times\Bigg(\sum_{k=1}^{3}\left(\mathbf{W}\times\hat{\mathbf{S}}_{j-1,k}+\mathbf{U}\times\hat{\mathbf{S}}_{j,k}+\overline{\mathbf{W}}\times\hat{\mathbf{S}}_{j+1,k}\right)\langle\hat{\mathbf{h}}_{0,N-1}\rangle_k\\
&+\sum_{k=1}^{3}\sum_{m=1}^{3}\{\mathbf{W}\times\hat{\mathbf{G}}_{j-1,mk}+\mathbf{U}\times\hat{\mathbf{G}}_{j,mk}+\overline{\mathbf{W}}\times\hat{\mathbf{G}}_{j+1,mk}\}\frac{\partial\langle\hat{\mathbf{h}}_{0,N-2}\rangle_k}{\partial X_m}\\
&+\{\mathbf{W}\times\hat{\mathbf{h}}'_{j-1,N-1}+\mathbf{U}\times\hat{\mathbf{h}}'_{j,N-1}+\overline{\mathbf{W}}\times\hat{\mathbf{h}}'_{j+1,N-1}\}\Bigg)+\sum_{m=0}^{N-2}\lambda_{N-m}\{\hat{\mathbf{h}}_{j,m}\}. \qquad (4.47)
\end{aligned}
$$

As in the case $N = 2$ (4.43) considered in Sect. 4.2.5, the structure of this equation implies that the solution $\{\hat{\mathbf{h}}_{j,N}\}$ is compatible with the expression (4.44), where vector fields $\hat{\mathbf{h}}'_{j,N}$ can be uniquely determined from the equations, obtained from (4.44) by replacing $\{\mathbf{h}_N\} \to \mathbf{h}'_N$ and setting to zero $\langle\hat{\mathbf{h}}_{0,N}\rangle_k$ and derivatives of $\langle\hat{\mathbf{h}}_{0,N-1}\rangle_k$ (the r.h.s. of this equation is known). Note that again there is a direct analogy with solution of the problem for a steady flow (in which the Fourier series in time with the coefficients $\hat{\mathbf{h}}'_{j,N}(\mathbf{x})$ is the analogue of the field $\mathbf{h}'_N(\mathbf{x}, t)$ in (3.39)). It is not difficult to verify that

$$
\hat{\mathbf{h}}'_{-j,N}=\overline{\hat{\mathbf{h}}'}_{j,N};\qquad \nabla_{\mathbf{x}}\cdot\hat{\mathbf{h}}'_{j,N}+\nabla_{\mathbf{X}}\cdot\left(\hat{\mathbf{h}}'_{j,N-1}+\sum_{k=1}^{3}\sum_{m=1}^{3}\hat{\mathbf{G}}_{j,mk}\frac{\partial\langle\hat{\mathbf{h}}_{0,N-2}\rangle_k}{\partial X_m}\right)=0
$$

(the latter relation implies that the condition (4.23), stemming from solenoidality of the magnetic mode, holds for $n = N$).

Thus, we have found a solution to the system of equations obtained from the series (4.25) at order ε^N, which reproduces all the required features of solutions to the systems for $n<N$. We have therefore constructed a complete formal asymptotic expansion of large-scale magnetic modes in a power series in the scale ratio for the time-periodic flow (4.1). Like for a steady flow, it easy to show by induction that all $\hat{\mathbf{h}}_{j,n}$ and $\hat{\mathbf{h}}'_{j,n}$ are parity-antiinvariant in the fast spatial variables for even n, and parity-invariant for odd n; in particular, (4.45) implies that $\langle\hat{\mathbf{h}}_{j,n}\rangle = 0$ for all odd n, and (4.46) that $\lambda_n = 0$ for all odd n, i.e. (4.8) is a power series in ε^2.

Suppose now we consider large-scale magnetic modes that are defined in the entire space of the slow variables and globally bounded. Then the domain of the operator of magnetic induction $\mathscr{M}$ splits into invariant subspaces of vector fields of the form $\mathbf{h}(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{X}}$, categorised by wave vectors $\mathbf{q}$, and it is natural to seek large-scale magnetic modes in such subspaces:

$$\hat{\mathbf{h}}_{j,n} = \hat{\mathbf{g}}_{j,n}(\mathbf{x})\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}. \tag{4.48}$$

We have established in the previous section that λ_2 is an eigenvalue of the 3×3 matrix (4.41) in the subspace of $\mathbb{C}^3$, orthogonal to $\mathbf{q}$ (4.42). Let λ'_2 and $\tilde{\mathbf{h}}'$ denote the second eigenvalue and the associated eigenvector in this subspace. We assume that $\lambda_2 \neq \lambda'_2$ and that the eigenvectors are normalised: $|\tilde{\mathbf{h}}| = |\tilde{\mathbf{h}}'| = 1$.

Substituting (4.48) into (4.46) for $j = 0$ and cancelling out the factor $\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}$, we obtain an equation in $\mathbb{C}^3$. Projecting it out parallel to $\tilde{\mathbf{h}}$, we find

$$\lambda_N = \mathrm{i}\tilde{\mathbf{h}}\cdot\mathscr{P}_{\tilde{\mathbf{h}}}\Big(\mathbf{q}\times(2\mathrm{Re}\langle\overline{\mathbf{W}}\times\hat{\mathbf{g}}'_{1,N-1}\rangle+\langle\mathbf{U}\times\hat{\mathbf{g}}'_{0,N-1}\rangle)\Big), \tag{4.49}$$

where $\hat{\mathbf{g}}'_{j,N-1}$ is defined by the identity $\hat{\mathbf{h}}'_{j,N-1} = \hat{\mathbf{g}}'_{j,N-1}(\mathbf{x})\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}$, and $\mathscr{P}_{\mathbf{u}}$ is the invariant projection parallel to a vector $\mathbf{u}$ for $\mathbf{u} = \tilde{\mathbf{h}}$ or $\tilde{\mathbf{h}}'$. Projection parallel to $\tilde{\mathbf{h}}'$ yields

$$\langle\hat{\mathbf{g}}_{0,N-2}\rangle = \frac{1}{\lambda'_2-\lambda_2}\Big(-\mathrm{i}\mathscr{P}_{\tilde{\mathbf{h}}'}(\mathbf{q}\times(2\mathrm{Re}\langle\overline{\mathbf{W}}\times\hat{\mathbf{g}}'_{1,N-1}\rangle+\langle\mathbf{U}\times\hat{\mathbf{g}}'_{0,N-1}\rangle)) + \sum_{m=1}^{N-3}\lambda_{N-m}\langle\hat{\mathbf{g}}_{0,m}\rangle\Big).$$

For $m > 0$, all $\langle\hat{\mathbf{g}}_{0,m}\rangle$ are parallel to $\tilde{\mathbf{h}}'$ (as we have pointed out above, the additive term parallel to $\tilde{\mathbf{h}}'$ can be set to zero as a normalisation condition).

Clearly, the dependence of $\langle\hat{\mathbf{h}}_{j,N}\rangle$ for $j \neq 0$ and $\mathbf{h}'_N$ on the slow spatial variables in the form of proportionality to $\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}$ (4.48) is reproduced by equations (4.46) for $j \neq 0$ and the fluctuating parts of the equations obtained from the series (4.25) at order ε^N.

4.4 Computation of the Tensor of Magnetic Eddy Diffusion

When a large-scale magnetic mode is defined in the entire three-dimensional space of the fast variables, and depends on the three slow spatial variables, computation of the tensor of eddy correction of magnetic diffusion via relations (4.14) requires to solve 12 auxiliary problems of the form $\mathscr{L}\mathbf{g} = \mathbf{f}$ (three problems of type I and nine of type II). We present here a method [327] for computation of the tensor, requiring to solve twice less auxiliary problems of the same complexity. It is also applicable for computation of magnetic eddy diffusivity in large-scale dynamo problems for steady flows $\mathbf{V}$, considered in the previous chapter.

Let $\mathbf{Z}_l$ be solutions with zero spatio-temporal means to auxiliary problems for the adjoint operator:

$$\mathscr{L}'^{*}\mathbf{Z}_l = \mathbf{V} \times \mathbf{e}_l, \tag{4.50}$$

whose spatial and temporal periods coincide with the ones of the flow $\mathbf{V}$. The solvability condition for this equation consists of orthogonality of the r.h.s. to the kernel of $\mathscr{L}'$. Evidently, for parity-invariant flows it is satisfied, because the r.h.s. of (4.50) is parity-invariant (and hence the fields $\mathbf{Z}_l$ are parity-invariant, since parity-invariant and parity-antiinvariant fields constitute invariant subspaces of the operators $\mathscr{L}'$ and $\mathscr{L}'^{*}$), and the kernel of $\mathscr{L}'$ is spanned by parity-antiinvariant fields $\mathbf{S}_k + \mathbf{e}_k$. The operator $\mathscr{L}'^{*}$ is not parabolic, but it becomes parabolic when time is reversed: $t \to -t$. The spectra of the operators $\mathscr{L}'$ and $\mathscr{L}'^{*}$ coincide, and hence numerical complexity of the auxiliary problems for the adjoint operator does not exceed the one of the auxiliary problems of type II.

Expressions (4.14) for the tensor of magnetic diffusion eddy correction and the statements of the auxiliary problems for the adjoint operator (4.50) and auxiliary problems of type II (4.35) imply

$$\begin{aligned}(D_{mk})_l &= \langle\!\langle \mathbf{V} \times \mathbf{G}_{mk} \rangle\!\rangle_l = -\langle\!\langle \mathscr{L}'^{*}\mathbf{Z}_l \cdot \mathbf{G}_{mk} \rangle\!\rangle = -\langle\!\langle \mathbf{Z}_l \cdot \mathscr{L}'\mathbf{G}_{mk} \rangle\!\rangle \\ &= \left\langle\!\!\left\langle \mathbf{Z}_l \cdot \left(2\eta \frac{\partial \mathbf{S}_k}{\partial x_m} + \mathbf{V}(S_k)_m - (\mathbf{S}_k + \mathbf{e}_k)V_m \right) \right\rangle\!\!\right\rangle. \end{aligned} \tag{4.51}$$

Thus, for computation of all $\mathbf{D}_{mk}$ it suffices to solve three auxiliary problems of type I (4.29) and three auxiliary problems for the adjoint operator (4.50).

The solenoidal part of the solution to the problem (4.50), $\mathbf{Z}'_l$, satisfies

$$\frac{\partial \mathbf{Z}'_l}{\partial t} + \eta\nabla^2\mathbf{Z}'_l - \mathscr{P}_{\mathrm{sol}}(\mathbf{V} \times (\nabla \times \mathbf{Z}'_l + \mathbf{e}_l)) = 0,$$

where $\mathscr{P}_{\mathrm{sol}}$ denotes projection onto the subspace of $\mathbf{L}$-periodic solenoidal fields with a vanishing spatial mean:

$$\mathscr{P}_{\mathrm{sol}}\mathbf{a} \equiv \mathbf{a} - \nabla(\nabla^{-2}(\nabla \cdot \mathbf{a})).$$

The solution to the auxiliary problem for the adjoint operator (4.50) can be expressed in the terms of $\mathbf{Z}'_l$:

$$\mathbf{Z}_l = \mathbf{Z}'_l + (\partial/\partial t + \eta\nabla^2)^{-1}\nabla\big(\nabla^{-2}(\nabla \cdot (\mathbf{V} \times (\nabla \times \mathbf{Z}'_l + \mathbf{e}_l)))\big). \tag{4.52}$$

The operator $(\partial/\partial t + \eta\nabla^2)^{-1}$ acts in the subspace of space- and time-periodic fields with a vanishing spatio-temporal mean, as well as the operator $(-\partial/\partial t + \eta\nabla^2)^{-1}$ (see Sect. 4.1.3). Due to solenoidality of $\mathbf{Z}'_l$, it is sought in a subspace of a reduced dimension, and this accelerates computations.[1] Numerical

[1] Similarly, it is advisable not to compute solutions to auxiliary problems of type II employing their statements in the form (4.35) directly, but to benefit from the knowledge of the potential part of the solutions (see (4.36)), and to transform (4.35) into an equation in solenoidal fields $\hat{\mathbf{G}}'_{j,mk} = \hat{\mathbf{G}}_{j,mk} + \nabla(\nabla^{-2}(\hat{S}_{j,k})_m)$.

inversion of the Laplacian ∇^2 and of the operator $\partial/\partial t + \eta\nabla^2$ are computationally simple procedures in the Fourier space.

The vector fields $\varepsilon_{pql}x_q\mathbf{e}_p$, where ε_{pql} is the standard unit antisymmetric tensor,

$$\mathbf{e}_p \times \mathbf{e}_q = \varepsilon_{pql}\mathbf{e}_l,$$

solve equation (4.50) (but they are not solutions to the auxiliary problem for the adjoint operator, since they do not satisfy the condition of periodicity in space). They can be used to derive alternative expressions for the tensor of magnetic diffusion eddy correction, where solutions to the auxiliary problem of type II enter only into surface integrals. Scalar multiplying equation (3.30), where the operator $\mathscr{L}$ defined in Sect. 4.1.3 is assumed (and hence (4.35) are components of the expansion of (3.30) in a Fourier series in time), by this function and considering the spatio-temporal mean of the product, we find

$$(D_{mk})_l = \left\langle\!\!\left\langle \eta\varepsilon_{pql}\left(2\delta^q_m(S_k)_p + \frac{\partial(G_{mk})_p}{\partial x_q}\right) + (\mathbf{V}\times\mathbf{G}_{mk})_l \right\rangle\!\!\right\rangle^{\perp q}$$
$$+\, \varepsilon_{pql}\langle\!\langle (V_p(S_k)_m - V_m((S_k)_p + \delta^k_p))x_q\rangle\!\rangle. \tag{4.53}$$

Here $l \neq p \neq q \neq l$, and $\langle\!\langle\cdot\rangle\!\rangle^{\perp q}$ denotes the mean over a time period and a rectangular periodicity box in the plane, normal to the Cartesian coordinate axis x_q. This mean in (4.53) is independent of x_q: if integration in x_q over the interval $\tilde{x}_q \leq x_q \leq \tilde{x}_q + L_q$ is carried out for computation of the means in (4.53), then the mean $\langle\!\langle\cdot\rangle\!\rangle^{\perp q}$ in this formula is computed at $x_q = \tilde{x}_q$, and all the remaining terms in (4.53) are independent of $\tilde{x}_q$.

Because of solenoidality of the mean field in the slow variables (see (4.22) for $j = n = 0$), the operator of magnetic eddy diffusion $\mathscr{E}$ (4.39) (as well as its steady-flow analogue (3.33)) is invariant with respect to addition of the same vector to all entries $\mathbf{D}_{mm}$. Thus, instead of the tensor (4.14), in the operator of magnetic eddy diffusion we can employ the tensor

$$\mathbf{D}'_{mk} = \langle\!\langle \mathbf{V}\times\mathbf{G}'_{mk}\rangle\!\rangle,$$

where $\mathbf{G}'_{mk}$ are solutions to the equation

$$\mathscr{L}\mathbf{G}'_{mk} = -2\eta\frac{\partial\mathbf{S}_k}{\partial x_m} - \mathbf{e}_m\times(\mathbf{V}\times(\mathbf{S}_k + \mathbf{e}_k)), \tag{4.54}$$

the r.h.s. of which differs from the r.h.s. of (3.30) by the additive term $-\delta^k_m\mathbf{V}$.

The entry $(D'_{mk})_l$ for $l \neq m$ can be expressed as a surface mean by setting $p = m$ in (4.53) (and modifying the vectors $\mathbf{D}_{mm}$ following the remark above). Alternatively, let us consider the vector field

$$\zeta_{mk,j} = \mathrm{e}^{\mathrm{i}jx_m}(\mathbf{S}_k + \mathbf{e}_k + \mathrm{i}j\mathbf{G}'_{mk})/j^2$$

for $j \neq 0$. From the statements of the auxiliary problems of type I (3.18) and modified type II (4.54) we obtain

$$\mathscr{L}\zeta_{mk,j} = \mathrm{e}^{\mathrm{i}jx_m}\left(\eta\left(\mathbf{S}_k + \mathbf{e}_k + \mathrm{i}j\mathbf{G}'_{mk} + 2\frac{\partial \mathbf{G}'_{mk}}{\partial x_m}\right) + \mathbf{e}_m \times (\mathbf{V} \times \mathbf{G}'_{mk})\right). \tag{4.55}$$

The spatio-temporal mean of the l.h.s. of (4.55) vanishes, and hence this identity yields

$$0 = \left\langle\!\!\left\langle \mathrm{e}^{\mathrm{i}jx_m}\left(\eta\left(\mathbf{S}_k + \mathbf{e}_k + \frac{\partial \mathbf{G}'_{mk}}{\partial x_m}\right) + \mathbf{e}_m \times (\mathbf{V} \times \mathbf{G}'_{mk})\right)\right\rangle\!\!\right\rangle, \tag{4.56}$$

i.e. the vector field

$$\left\langle\!\!\left\langle \eta\left(\mathbf{S}_k + \mathbf{e}_k + \frac{\partial \mathbf{G}'_{mk}}{\partial x_m}\right) + \mathbf{e}_m \times (\mathbf{V} \times \mathbf{G}'_{mk})\right\rangle\!\!\right\rangle^{\perp m} \equiv \mathbf{C}_{mk} \tag{4.57}$$

is a constant vector (because all Fourier coefficients of the function of x_m in the l.h.s. of (4.57) vanish for $j \neq 0$ by virtue of (4.56)). Averaging (4.57) over x_m, we find

$$\mathbf{C}_{mk} = \eta\mathbf{e}_k + \mathbf{e}_m \times \mathbf{D}'_{mk},$$

whereby

$$\mathbf{e}_m \times \mathbf{D}'_{mk} = \left\langle\!\!\left\langle \eta\left(\mathbf{S}_k + \frac{\partial \mathbf{G}'_{mk}}{\partial x_m}\right) + \mathbf{e}_m \times (\mathbf{V} \times \mathbf{G}'_{mk})\right\rangle\!\!\right\rangle^{\perp m}. \tag{4.58}$$

(The mth component of the r.h.s. of (4.58) vanishes due to (3.31).)

The number of auxiliary problems to be solved in order to determine the tensor of eddy correction of magnetic diffusion can be, in principle, further halved. Application of the curl to (4.50) yields

$$\frac{\partial \mathbf{R}_l}{\partial t} + \eta\nabla^2\mathbf{R}_l - \nabla \times (\mathbf{V} \times \mathbf{R}_l) - \frac{\partial \mathbf{V}}{\partial x_l} = 0, \tag{4.59}$$

where $\mathbf{R}_l = \nabla \times \mathbf{Z}'_l$. From (4.59) and the statement of the auxiliary problems of type I,

$$-\frac{\partial \mathbf{S}_l}{\partial t} + \eta\nabla^2\mathbf{S}_l + \nabla \times (\mathbf{V} \times \mathbf{S}_l) + \frac{\partial \mathbf{V}}{\partial x_l} = 0 \tag{4.60}$$

(this equation coincides with (3.18), except for the operator $\mathscr{L}$ defined in Sect. 4.1.3 is assumed in (4.60)), we find

$$-\frac{\partial \mathbf{B}_l}{\partial t} + \eta\nabla^2\mathbf{A}_l + \nabla \times (\mathbf{V} \times \mathbf{B}_l) = 0, \tag{4.61}$$

$$-\frac{\partial \mathbf{A}_l}{\partial t} + \eta\nabla^2\mathbf{B}_l + \nabla \times (\mathbf{V} \times \mathbf{A}_l) = -\frac{\partial \mathbf{V}}{\partial x_l}, \tag{4.62}$$

where it is denoted

$$\mathbf{A}_l = \frac{1}{2}(\mathbf{S}_l + \mathbf{R}_l), \quad \mathbf{B}_l = \frac{1}{2}(\mathbf{S}_l - \mathbf{R}_l). \tag{4.63}$$

From (4.61) we obtain

$$\mathbf{A}_l = \frac{1}{\eta}\nabla^{-2}\left(\frac{\partial}{\partial t} - \nabla \times (\mathbf{V} \times \cdot)\right)\mathbf{B}_l. \tag{4.64}$$

Substitution of this expression into (4.62) yields the desired closed equation in $\mathbf{B}_l$:

$$-\mathbf{B}_l + \left(\frac{1}{\eta}\nabla^{-2}\left(\frac{\partial}{\partial t} - \nabla \times (\mathbf{V} \times \cdot)\right)\right)^2 \mathbf{B}_l = \frac{1}{\eta}\nabla^{-2}\left(\frac{\partial \mathbf{V}}{\partial x_l}\right). \tag{4.65}$$

After $\mathbf{B}_l$ is obtained from this equation, $\mathbf{A}_l$ is determined by (4.64), $\mathbf{S}_l$ and $\mathbf{R}_l$ can be found from (4.63), and $\mathbf{Z}_l$ from (4.52), where $\mathbf{Z}'_l = -\nabla^{-2}(\nabla \times \mathbf{R}_l)$. However, the fourth order equation (4.65) may prove significantly computationally more intensive than any of the auxiliary problems that we have introduced in this chapter. Thus, it is unclear whether it is computationally efficient to employ the reduction of the auxiliary problems to the three equations (4.65).

If the flow possesses a translation symmetry with the time reversal:

$$\mathbf{V}(\mathbf{x}, t) = -\mathbf{V}(\mathbf{x} + \mathbf{x}_0, t_0 - t),$$

for a constant vector $\mathbf{x}_0$ (for $\mathbf{L}$-periodic fields $\mathbf{V}$, $\mathbf{x}_0 = (k_1 L_1, k_2 L_2, k_3 L_2)/2$, where each k_j takes values 0 or 1) and a scalar t_0, then solutions to the problem (4.60) can be expressed in the terms of solutions to (4.50):

$$\mathbf{S}_l(\mathbf{x}, t) = \nabla \times \mathbf{Z}_l(\mathbf{x} + \mathbf{x}_0, t_0 - t).$$

4.5 Is Periodic Time Dependence of Flows Favourable for Negative Magnetic Eddy Diffusivity?

Simulation of large-scale magnetic modes defined in the entire space of the slow variables were carried out for the flow (4.1). The synthesis of solenoidal parity-invariant zero-mean flow components $\mathbf{U}$, $\mathbf{V}_c$ and $\mathbf{V}_s$, 2π-periodic in the fast spatial variables, was performed by the procedure outlined in the previous chapter: (i) imaginary parts (Fourier coefficients of a parity-invariant vector field are imaginary) of pairs of complex conjugate (ensuring that the synthetic field is real) coefficients are random generated in the interval [–0.5, 0.5]; (ii) the potential part of the field is removed by projection of the coefficients on the planes normal to the respective wave vectors; (iii) coefficients in every spherical shell of wave vectors are normalised so that the resultant field has a desirable energy spectrum and the prescribed total energy. The spectrum of the flows employed in computations falls off exponentially by 6 orders of magnitude, the Fourier series being cut off after the wave number 10. Vector fields are normalised so that $\langle\!\langle |\mathbf{V}|^2 \rangle\!\rangle = 1$ for $\omega = 1$, i.e.

$$\langle |\mathbf{U}|^2 \rangle + \frac{1}{2}\langle |\mathbf{V}_c|^2 + |\mathbf{V}_s|^2 \rangle = 1 \tag{4.66}$$

(and hence the magnetic Reynolds number can be estimated as $R_m = \eta^{-1}$). The energies of the two time-dependent terms in (4.1) were kept equal:

$$\langle |\mathbf{V}_c|^2 \rangle = \langle |\mathbf{V}_s|^2 \rangle. \tag{4.67}$$

We will call the *set of profiles defining the flow* the three fields $\mathbf{U}(\mathbf{x}), \mathbf{V}_c(\mathbf{x})$ and $\mathbf{V}_s(\mathbf{x})$ (considered without taking into account their amplitudes).

Solutions to auxiliary problems (4.29) and (4.35) were computed as Fourier series with the resolution of 64^3 Fourier harmonics in the spatial variables and 8 Fourier harmonics in time. The energy spectra of the solutions found in computations decay at least by 10 orders of magnitude in the spatial variables, and by 4–5 orders in time. After computation of the tensor of magnetic diffusion eddy correction (4.14), the minimum magnetic eddy diffusivity

$$\eta_{\text{eddy}} = \min_{|\mathbf{q}|=1}(-\lambda_2)$$

was determined.

Because of a considerable numerical complexity of the problem, computations were carried out for a single value of magnetic molecular diffusivity $\eta = 0.1$ only. It was chosen for the following reasons: in the study of generation of large-scale magnetic fields by steady flows (see the previous chapter) it was established that, on the one hand, for this η short-scale magnetic fields are not generated, and, on the other, the fraction of flows exhibiting negative eddy diffusivity is the largest for this value of η among the three values employed in this study (see Fig. 3.2). We have checked numerically that none of the flows (4.1), employed for computation of magnetic eddy diffusivity, generates a short-scale magnetic field. Computation of the dominant eigenvalues (satisfying the normalisation condition (4.6)) in the short-scale Floquet problems was performed by an adapted version of the code [325].

The following questions were considered:

(i) How does the minimum magnetic eddy diffusivity η_{eddy} change under a small variation of a steady flow introducing a weak dependence on time of the form (4.1)? Eddy diffusivity η_{eddy} was computed for $\omega = 1$ and $\eta = 0.1$ for an ensemble of 30 flows (4.1), satisfying (4.66) and (4.67), in which the ratio of the time-averaged energy of the time-dependent part,

$$E_{\text{osc}} = \frac{1}{2}\langle |\mathbf{V}_c|^2 + |\mathbf{V}_s|^2 \rangle$$

to the energy of the steady part is small:

$$E_{\text{osc}} / \langle |\mathbf{U}|^2 \rangle = 1/400.$$

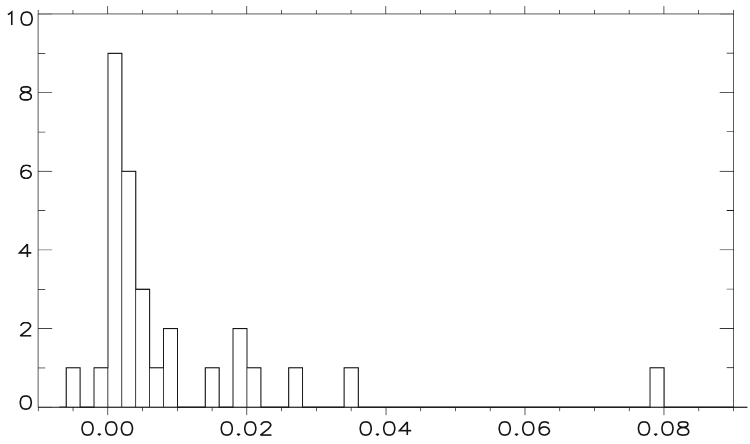

Fig. 4.1 Histogram of $\delta\eta_{\text{eddy}}$, the changes in the minimum magnetic eddy diffusivities due to emergence in the flow of a weak time dependence of the form (4.1)

Let $\delta\eta_{\text{eddy}}$ denote the difference between the minimum magnetic eddy diffusivity for the flow (4.1) with such energy distribution and the minimum magnetic eddy diffusivity for the steady flow $\mathbf{U}$. The histogram of the computed values of $\delta\eta_{\text{eddy}}$ is shown in Fig. 4.1. Only in two cases out of 30 introduction into the flow of the weak time dependence of the form (4.1) has resulted in a decrease of η_{eddy}.

(ii) How does the minimum magnetic eddy diffusivity η_{eddy} depend on the shares of kinetic energy of the steady and time-dependent parts of the flow? We computed η_{eddy} for $\omega = 1$ for three families of flows (4.1). The set of profiles defining the flow was fixed in each family (the three sets were also employed in the ensemble of flows, for which the data shown on the histogram in Fig. 4.1 was computed). In each family, amplitudes of the fields $\mathbf{U}(\mathbf{x}), \mathbf{V}_c(\mathbf{x})$ and $\mathbf{V}_s(\mathbf{x})$ were varied respecting the conditions (4.66) and (4.67). The plots of η_{eddy} as a function of the part (relative the total kinetic energy E_{total}) of energy E_{osc} stored in the time-dependent parts of the flow is shown in Fig. 4.2. While the plots differ significantly in details, they reflect the general tendency: on the relative increase of the energy of the time-dependent part of the flow, the minimum magnetic eddy diffusivity η_{eddy} in general increases, although not necessarily monotonically.

Although it may be expected that a relative increase of the amplitude of the time-dependent part of the velocity amplifies chaotic properties of the flow (which are a prerequisite for a fast magnetic field generation), the numerical results shown here indicate, that dependence of the flow on time (at least of the form (4.1)) is not beneficial for generation of large-scale magnetic fields. (This observation does not involve a paradox, since the large-scale dynamos that we consider are slow).

(iii) How does the minimum magnetic eddy diffusivity depend on the temporal frequency of the flow, ω? We have computed η_{eddy} for the flows (4.1) for a fixed set of defining profiles for two ratios of energies $E_{\text{osc}}/E_{\text{total}}$ (see Fig. 4.3). We have employed the set of fields $\mathbf{U}$, $\mathbf{V}_c$ and $\mathbf{V}_s$, for which the minimum eddy diffusivities

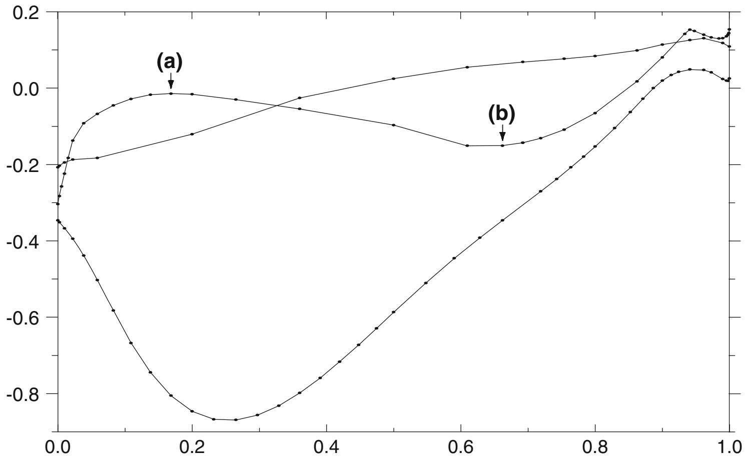

Fig. 4.2 Minimum magnetic eddy diffusivity η_{eddy} (*the vertical axis*) as a function of the relative share $E_{\mathrm{osc}}/E_{\mathrm{total}}$ of kinetic energy of the time-dependent part of the flow (*the horizontal axis*), for three families of flows (4.1) for $\omega = 1$. *Dots* show the computed values of η_{eddy}. For two flows marked by *arrows* further numerical results are shown in Fig. 4.3

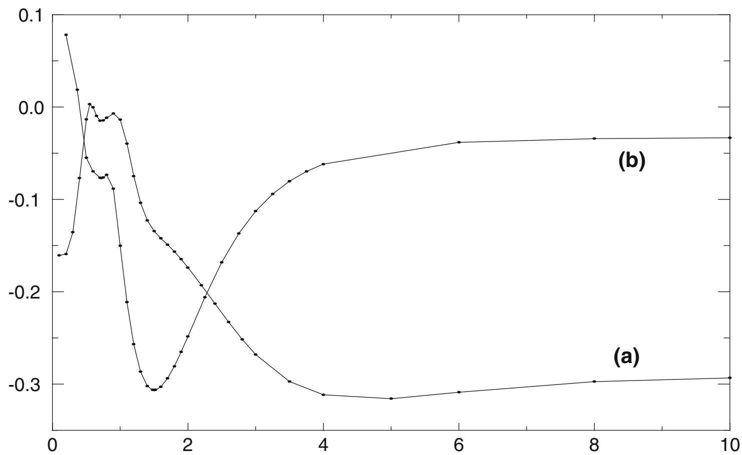

Fig. 4.3 Minimum magnetic eddy diffusivity η_{eddy} (*the vertical axis*) as a function of temporal frequency ω (*the horizontal axis*) of the flow. *Dots* show the computed values η_{eddy}. The two curves **a** and **b** are represented by the points labelled (**a**) and (**b**) in the plot Fig. 4.2

are shown on the plot in Fig. 4.2 by two points marked (a) and (b); in agreement with this labelling the two curves computed for the set of profiles $\mathbf{U}$, $\mathbf{V}_c$ and $\mathbf{V}_s$ and the two energy ratios $E_{\mathrm{osc}}/E_{\mathrm{total}}$ are labelled (a) and (b) in Fig. 4.3. The details of the dependence on the frequency ω revealed by the two curves are essentially different. For $\omega \to \infty$, magnetic eddy diffusivity tends to two distinct limits, which can be much larger than the minimum value $\min_\omega \eta_{\text{eddy}}(\omega)$, as happens in the case (b) (in this case the time-averaged energy of the time-dependent part of the flow is relatively higher). The minima of η_{eddy} are attained in both cases for frequencies ω of order 1.

The typical structures observed in magnetic modes are shown in Figs. 4.4 and 4.5 for the mode generated by the flow, for which the curve (b) in Fig. 4.3 is computed, for the temporal frequency $\omega = 1.5$ at which the minimum of magnetic eddy diffusivity is attained. The main term in the expansion of a magnetic mode into a power series in the scale ratio ε is

$$\mathbf{h}_0 = \mathrm{e}^{\mathrm{i}\varepsilon\mathbf{q}\cdot\mathbf{x}} \left(\widetilde{\mathbf{h}} + \sum_{k=1}^{3} \widetilde{h}_k \mathbf{S}_k(\mathbf{x}, t) \right)$$

(see (4.21), (4.28) and (4.39)), where $\mathbf{S}_k(\mathbf{x}, t)$ are solutions to auxiliary problems of type I, and $\widetilde{\mathbf{h}}$ is found from the eigenvalue problem (4.39)–(4.42). We visualise the fluctuating part of the factor prescribing the dependence of the leading term on the fast variables,

$$\{\mathbf{h}_0\} = \sum_{k=1}^{3} \widetilde{h}_k \mathbf{S}_k(\mathbf{x}, t),$$

ignoring the harmonic amplitude modulation resulting from the dependence on the slow variables. Isosurfaces of the magnetic energy density $|\{\mathbf{h}_0\}|^2$ are shown in Figs. 4.4 and 4.5 in one periodicity cell step $\overline{T}/6$ (where $\overline{T}$ is the temporal period of the flow). Most of cigar-like magnetic structures seen on this figures are apparently associated with stagnation points of the flow (4.1), despite the magnetic flux rope solutions found in [111, 324] for steady states have not been, to the best of our knowledge, generalised to encompass the case of time-dependent flows. (Recall that $\mathbf{L}$-periodicity and parity invariance (4.2) imply that at the points $(m_1L_1/2,\ m_2L_2/2,\ m_3L_3/2)$, where $m_1,\ m_2,\ m_3$ are arbitrary integers, the flow velocity vanishes). The sharpness and the shape of the cigar-like structures considerably varies in time.

4.6 The Limit of High Frequencies for Flows of the Simple Structure

The plots in Fig. 4.3 suggest that in the limit $\omega \to \infty$ the contribution of the time-dependent part of the flow (4.1) to magnetic eddy diffusivity is finite. In this section we consider the limit of high temporal frequencies and prove this conjecture.

It is convenient to express the flow (4.1) in the form

$$\mathbf{V}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x}) + \sqrt{\omega}\left(\mathbf{W}'(\mathbf{x})\mathrm{e}^{\mathrm{i}\omega t} + \overline{\mathbf{W}}'(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\right), \qquad (4.68)$$

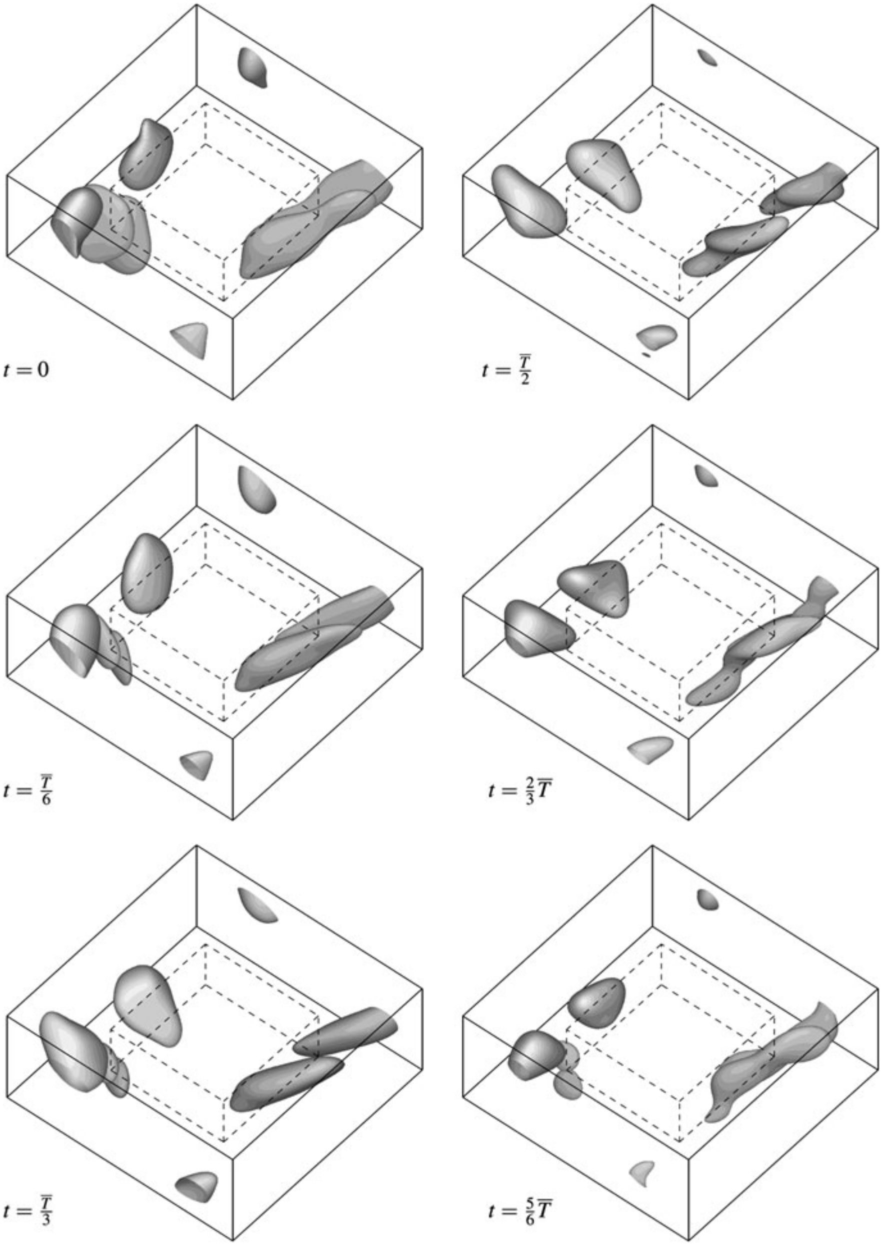

Fig. 4.4 Isosurfaces of the energy density $|\{\mathbf{h}_0\}|^2$of the fluctuating part of the leading term of expansion of a large-scale magnetic mode at the level of 0.4 of the maximum. One periodicity cell of the flow is shown (*solid lines*) step 1/6 of the temporal period $\overline{T}$. The lowest vertex of the cube is the point $(-\pi/2, -\pi/2, -\pi/2)$. *Dashed lines* show the elementary cube of the grid of the flow stagnation points, $(m_1\pi, m_2\pi, m_3\pi)$

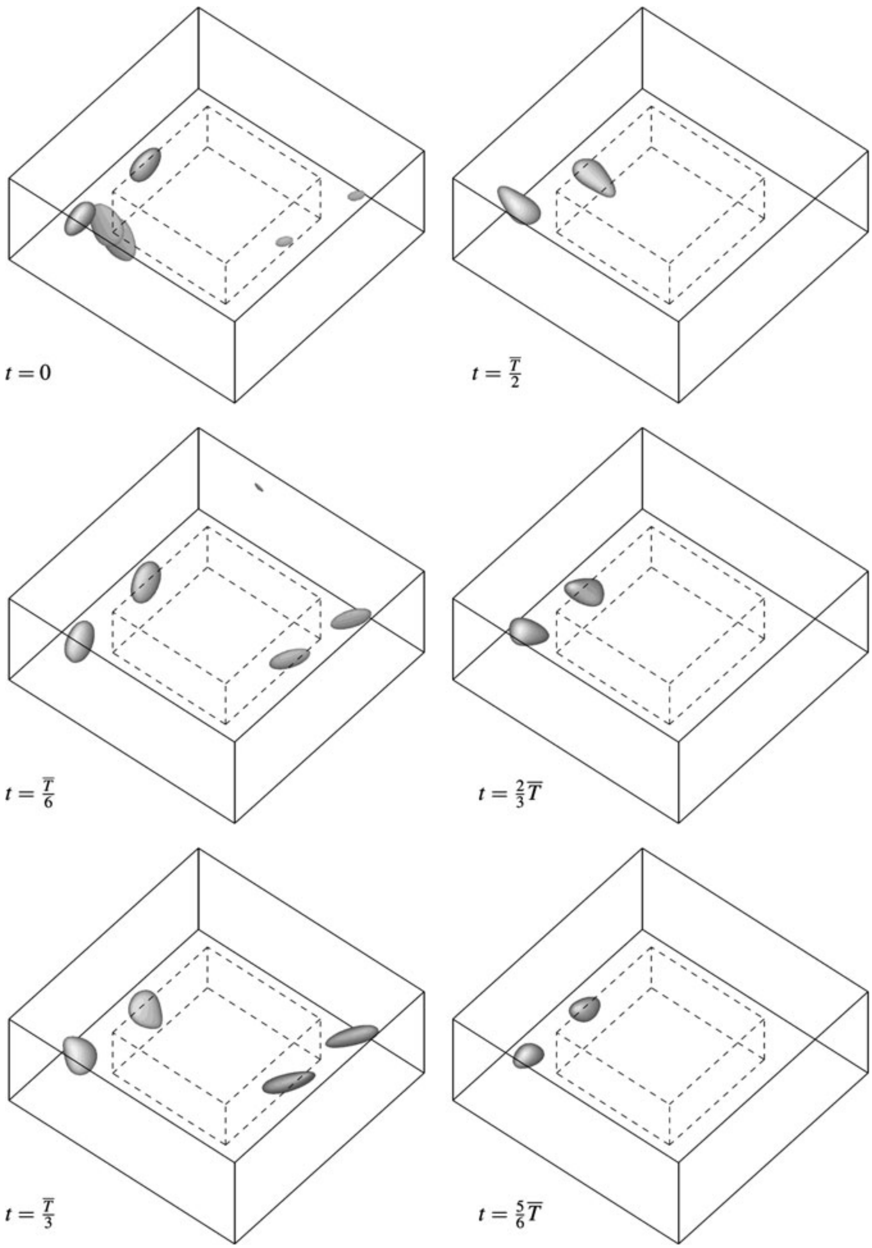

Fig. 4.5 Same as in Fig. 4.4, but the isosurfaces are at the level of 0.75 of the maximum energy density

where $\mathbf{W}' = (\mathbf{V}_c - i\mathbf{V}_s)/2$. For the flow (4.68), the operators $\mathscr{L}_j$ yielding the coefficients $\mathscr{L}_j\mathbf{f}$ of the expansion of the field $\mathscr{L}\mathbf{f}$ in a Fourier series in time (see (4.24)) take the form

$$\mathscr{L}_j\mathbf{f} = -\mathrm{i}j\omega\mathbf{f}_j + \eta\nabla^2\mathbf{f}_j + \nabla\times\Big(\mathbf{U}\times\mathbf{f}_j + \sqrt{\omega}(\mathbf{W}'\times\mathbf{f}_{j-1} + \overline{\mathbf{W}}'\times\mathbf{f}_{j+1})\Big), \quad (4.69)$$

where

$$\mathbf{f}(\mathbf{x},t) = \sum_{j=-\infty}^{\infty}\mathbf{f}_j(\mathbf{x})\mathrm{e}^{\mathrm{i}j\omega t}.$$

Auxiliary problems of type I (4.29) for the flow (4.68) reduce to

$$\mathscr{L}_j\mathbf{S}_k = -\frac{\partial}{\partial x_k}\Big(\delta_0^j\mathbf{U} + \sqrt{\omega}(\delta_1^j\mathbf{W}' + \delta_{-1}^j\overline{\mathbf{W}}')\Big), \quad (4.70)$$

and auxiliary problems of type II (4.35) reduce to

$$\begin{aligned}\mathscr{L}_j\mathbf{G}_{mk} = &-2\eta\frac{\partial\hat{\mathbf{S}}_{j,k}}{\partial x_m} + U_m(\hat{\mathbf{S}}_{j,k} + \delta_0^j\mathbf{e}_k) - \mathbf{U}(\hat{S}_{j,k})_m + \sqrt{\omega}\Big(W'_m(\hat{\mathbf{S}}_{j-1,k} + \delta_1^j\mathbf{e}_k)\\ &+ \overline{W}'_m(\hat{\mathbf{S}}_{j+1,k} + \delta_{-1}^j\mathbf{e}_k) - \mathbf{W}'(\hat{S}_{j-1,k})_m - \overline{\mathbf{W}}'(\hat{S}_{j+1,k})_m\Big). \quad (4.71)\end{aligned}$$

Suppose, the coefficients $\hat{\mathbf{S}}_{j,k}$ and $\hat{\mathbf{G}}_{j,mk}$ of expansions of the fields $\mathbf{S}_k$ and $\mathbf{G}_{mk}$, respectively, in the Fourier series in time have a polynomial asymptotics at large ω. It is easy to verify that

$$\hat{\mathbf{S}}_{j,k} = \omega^{-|j|/2}\mathbf{s}_{j,k} + \mathrm{O}(\omega^{-(|j|+1)/2}),$$

$$\hat{\mathbf{G}}_{j,mk} = \omega^{-|j|/2}\mathbf{g}_{j,mk} + \mathrm{O}(\omega^{-(|j|+1)/2})$$

is compatible with equations (4.69)–(4.71), implying that the first two leading terms of the expansion of the Fourier coefficients of $\mathbf{S}_k$ and $\mathbf{G}_{mk}$ in power series in $\omega^{-1/2}$ satisfy the equations

$$\eta\nabla^2\mathbf{s}_{0,k} + \nabla_\mathbf{x}\times\Big(2\mathrm{Re}(\overline{\mathbf{W}}'\times\mathbf{s}_{1,k}) + \mathbf{U}\times\mathbf{s}_{0,k}\Big) = -\frac{\partial\mathbf{U}}{\partial x_k}, \quad (4.72)$$

$$\mathbf{s}_{1,k} = -\mathrm{i}\left(\frac{\partial\mathbf{W}'}{\partial x_k} + \nabla_\mathbf{x}\times(\mathbf{W}'\times\mathbf{s}_{0,k})\right); \quad (4.73)$$

$$\begin{aligned}\eta\nabla^2\mathbf{g}_{0,mk} &+ \nabla_\mathbf{x}\times\Big(2\mathrm{Re}(\overline{\mathbf{W}}'\times\mathbf{g}_{1,mk}) + \mathbf{U}\times\mathbf{g}_{0,mk}\Big)\\ &= -2\eta\frac{\partial\mathbf{s}_{0,k}}{\partial x_m} + 2\mathrm{Re}(\overline{W}'_m\mathbf{s}_{1,k} - \overline{W}'(s_{1,k})_m), \quad (4.74)\end{aligned}$$

$$\mathbf{g}_{1,mk} = -\mathrm{i}(W'_m(\mathbf{s}_{0,k} + \mathbf{e}_k) - \mathbf{W}'(s_{0,k})_m + \nabla_{\mathbf{x}} \times (\mathbf{W}' \times \mathbf{g}_{0,mk})). \tag{4.75}$$

Thus, for $\omega \to \infty$ the tensor of magnetic diffusion eddy correction (4.40) has the asymptotics

$$\mathbf{D}_{mk} = 2\mathrm{Re}\langle \overline{\mathbf{W}}' \times \mathbf{g}_{1,mk}\rangle + \langle \mathbf{U} \times \mathbf{g}_{0,mk}\rangle + \mathrm{O}(\omega^{-1/2}). \tag{4.76}$$

The limit values $\mathbf{D}_{mk}$ are invariant with respect to phase shifts: (4.72)–(4.75) imply, that if $\mathbf{W}'$ is changed to $\mathrm{e}^{\mathrm{i}\alpha}\mathbf{W}'$, where α is a real constant, then $\mathbf{s}_{\pm1,k}$ change to $\mathrm{e}^{\pm\mathrm{i}\alpha}\mathbf{s}_{\pm1,k}, \mathbf{g}_{\pm1,mk}$ to $\mathrm{e}^{\pm\mathrm{i}\alpha}\mathbf{g}_{\pm1,mk}$, and $\mathbf{s}_{0,k}, \mathbf{g}_{0,mk}$ and hence $\mathbf{D}_{mk}$ remain unaltered. $\mathbf{s}_{0,k}$ and $\mathbf{g}_{0,mk}$ are real. If $\mathbf{W}'$ is also real (i.e., if $\mathbf{V}_s = 0$), then $\mathbf{s}_{\pm1,k}$ and $\mathbf{g}_{\pm1,mk}$ are imaginary, and therefore

$$\mathrm{Re}(\overline{\mathbf{W}}' \times \mathbf{s}_{1,k}) = \mathrm{Re}(\overline{\mathbf{W}}' \times \mathbf{g}_{1,mk}) = 0,$$

i.e., at high temporal frequencies the contribution of the time-dependent part of the flow (4.68) to magnetic eddy diffusivity is zero. The two statements together imply that in the limit $\omega \to \infty$ the order O(1) contribution of the time-dependent part of the flow (4.1) does not vanish only, if the fields $\mathbf{V}_c$ and $\mathbf{V}_s$ are linearly independent.

As in the previous section, we made computations for magnetic molecular diffusivity $\eta = 0.1$ only. Coefficients in the expansion of short-scale magnetic modes in the Fourier series in time have the same asymptotics in the high frequency ω limit, as solutions to the auxiliary problems. The leading term in the expansion of a short-scale mode in a power series in $\omega^{-1/2}$ is an eigenfunction of the limit magnetic induction operator defined by the homogeneous parts of (4.72), (4.73):

$$\eta\nabla^2\mathbf{h}'_{0,k} + \nabla_{\mathbf{x}} \times \left(2\mathrm{Re}(\overline{\mathbf{W}}' \times \mathbf{h}'_{1,k}) + \mathbf{U} \times \mathbf{h}'_{0,k}\right) = \lambda'_0\mathbf{h}'_{0,k},$$

$$\mathbf{h}'_{1,k} = -\mathrm{i}\nabla_{\mathbf{x}} \times (\mathbf{W}' \times \mathbf{h}'_{0,k})$$

(here $\mathbf{h}'_{0,k}$ and $\mathbf{h}'_{1,k}$ are the leading terms of expansions in power series in $\omega^{-1/2}$ of the respective coefficients in the Fourier series in time for a short-scale magnetic mode). This gives an opportunity to examine numerically stability of short-scale magnetic modes in the limit $\omega \to \infty$. For every instance of the flow (4.1) employed for construction of Figs. 4.6 and 4.7 we have computed the dominant eigenvalue λ'_0 of the limit magnetic induction operator in the space of solenoidal zero-mean fields, 2π-periodic in the fast spatial variables, and verified that the growth rates of short-scale limit modes are negative, i.e. there is no generation of short-scale magnetic fields at high frequencies ω.

We conducted the following numerical experiments:

(i) The limit minimum magnetic eddy diffusivity η_{eddy} (see Fig. 4.6) was computed using equations (4.72)–(4.75) for the three sets of profiles defining a

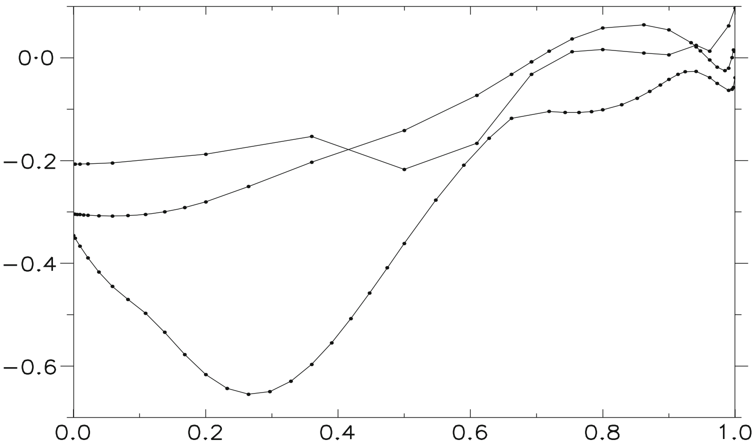

Fig. 4.6 Minimum magnetic eddy diffusivity η_{eddy} (*the vertical axis*) in the limit $\omega \to \infty$ as a function of the relative energy share of the time-dependent part of the flow, $E_{\mathrm{osc}}/E_{\mathrm{total}}$ (*the horizontal axis*), for three families of flows (4.1) for the sets of defining profiles employed for construction of the three curves in Fig. 4.2. *Dots* show the computed values of η_{eddy}

flow (4.1), for which computations for $\omega = 1$ were performed (see numerical experiment (ii) in Sect. 4.5 and Fig. 4.2). The left-most points (for $E_{\mathrm{osc}} = 0$) show the same values in Figs. 4.2 and 4.6 for the same sets of defining profiles; this identifies the pairs of plots constructed for the same sets. As for $\omega = 1$, the minimum magnetic eddy diffusivity η_{eddy} in general increases on increasing the relative amount of kinetic energy contained in the time-dependent part of the flow, although the dependence is not monotonic. However, the influence of the time-dependent part of the flow (4.1) has weakened.

(ii) The distribution of the minimum magnetic eddy diffusivity η_{eddy} was investigated in the high-frequency limit for flows with a vanishing temporal mean: $\mathbf{U} = 0$. According to the results discussed above, these ones are the least advantageous conditions for generation of large-scale magnetic fields. A histogram of the limit values of η_{eddy}, computed for 45 instances of flows, satisfying (4.66), (4.67) and $\mathbf{U} = 0$, is shown in Fig. 4.7 ; $\lim_{\omega \to \infty} \eta_{\mathrm{eddy}}$ is negative only in two cases out of 45.

4.7 Conclusions

1. We have constructed in this chapter asymptotic expansions of large-scale magnetic modes generated by short-scale parity-invariant time- and space-periodic flows, and of their growth rates in power series in the spatial scale ratio ε. Mean-field equations and expressions for the tensor of eddy correction of magnetic diffusion have the same structure as for steady flows, except for averaging over time must be performed now in addition to averaging over the

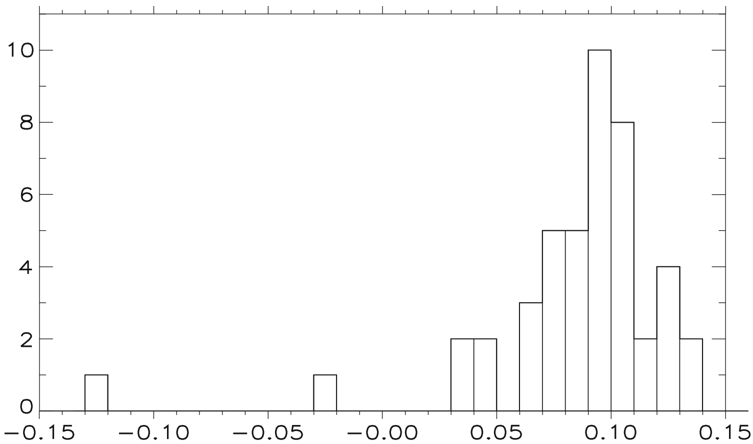

Fig. 4.7 Histogram of the values of $\lim_{\omega\to\infty}\eta_{\rm eddy}$ for 45 instances of flows (4.1) with zero temporal means ($\mathbf{U}=0$)

fast spatial variables. (In particular, like for steady flows, magnetic eddy diffusivity can take anomalously high in absolute value negative values, if the magnetic Reynolds number is close to the critical one for the onset of generation of short-scale magnetic fields).

2. The problem is considered in more details for flows of the simple structure (4.1) in the terms of coefficients of a large-scale magnetic mode expanded in a Fourier series in time. The operator of magnetic eddy diffusion (4.39) has been derived. The spatial mean of the leading term of the expansion of the mode in the power series in ε is found to be independent of time. Complete formal expansions in the spatial scale ratio ε of the large-scale magnetic mode and the associated eigenvalue have been constructed. Asymptotics of the Fourier coefficients in the limit of high temporal frequencies has been determined, and the limit tensor of eddy correction of magnetic diffusion (4.76) has been derived.
3. Minimum magnetic eddy diffusivity $\eta_{\rm eddy}$ was computed for magnetic molecular diffusivity $\eta = 0.1$ for parity-invariant flows (4.1) with random-generated Fourier coefficients and an exponential fall off of the energy spectrum of the profiles $\mathbf{U}, \mathbf{V}_c$ and $\mathbf{V}_s$. A histogram of changes in the minimum magnetic eddy diffusivities due to introduction into flows of small-amplitude terms, periodic in time, was constructed for an ensemble of 30 flows. The dependence of $\eta_{\rm eddy}$ on, $E_{\rm osc}/E_{\rm total}$, the relative share of kinetic energy of the time-dependent part of the flow (4.1), was numerically investigated in details for $\omega = 1$ and in the limit of high temporal frequencies ω for three sets of profiles defining the flow, as well as the dependencies of $\eta_{\rm eddy}$ on the frequency ω for one set and two energy ratios $E_{\rm osc}/E_{\rm total}$. The dependencies are non-monotonic, but the results reveal a general tendency: periodic dependence of flows on time, at least of the form (4.1), is not beneficial for generation of large-scale magnetic fields.

Nevertheless, in an ensemble comprised of 45 instances of flows (4.1), in two cases the minimum magnetic eddy diffusivity η_{eddy} is negative under the least advantageous conditions for generation of large-scale magnetic fields: the flows have vanishing steady (i.e., mean over time) profiles, and $\omega \to \infty$.

Chapter 5
Convective Plan Form Dynamos in a Layer

We have explored in Chaps. 3 and 4 kinematic generation of the large-scale magnetic field by flows with a prescribed energy spectrum, that are synthesised from random-amplitude Fourier harmonics and model turbulent flows. In this chapter, following [327], we consider flows, which are, perhaps, more natural for a physicist and an experimenter—convective plan forms in an infinite horizontal plane layer of fluid. Such flows set in, when the difference of temperatures at the heated lower boundary and the upper boundary exceeds the critical value for the onset of thermal convection. We consider a finite-depth layer of fluid, and hence only horizontal slow variables can be introduced. Nevertheless, the tensor of magnetic eddy diffusion can be derived by a technique similar to the one employed in previous chapters, with modifications required to take into account the boundedness in the vertical direction of the volume where the operator of magnetic induction acts. Insignificance of the α-effect in these flows, which is necessary for observability of magnetic eddy diffusion, is guaranteed by their symmetry about the vertical axis (and not by parity invariance, as in the flows that were considered in previous chapters). However, the α-effect does not disappear entirely in flows possessing such a symmetry (see Sect. 5.2.2).

We solve the kinematic dynamo problem in an infinite plane layer of electrically conducting fluid, whose horizontal boundaries are perfectly electrically conducting. An analysis of bifurcations in a space-periodic convective MHD system with symmetries has been performed in [27], and it was shown formally that at the onset of convective flows (the primary bifurcation), the onset of magnetic field generation (the secondary bifurcation) can be studied (up to infinitely small terms that can be neglected) in the framework of a kinematic magnetic dynamo problem for the respective plan form. (Numerical results presented in this paper are incorrect [178, 334], since the computations were carried out with an insufficient spatial resolution). Generation of a short-scale magnetic field by hexagonal convective plan forms (the so-called Christopherson flow [61]) was observed in [334] in the context of investigation of the solar dynamo (conse-

V. Zheligovsky, *Large-Scale Perturbations of Magnetohydrodynamic Regimes*, Lecture Notes in Physics, 829, DOI: 10.1007/978-3-642-18170-2_5,

quently, different boundary conditions were assumed: the half-space over the upper boundary filled in by a dielectric was considered). In the absence of rotation, short-scale magnetic field is not generated by convective plan forms in a layer with perfectly conducting boundaries, but generation takes place, if the layer is rotating about an inclined axis [178]. In a certain window of Rayleigh numbers, the short-scale generation does not die out in the subsequent nonlinear regime.

5.1 Statement of the Problem

Assuming that the flow $\mathbf{V}(\mathbf{x})$ is steady, like in Chap. 3 we reduce the study of the kinematic magnetic field generation to an eigenvalue problem for the magnetic induction operator

$$\mathscr{L}'\mathbf{h} \equiv \eta\nabla^2\mathbf{h} + \nabla \times (\mathbf{V} \times \mathbf{h}) = \lambda\mathbf{h}, \tag{5.1}$$

defined in the space of solenoidal vector fields,

$$\nabla \cdot \mathbf{h} = 0, \tag{5.2}$$

satisfying the desired boundary conditions. The goal of this section is to examine the operator and to state formally the problem to be addressed.

5.1.1 Boundary Conditions

We consider the kinematic dynamo problem in an infinite layer with perfectly electrically conducting horizontal boundaries. Mathematically, they are described by the following boundary conditions for the magnetic field $\mathbf{h}(\mathbf{x}, t)$:

$$\left.\frac{\partial h_1}{\partial x_3}\right|_{x_3=0,\pi} = \left.\frac{\partial h_2}{\partial x_3}\right|_{x_3=0,\pi} = 0, \qquad h_3\Big|_{x_3=0,\pi} = 0. \tag{5.3}$$

(We assume that x_3 is the vertical Cartesian coordinate, and that the dimensionless width of the layer is π). These boundary conditions are the most convenient ones for computations (since they give an opportunity to employ the Fourier–Galerkin basis consisting of trigonometric functions, see Sect. 5.6). However, they represent a mathematical idealisation of the real physical processes at the fluid–conductor boundary (see [255]).

The flow is supposed to be periodic in horizontal directions,

$$\mathbf{V}(\mathbf{x}) = \mathbf{V}(x_1 + L_1, x_2, x_3) = \mathbf{V}(x_1, x_2 + L_2, x_3), \tag{5.4}$$

and the horizontal boundaries of the layer are impenetrable for the fluid:

$$V_3\Big|_{x_3=0,\pi} = 0. \tag{5.5}$$

5.1.2 Neutral Magnetic Modes

The operator, adjoint to the magnetic induction operator, is

$$\mathscr{L}'^{*}\mathbf{h} = \eta\nabla^2\mathbf{h} - \mathscr{P}_{\mathrm{sol}}(\mathbf{V} \times (\nabla \times \mathbf{h})),$$

where $\mathscr{P}_{\mathrm{sol}}$ is the projection onto the subspace of zero-mean solenoidal vector fields, periodic in the horizontal directions. Consequently, $\mathscr{L}'^{*}\mathbf{h} = 0$ for any constant vector $\mathbf{h}$. Thus, any constant vector belongs to the kernel of the adjoint operator, provided the vector satisfies the boundary conditions for the adjoint operator. These conditions can be found performing integration by parts in the identity

$$(\mathscr{L}'\mathbf{h}_1, \mathbf{h}_2) = (\mathbf{h}_1, \mathscr{L}'^{*}\mathbf{h}_2), \tag{5.6}$$

that defines the adjoint operator, demanding that all the emerging surface integrals vanish for any vector fields $\mathbf{h}_1(\mathbf{x})$ and $\mathbf{h}_2(\mathbf{x})$ from the domains of $\mathscr{L}'$ and $\mathscr{L}'^{*}$, respectively. (We assume the scalar product $(\cdot,\cdot)$ in the Lebesgue space $\mathbb{L}_2(\mathscr{Q})$ of functions defined in the parallelepiped of periodicity, $\mathscr{Q} = [0, L_1] \times [0, L_2] \times [0, \pi]$). The fields from the domain of the magnetic induction operator are supposed to have the same periods L_1 and L_2, as the flow,

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}(x_1 + L_1, x_2, x_3) = \mathbf{h}(x_1, x_2 + L_2, x_3) \tag{5.7}$$

(we will call *short* $-$ *scale* the fields, satisfying (5.4)). If they satisfy the boundary conditions (5.3), a simple application of vector analysis identities reveals that the boundary conditions for vector fields from the domains of $\mathscr{L}$ and $\mathscr{L}^{*}$ are the same.

For the conditions (5.3), a constant vector belongs to the kernel of $\mathscr{L}'^{*}$ as long as its vertical component is zero, implying

$$\dim\ker\mathscr{L}'^{*} \geq 2.$$

We will assume in this chapter that the kernel of $\mathscr{L}'^{*}$ is two-dimensional, and therefore there exist two short-scale neutral magnetic modes, i.e., magnetic fields $\mathbf{h}$, satisfying (5.1) for $\lambda = 0$ and the periodicity (5.3) and boundary (5.7) conditions. This assumption holds true for all magnetic molecular diffusivities $\eta > 0$, except for at most a countable number of them.

5.1.3 The Symmetry About a Vertical Axis

In addition to (5.4) and (5.5), we restrict our attention to flows that are solenoidal,

$$\nabla \cdot \mathbf{V} = 0, \tag{5.8}$$

and symmetric about a vertical axis[1]:

[1] Evidently, this symmetry is distinct from axisymmetry.

$$\begin{aligned} V_1(-x_1,-x_2,x_3) &= -V_1(x_1,x_2,x_3), \\ V_2(-x_1,-x_2,x_3) &= -V_2(x_1,x_2,x_3), \\ V_3(-x_1,-x_2,x_3) &= V_3(x_1,x_2,x_3). \end{aligned} \tag{5.9}$$

As parity invariance, this symmetry is compatible with the solenoidality condition (5.8), the Navier–Stokes equation and the equations of thermal convection (provided external forces, if any, are also symmetric about the axis). We will see (Sect. 5.5) that convective plan forms possess this symmetry. Examples of the fully nonlinear convective hydromagnetic regimes, symmetric about a vertical axis, can be found in [55, 333].

It is straightforward to check that the symmetry of the flow $\mathbf{V}$ about a vertical axis implies, that the domain of the magnetic induction operator $\mathscr{L}'$ (as well as the domain of the adjoint operator) splits into a direct sum of two invariant subspaces—fields, *symmetric about the vertical axis*:

$$\begin{aligned} f_1(-x_1,-x_2,x_3) &= -f_1(x_1,x_2,x_3), \\ f_2(-x_1,-x_2,x_3) &= -f_2(x_1,x_2,x_3), \\ f_3(-x_1,-x_2,x_3) &= f_3(x_1,x_2,x_3) \end{aligned}$$

and fields, *antisymmetric about the vertical axis*:

$$\begin{aligned} f_1(-x_1,-x_2,x_3) &= f_1(x_1,x_2,x_3), \\ f_2(-x_1,-x_2,x_3) &= f_2(x_1,x_2,x_3), \\ f_3(-x_1,-x_2,x_3) &= -f_3(x_1,x_2,x_3). \end{aligned}$$

(Without any loss of generality, here and in (5.9) we assume that the axis coincides with the Cartesian coordinate x_3).

5.1.4 Asymptotic Expansion of Large-Scale Magnetic Modes and the Hierarchy of Equations for Large-Scale Magnetic Modes

A large-scale magnetic mode $\mathbf{h}(\mathbf{X},\mathbf{x})$ depends on the *fast* spatial variables $\mathbf{x} \in \mathbb{R}^3$ and *slow* horizontal variables $\mathbf{X} = \varepsilon(x_1,x_2)$. The scale ratio $\varepsilon > 0$ is regarded as a small parameter of the problem. We seek a solution to the eigenvalue problem (5.1) in the form of power series in ε:

$$\mathbf{h} = \sum_{n=0}^{\infty} \mathbf{h}_n(\mathbf{X},\mathbf{x})\varepsilon^n, \tag{5.10}$$

$$\lambda = \sum_{n=0}^{\infty} \lambda_n \varepsilon^n. \tag{5.11}$$

The mode is supposed to have periods L_j in $x_j (j = 1, 2)$ and to satisfy the condition (5.3) for a perfect electric conductor at the horizontal boundaries $x_3 = 0, \pi$. Consequently, all $\mathbf{h}_n$ satisfy the periodicity (5.7) and boundary (5.3) conditions.

By the chain rule, the dependence on the fast and slow variables requires to modify the gradients

$$\nabla \to \nabla_{\mathbf{x}} + \varepsilon \nabla_{\mathbf{X}}, \qquad \nabla_{\mathbf{X}} \equiv \left(\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, 0 \right) \tag{5.12}$$

in the magnetic induction operator in the eigenvalue equation (5.1) and in the solenoidality condition (5.2). Substitution of (5.11) and (5.12) into (5.1) and expansion in a power series in ε yields equation (3.13). This gives rise to the hierarchy of equations

$$\mathscr{L}'\mathbf{h}_n + \eta(2(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\mathbf{h}_{n-1} + \nabla^2_{\mathbf{X}}\mathbf{h}_{n-2}) + \nabla_{\mathbf{X}} \times (\mathbf{V} \times \mathbf{h}_{n-1}) - \sum_{j=0}^{n} \lambda_{n-j}\mathbf{h}_j = 0. \tag{5.13}$$

We assume henceforth in this chapter that differentiation in the fast variables only is performed in the magnetic induction operator $\mathscr{L}'$.

As previously, $\langle \cdot \rangle$ and $\{\cdot\}$ denote the spatial mean and fluctuating parts, respectively, of a scalar or a vector field:

$$\langle f \rangle \equiv \frac{1}{\pi L_1 L_2} \int_0^{\pi} \int_{-L_2/2}^{L_2/2} \int_{-L_1/2}^{L_1/2} f(\mathbf{X}, \mathbf{x}) \mathrm{d}\mathbf{x}, \qquad \{f\} \equiv f - \langle f \rangle.$$

Let $\langle \cdot \rangle_h$ and $\{\cdot\}_h$ denote the mean part of the horizontal component of a field and the fluctuating compliment:

$$\langle \mathbf{f} \rangle_h \equiv \sum_{j=1}^{2} \langle \mathbf{f} \cdot \mathbf{e}_j \rangle \mathbf{e}_j, \qquad \{\mathbf{f}\}_h \equiv \mathbf{f} - \langle \mathbf{f} \rangle_h, \tag{5.14}$$

and $\langle \cdot \rangle_v$ and $\{\cdot\}_v$ denote the mean part of the vertical component of a vector field and the fluctuating compliment:

$$\langle \mathbf{f} \rangle_v \equiv \sum_{j=1}^{2} \langle \mathbf{f} \cdot \mathbf{e}_3 \rangle \mathbf{e}_3, \qquad \{\mathbf{f}\}_v \equiv \mathbf{f} - \langle \mathbf{f} \rangle_v; \tag{5.15}$$

here $\mathbf{e}_j$ is the unit vector collinear with the Cartesian coordinate axis x_j.

5.1.5 Solvability of Auxiliary Problems

We denote by $\mathscr{L}$ the restriction of the magnetic induction operator $\mathscr{L}'$ to the subspace of vector fields from the domain of $\mathscr{L}'$ (i.e., satisfying the periodicity

(5.7) and boundary (5.3) conditions), whose horizontal component is zero-mean. We consider in this subsection solvability of a problem

$$\mathscr{L}\mathbf{g} = \mathbf{f} \tag{5.16}$$

in the layer. Averaging of the horizontal component of this equation over the fast spatial variables shows that $\langle \mathbf{f} \rangle_h = 0$ is a necessary condition for the solvability. It is simple to show that the Laplacian is a self-adjoint operator in the space of vector fields satisfying (5.7) and (5.3), whose kernel consists of constant vector fields with a vanishing vertical component. It is invertible in the subspace of fields with zero-mean horizontal components; we denote the inverse Laplacian by ∇^{-2}. Suppose now the condition $\langle \mathbf{f} \rangle_h = 0$ is satisfied. Then (5.16) is equivalent to the equation

$$(1/\eta)\nabla^{-2}\mathscr{L}\mathbf{g} = \nabla^{-2}\mathbf{f}/\eta. \tag{5.17}$$

By our assumption that the kernel of $\mathscr{L}'^{*}$ is two-dimensional, the kernel of the operator $(1/\eta)\mathscr{L}^{*}\nabla^{-2}$, adjoint to the operator in the l.h.s. of this equation, is trivial. The operator in the l.h.s. of (5.17) is a sum of the identity and a compact operator; hence, the Fredholm alternative theorem [154, 171] is applicable to (5.17). By this theorem, it has a unique solution in the subspace of fields with a vanishing mean horizontal component (satisfying the periodicity and boundary conditions (5.7) and (5.3)).

Consequently, supposing that the kernel of $\mathscr{L}'$ is two-dimensional, we find that $\langle \mathbf{f} \rangle_h = 0$ is the necessary and sufficient solvability condition for the problem (5.16) under the conditions (5.7) and (5.3).

Evidently, $\langle \mathbf{h}_n \rangle_h$ satisfies (5.7) and (5.3), and hence $\{\mathbf{h}_n\}_h$ must also satisfy these conditions. Using the above criterion for solvability, we will solve successively equations (5.13) for all n in two steps:

1°. Consider the mean horizontal component of (5.13) as a partial differential equation in the slow variables, and solve it (for $n \geq 2$) in $\langle \mathbf{h}_{n-2} \rangle_h$.
2°. The solvability condition for the PDE in the fast spatial variables is satisfied in 1°; solve it in $\{\mathbf{h}_n\}_h$.

Substituting the series (5.10) and the modified gradient (5.12) into the solenoidality condition (5.2), we also obtain an equation in the form of an equality of power series in ε. Equating the terms at each order ε^n and considering separately the horizontal means and the remaining fluctuating parts, we find conditions for solenoidality of large-scale magnetic modes

$$\nabla_{\mathbf{X}} \cdot \langle \mathbf{h}_n \rangle_h = 0, \tag{5.18}$$

$$\nabla_{\mathbf{x}} \cdot \{\mathbf{h}_n\}_h + \nabla_{\mathbf{X}} \cdot \{\mathbf{h}_{n-1}\}_h = 0 \tag{5.19}$$

for all $n \geq 0$.

5.2 Magnetic Eddy Diffusion in Flows, Symmetric About a Vertical Axis

5.2.1 *Solution of Order ε⁰ Equations*

Step 1° for $n = 0$. The equation (5.13) for $n = 0$ is

$$\mathscr{L}\{\mathbf{h}_0\}_h + (\langle \mathbf{h}_0 \rangle_h \cdot \nabla_{\mathbf{x}})\mathbf{V} = \lambda_0 \mathbf{h}_0.$$

Upon averaging over the fast spatial variables, its horizontal component reduces to $0 = \lambda_0 \langle \mathbf{h}_0 \rangle_h$. We choose the possibility $\lambda_0 = 0$ (following the arguments that were discussed in Chap. 3).

Step 2° for $n = 0$. Consequently, the equation has solutions of the form

$$\{\mathbf{h}_0\}_h = \sum_{k=1}^{2} \mathbf{S}_k \langle \mathbf{h}_0 \rangle_k, \tag{5.20}$$

where $\mathbf{S}_k(\mathbf{x})$ are solutions to auxiliary problems of type I:

$$\mathscr{L}\mathbf{S}_k = -\frac{\partial \mathbf{V}}{\partial x_k} \qquad (k = 1, 2), \tag{5.21}$$

satisfying the periodicity (5.7) and boundary (5.3) conditions; horizontal components of $\mathbf{S}$ have vanishing spatial means.

The solvability condition for Eq. (5.23) is, evidently, satisfied by virtue of the periodicity of the flow in the horizontal directions. Eigenfunctions or generalised eigenfunctions satisfying the boundary conditions under consideration have a non-zero mean horizontal component, if and only if they are associated with the zero eigenvalue. This can be shown by the same methods, as were used in Chap. 3 to assert the same for fields that are periodic in the three-dimensional space. Also note that for the conditions (5.7) and (5.3) assumed in this chapter, the horizontal mean of the magnetic field does not change in time (see, e.g., [142]), and hence only vector fields from the kernel of $\mathscr{L}'$ can possess non-zero mean horizontal components. The reverse is true, if the kernel of $\mathscr{L}'$ is two-dimensional. Therefore, we obtain by these arguments an independent proof of solvability of (5.21), which is equivalent to

$$\mathscr{L}'(\mathbf{S}_k + \mathbf{e}_k) = 0.$$

Since symmetric and antisymmetric vector fields are invariant subspaces of the operator $\mathscr{L}$, the antisymmetry of the r.h.s. of (5.21) implies that $\mathbf{S}_k$ are antisymmetric about the vertical axis.

Taking the divergence of (5.21), we find

$$\nabla^2(\nabla \cdot \mathbf{S}_k) = 0. \tag{5.22}$$

By virtue of the boundary conditions (5.3) for $\mathbf{S}_k$ and (5.5) for the flow $\mathbf{V}$, we find from the vertical component of (5.21) that

$$\left.\frac{\partial^2 (S_k)_3}{\partial^2 x_3}\right|_{x_3=0,\pi} = 0,$$

and thus $\nabla \cdot \mathbf{S}_k$ satisfies

$$\left.\frac{\partial}{\partial x_3} \nabla \cdot \mathbf{S}_k\right|_{x_3=0,\pi} = 0$$

on the horizontal boundaries. Evidently, the divergence of $\mathbf{S}_k$ has the same periodicity in the horizontal directions, as $\mathbf{V}$, and therefore (5.22) implies that $\nabla \cdot \mathbf{S}_k$ is a constant. Integrating the identity $\nabla \cdot \mathbf{S}_k = C$ over a parallelepiped of periodicity, we find

$$\nabla \cdot \mathbf{S}_k = 0. \tag{5.23}$$

5.2.2 *The Solvability Condition for Order ε^1 Equations: Insignificance of the α-Effect*

The equation (5.13) for $n = 1$ is

$$\begin{aligned} \mathscr{L}\{\mathbf{h}_1\}_h + 2\eta(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\{\mathbf{h}_0\}_h + (\langle \mathbf{h}_1 \rangle_h \cdot \nabla_{\mathbf{x}})\mathbf{V} \\ + \nabla_{\mathbf{X}} \times (\mathbf{V} \times \{\mathbf{h}_0\}_h) - (\mathbf{V} \cdot \nabla_{\mathbf{X}})\langle \mathbf{h}_0 \rangle_h \\ = \lambda_1 \mathbf{h}_0. \end{aligned} \tag{5.24}$$

<u>Step 1° for $n = 1$</u>. Averaging of the horizontal component of (5.24) over the fast spatial variables yields

$$\nabla_{\mathbf{X}} \times \sum_{k=1}^{2} \langle \mathbf{V} \times \mathbf{S}_k \rangle_v \langle \mathbf{h}_0 \rangle_k = \lambda_1 \langle \mathbf{h}_0 \rangle_h. \tag{5.25}$$

Since $\mathbf{V}$ is symmetric about the vertical axis and $\mathbf{S}_k$ is antisymmetric, we find

$$\langle \mathbf{V} \times \mathbf{S}_k \rangle_v = 0,$$

and hence (5.25) implies

$$\lambda_1 = 0.$$

Thus, although only the vertical part of the magnetic α-effect tensor,$\langle \mathbf{V} \times \mathbf{S}_k \rangle$, vanishes, in the leading order the α-effect is insignificant in the sense that it does not influence directly generation of large-scale mean magnetic fields in the convective hydromagnetic dynamo that we consider—its influence is felt only via the short-scale solutions to auxiliary problems of type II discussed in the next subsection.

5.2.3 Solution of Order ε[1] Equations

Step 2° for $n = 1$. After (5.20) is substituted, Eq. 5.24 becomes

$$\mathscr{L}\{\mathbf{h}_1\}_h = -(\langle \mathbf{h}_1\rangle_h \cdot \nabla_{\mathbf{x}})\mathbf{V} - \sum_{k=1}^{2}\sum_{m=1}^{2}\left(2\eta\frac{\partial \mathbf{S}_k}{\partial x_m} + \mathbf{e}_m \times (\mathbf{V} \times (\mathbf{S}_k + \mathbf{e}_k))\right)\frac{\partial \langle \mathbf{h}_0\rangle_k}{\partial X_m}. \tag{5.26}$$

By linearity, its solution has the following structure:

$$\{\mathbf{h}_1\}_h = \sum_{k=1}^{2}\mathbf{S}_k\langle \mathbf{h}_1\rangle_k + \sum_{k=1}^{2}\sum_{m=1}^{2}\mathbf{G}_{mk}\frac{\partial \langle \mathbf{h}_0\rangle_k}{\partial X_m}, \tag{5.27}$$

where $\mathbf{G}_{mk}(\mathbf{x})$ are solutions with zero-mean horizontal components to auxiliary problems of type II:

$$\mathscr{L}\mathbf{G}_{mk} = -2\eta\frac{\partial \mathbf{S}_k}{\partial x_m} - \mathbf{e}_m \times (\mathbf{V} \times (\mathbf{S}_k + \mathbf{e}_k)) \qquad (m, k = 1, 2), \tag{5.28}$$

which satisfy the periodicity and boundary conditions (5.7) and (5.3). The symmetry of $\mathbf{V}$ and antisymmetry of $\mathbf{S}_k$ about the vertical axis imply that the solvability condition is satisfied for (5.28). The r.h.s. of this equation is symmetric about the vertical axis; since symmetric fields are invariant subspaces of the operator $\mathscr{L}$, $\mathbf{G}_{mk}(\mathbf{x})$ are symmetric about the vertical axis.

Let us demonstrate the identity

$$\nabla \cdot \mathbf{G}_{mk} + (S_k)_m = 0. \tag{5.29}$$

Denote by $\mathbf{\Phi}$ the vector potential of $\mathbf{V}$:

$$\mathbf{V} = \nabla \times \mathbf{\Phi}.$$

From (5.21), we find

$$\mathbf{V} \times \mathbf{S}_k = \eta\nabla \times \mathbf{S}_k - \frac{\partial \mathbf{\Phi}}{\partial x_k} + \nabla\psi_k + \mathbf{C}_k. \tag{5.30}$$

Here $\mathbf{\Phi}$ and ψ_k have the short-scale periodicity (5.4) of the flow $\mathbf{V}$, and vectors $\mathbf{C}_k$ are constant. Upon substitution of (5.30), Eq. 5.28 takes the form

$$\mathscr{L}\mathbf{G}_{mk} + \eta\left(\frac{\partial \mathbf{S}_k}{\partial x_m} + \nabla(S_k)_m\right) + \nabla \times ((\mathbf{\Phi}_k - \psi_k)\mathbf{e}_m) + \mathbf{e}_m \times \mathbf{C}_k = 0.$$

Taking the divergence, we obtain

$$\nabla^2 \phi_{mk} = 0, \tag{5.31}$$

where we have denoted

$$\phi_{mk} \equiv \nabla \cdot \mathbf{G}_{mk} + (S_k)_m.$$

By virtue of the boundary conditions (5.3) for $\mathbf{S}_k$ and $\mathbf{G}_{mk}$, and of the impenetrability condition (5.5) for the flow, we find from the vertical component of (5.28) that

$$\left.\frac{\partial^2 (G_{mk})_3}{\partial^2 x_3}\right|_{x_3=0,\pi} = 0 \qquad \Rightarrow \qquad \left.\frac{\partial \phi_{mk}}{\partial x_3}\right|_{x_3=0,\pi} = 0.$$

A solution to the Laplace equation (5.31), satisfying this condition at the horizontal boundaries and possessing the periodicity (5.7) of the flow, is, evidently, a constant. Averaging over the fast spatial variables the equation $\phi_{mk} = C$, we obtain (5.29), as required.

5.2.4 *The Solvability Condition for Order ε^2 Equations: The Operator of Magnetic Eddy Diffusion*

Step 1° for $n = 2$. For $n = 2$, Eq. 5.13 is

$$\mathscr{L}\{\mathbf{h}_2\}_h + \eta\big(2(\nabla_\mathbf{x} \cdot \nabla_\mathbf{X})\{\mathbf{h}_1\}_h + \nabla^2_\mathbf{X}\mathbf{h}_0\big)$$
$$+ (\langle\mathbf{h}_2\rangle_h \cdot \nabla_\mathbf{x})\mathbf{V} + \nabla_\mathbf{X} \times (\mathbf{V} \times \{\mathbf{h}_1\}_h) - (\mathbf{V} \cdot \nabla_\mathbf{X})\langle\mathbf{h}_1\rangle_h = \lambda_2 \mathbf{h}_0. \tag{5.32}$$

The horizontal component, averaged over the fast spatial variables,

$$\mathscr{E}\langle\mathbf{h}_0\rangle_h \equiv \eta\nabla^2_\mathbf{X}\langle\mathbf{h}_0\rangle_h + \nabla_\mathbf{X} \times \sum_{k=1}^{2}\sum_{m=1}^{2}\langle\mathbf{V} \times \mathbf{G}_{mk}\rangle_v \frac{\partial\langle\mathbf{h}_0\rangle_k}{\partial X_m} = \lambda_2\langle\mathbf{h}_0\rangle_h, \tag{5.33}$$

is an eigenvalue problem for the magnetic eddy diffusion operator $\mathscr{E}$. Solving this problem, we determine the leading terms of the expansions of the magnetic mode, $\langle\mathbf{h}_0\rangle_h$, and of the associated eigenvalue, λ_2. As in the previous chapters, one might consider this problem in a bounded region in the plane of the slow variables with suitable conditions prescribed at its boundary. Here, our interest is in magnetic modes, defined and globally bounded in the entire plane of the slow variables. $\mathscr{E}$ is a partial differential operator of the second order with constant coefficients. Consequently, solutions to the magnetic mean-field equation (5.33) are Fourier harmonics

$$\langle\mathbf{h}_0\rangle_h = \hat{\mathbf{h}}e^{i\mathbf{q}\cdot\mathbf{X}}, \tag{5.34}$$

where $\mathbf{q} \in \mathbb{R}^2$ is an arbitrary constant wave vector, and $\hat{\mathbf{h}} = (\hat{h}_1, \hat{h}_2, 0)$ a constant vector satisfying

$$-\eta|\mathbf{q}|^2\hat{\mathbf{h}} - \left((q_1, q_2, 0) \times \sum_{k=1}^{2}\sum_{m=1}^{2}\langle \mathbf{V} \times \mathbf{G}_{mk}\rangle_v q_m \hat{h}_k\right) = \lambda_2\hat{\mathbf{h}}. \tag{5.35}$$

The solenoidality condition (5.18) for $n = 0$ implies

$$(q_1, q_2, 0) \cdot \hat{\mathbf{h}} = 0.$$

We obtain from this equation, together with (5.35), the solution

$$\hat{\mathbf{h}} = (-q_2, q_1, 0),$$

$$\lambda_2(\mathbf{q}) = -\eta|\mathbf{q}|^2 + \sum_{k=1}^{2}\sum_{m=1}^{2}(-1)^k\langle \mathbf{V} \times \mathbf{G}_{mk}\rangle_3\, q_m q_{3-k}. \tag{5.36}$$

Thus, we have found $\langle \mathbf{h}_0 \rangle_h$ and $\{\mathbf{h}_0\}_h$ determined by (5.20), i.e. $\mathbf{h}_0$ is now known. The solvability condition for the equation (5.32) in $\{\mathbf{h}_2\}_h$ is satisfied. The equation can be solved similarly to how the analogous equations are solved for $n > 2$, as discussed in the next section.

5.3 Complete Asymptotic Expansion of Large-Scale Magnetic Modes

We examine in this section how equations (5.13) can be successively solved for $n > 2$. Like in Sect. 5.2.4 we focus on the case, where the magnetic mode is defined and globally bounded in the entire plane of the slow variables. Sets of vector fields $\mathbf{h}(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{X}}$, categorised by wave vectors $\mathbf{q} \in \mathbb{R}^2$, constitute invariant subspaces of the magnetic induction operator. It is natural to seek magnetic modes in such a subspace: magnetic modes involving Fourier harmonics for different wave vectors are superpositions of modes considered here.

Suppose the following information has been determined from the equations (5.13) for $n \le N - 1$:

- Vector fields

$$\mathbf{h}_n(\mathbf{X}, \mathbf{x}) = \mathbf{g}_n(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{X}} \tag{5.37}$$

for $n < N - 2$, such that

$$\langle \mathbf{h}_n \rangle_h = 0 \quad \text{for } 0 < n < N - 2; \tag{5.38}$$

- Relations of the form

$$\{\mathbf{h}_n\}_h = \sum_{k=1}^{2} \mathbf{S}_k \langle \mathbf{h}_n \rangle_k + \sum_{k=1}^{2}\sum_{m=1}^{2} \mathbf{G}_{mk} \frac{\partial \langle \mathbf{h}_{n-1} \rangle_k}{\partial X_m} + \mathbf{h}'_n \tag{5.39}$$

for $n = N-1$ and $n = N-2$ with known fields

$$\mathbf{h}'_n(\mathbf{X}, \mathbf{x}) = \mathbf{g}'_n(\mathbf{x}) \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}, \quad \langle \mathbf{h}'_n \rangle_h = 0$$

(relations (5.20) and (5.27) are particular cases of (5.39), where $\mathbf{h}'_0 = \mathbf{h}'_1 = 0$);
- λ_n for all $n < N$.

Step 1° for $n = N$. By virtue of (5.38) and (5.39), the horizontal component of Eq. 5.13 for $n = N$, averaged over the fast spatial variables, takes the form

$$(\mathscr{E} - \lambda_2)\langle \mathbf{h}_{N-2} \rangle_h - \lambda_N \langle \mathbf{h}_0 \rangle_h = -\nabla_{\mathbf{X}} \times \langle \mathbf{V} \times \mathbf{h}'_{N-1} \rangle_v. \tag{5.40}$$

The r.h.s. of this equation (which is already known at this stage) is parallel to the vector $\hat{\mathbf{h}}$. We cancel out the factor $\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}$ and solve this equation to find

$$\langle \mathbf{h}_{N-2} \rangle_h = 0, \qquad \lambda_N = -\mathrm{i}\langle \mathbf{V} \times \mathbf{g}'_{N-1} \rangle_3. \tag{5.41}$$

Evidently, λ_N is now uniquely determined. In view of the solenoidality condition (5.18) for $n = N-2$, a horizontal vector field $\langle \mathbf{h}_{N-2} \rangle_h$ is determined in the invariant subspace under consideration (comprised of harmonic waves in the slow variables with $\mathbf{x}$-dependent amplitudes) up to an additive term, equal to (5.34) multiplied by a constant factor. We set this term to zero, as a normalisation condition.

Step 2° for $n = N$. After the solution (5.41) is found, the equation (5.13) for $n = N$ takes the form

$$\begin{aligned}
\mathscr{L}\{\mathbf{h}_N\}_h = &-\eta\Big(2(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\mathbf{h}'_{N-1} + \nabla^2_{\mathbf{X}}\{\mathbf{h}_{N-2}\}_h \\
&+ 2\sum_{k=1}^{2}\sum_{m=1}^{2}\left(\frac{\partial \mathbf{S}_k}{\partial x_m}\frac{\partial \langle \mathbf{h}_{N-1}\rangle_k}{\partial X_m} + \sum_{l=1}^{2}\frac{\partial \mathbf{G}_{mk}}{\partial x_l}\frac{\partial^2 \langle \mathbf{h}_{N-2}\rangle_k}{\partial X_l \partial X_m}\right)\Big) \\
&- \nabla_{\mathbf{X}} \times \left(\sum_{k=1}^{2}(\mathbf{V} \times \mathbf{S}_k)\langle \mathbf{h}_{N-1}\rangle_k + \sum_{k=1}^{2}\sum_{m=1}^{2}\{\mathbf{V} \times \mathbf{G}_{mk}\}_v \frac{\partial \langle \mathbf{h}_{N-2}\rangle_k}{\partial X_m} + \{\mathbf{V} \times \mathbf{h}'_{N-1}\}_v\right) \\
&- (\langle \mathbf{h}_N \rangle_h \cdot \nabla_{\mathbf{x}})\mathbf{V} + (\mathbf{V} \cdot \nabla_{\mathbf{X}})\langle \mathbf{h}_{N-1}\rangle_h + \sum_{j=0}^{N-2}\lambda_{N-j}\{\mathbf{h}_j\}_h.
\end{aligned} \tag{5.42}$$

$\langle \mathbf{h}_N \rangle_h$ and derivatives of $\langle \mathbf{h}_{N-1} \rangle_k$ are not yet known, but the $\mathbf{x}$-dependent factors in the terms in (5.42) involving these quantities are identical to the factors in the terms in the r.h.s. of (5.26) involving $\langle \mathbf{h}_1 \rangle_h$ and derivatives of $\langle \mathbf{h}_0 \rangle_k$, respectively. Consequently, $\{\mathbf{h}_N\}_h$ can be expressed as (5.39). An equation for $\mathbf{h}'_N$ is obtained by assuming $\mathbf{h}'_N$ in place of $\{\mathbf{h}_N\}_h$ in (5.42) and removing from (5.42) all terms involving $\langle \mathbf{h}_N \rangle_h$ and derivatives of $\langle \mathbf{h}_{N-1} \rangle_k$. The solvability condition is, evidently, satisfied for this equation.

It is easy to check by induction, that $\mathbf{h}'_N$ and $\{\mathbf{h}_N\}_h$ are symmetric about the vertical axis for odd N, and antisymmetric for even N. Consequently, for odd N,

$$\langle \mathbf{V} \times \mathbf{h}'_{N-1} \rangle_v = 0,$$

and hence (5.41) implies

$$\lambda_N = 0 \qquad \text{for odd } N.$$

Thus, we have constructed complete formal asymptotic expansions of magnetic modes (5.10) and the associated eigenvalues (5.11), which satisfy relations (5.34), (5.36), (5.37) and (5.38).

5.4 Enhancement of Molecular Diffusion in Plane-Parallel Flows in a Layer

By the Zeldovich antidynamo theorem [314] (see also [186]), *plane-parallel* flows (i.e., solenoidal flows $\mathbf{V}(\mathbf{x})$, everywhere orthogonal to a constant vector $\mathbf{k}$) cannot generate magnetic field. We prove here a stronger, in a sense, statement for a narrower class of flows: eddy corrections of magnetic diffusivity due to short-scale plane-parallel flows in a layer, satisfying the conditions listed in Sects. 5.1.1 and 5.1.3, are always non-negative. In other words, plane-parallel symmetric flows can only enhance magnetic molecular diffusion. Without any loss of generality, we assume in this section that the flow is independent of x_2, and $\mathbf{V}_2 = 0$ (i.e., the flow is plane-parallel for $\mathbf{k} = \mathbf{e}_2$).

Let us consider the auxiliary problems for flows of this kind.

$k = 1$. Equation (5.21) implies

$$(S_1)_2 = 0 \quad \Rightarrow \quad \mathbf{V} \times (\mathbf{S}_1 + \mathbf{e}_1) = R(\mathbf{x})\mathbf{e}_2.$$

Consequently, for $m = 1$ the r.h.s. of (5.28) is equal to $-2\eta \dfrac{\partial \mathbf{S}_1}{\partial x_1} - R(\mathbf{x})\mathbf{e}_3$, whereby

$$(G_{11})_2 = 0 \quad \Rightarrow \quad \langle \mathbf{V} \times \mathbf{G}_{11} \rangle_v = 0.$$

For $m = 2$, the r.h.s. of (5.28) vanishes, and hence

$$\mathbf{G}_{21} = 0 \quad \Rightarrow \quad \langle \mathbf{V} \times \mathbf{G}_{21} \rangle_v = 0.$$

$k = 2$. From (5.21), we find $\mathbf{S}_2 = 0$. Consequently, for $m = 1$ the r.h.s. of (5.28) is equal to $V_1\mathbf{e}_2$, and thus $\mathbf{G}_{12} = G\mathbf{e}_2$, where G solves the equation

$$\eta \nabla^2 \boldsymbol{\Gamma} - (\mathbf{V} \cdot \nabla)\boldsymbol{\Gamma} = V_1.$$

A similar equation emerges as the auxiliary problem in analysis of large-scale passive scalar transport (see Chap. 2). The respective coefficient of the tensor of eddy correction of magnetic diffusion is non-positive:

$$\langle \mathbf{V} \times \mathbf{G}_{12} \rangle_3 = \langle V_1 \boldsymbol{\Gamma} \rangle = -\eta \langle |\nabla \boldsymbol{\Gamma}|^2 \rangle \le 0$$

(it is strictly negative for $V_1 \neq 0$).

For $m = 2$, the r.h.s. of (5.28) is equal to $-\mathbf{V}$, and therefore

$$(G_{22})_2 = 0 \quad \Rightarrow \quad \langle \mathbf{V} \times \mathbf{G}_{22} \rangle_v = 0.$$

Summing up these results, we find from (5.36):

$$-\lambda_2(\mathbf{q}) = \eta |\mathbf{q}|^2 - \langle \mathbf{V} \times \mathbf{G}_{12} \rangle_3 \, q_1^2 \ge \eta |\mathbf{q}|^2,$$

i.e. the minimum of magnetic eddy diffusivity over unit wave vectors is attained for a wave vector, normal to the flow, i.e. for $\mathbf{q} = \mathbf{e}_2$.

5.5 Convective Plan Forms in a Layer of a Non-rotating Fluid

The findings of the previous sections will be used for the study of magnetic eddy diffusion in convective plan forms. The properties of these flows, relevant for our goals, are summarised in this section. A comprehensive reference is [53].

5.5.1 *Analytic Expressions*

Thermal convection in a horizontal non-rotating layer of fluid due to heating of the lower boundary is considered (both boundaries of the layer are held at constant temperatures). Plan forms are the flow components of the instability modes near the bifurcation of the trivial steady state, at which the fluid is at rest, heat is conducted by molecular diffusion, and no magnetic field is present—in other words, near the onset of the fluid flow, when the difference in temperature at the lower and upper boundary exceed the critical value. The flows are *poloidal*:

$$\mathbf{V}(\mathbf{x}) = \nabla \times \nabla \times (P(\mathbf{x}) \, \mathbf{e}_3). \tag{5.43}$$

In the most general form the potential $P(\mathbf{x})$ takes the form [22]

$$P(\mathbf{x}) = (\alpha_1 \cos(K_1 x_1) \cos(K_2 x_2) + \alpha_2 \cos(p K_2 x_2)) \, w(x_3). \tag{5.44}$$

Here α_1 and α_2 are constants, p is an integer, $K_i = 2\pi/L_i$ $(i = 1, 2)$ are wave numbers, associated with the period L_i along the axis x_i. Furthermore,

$$K_1/K_2 = \sqrt{p^2 - 1}, \quad \text{if } \alpha_2 \neq 0.$$

Evidently, a plan form has all the prerequisite properties of periodicity in the horizontal directions (5.4), solenoidality (5.8) and symmetry about a vertical axis

(5.9). The remaining condition—impenetrability of the horizontal boundaries of the layer for the fluid (5.5)—is equivalent to the requirement

$$w(x_3)\Big|_{x_3=0,\pi} = 0. \tag{5.45}$$

If, in addition,

$$\frac{\partial^{2m} w}{\partial x_3^{2m}}\Big|_{x_3=0,\pi} = 0$$

for all integer $m > 0$, the plan form satisfies the condition for the *stress-free boundary*:

$$\frac{\partial V_1}{\partial x_3}\Big|_{x_3=0,\pi} = \frac{\partial V_2}{\partial x_3}\Big|_{x_3=0,\pi} = 0, \qquad V_3\Big|_{x_3=0,\pi} = 0.$$

Vertical profiles of instability modes in the case of stress-free boundaries are simple harmonics,

$$w_n(x_3) = \sin nx_3, \tag{5.46}$$

where $n > 0$ is an integer. When the Rayleigh number (proportional to the difference of temperatures at the horizontal boundaries) exceeds the critical value, the mode with $n = 1$ becomes unstable. For rigid boundaries, the vertical profiles of instability modes have a more complex structure (see [53]).

5.5.2 *Additional Symmetries*

Plan forms have two additional reflection symmetries:

about the plane $x_1 = 0$:

$$\begin{aligned} V_1(-x_1, x_2, x_3) &= -V_1(x_1, x_2, x_3), \\ V_2(-x_1, x_2, x_3) &= V_2(x_1, x_2, x_3), \\ V_3(-x_1, x_2, x_3) &= V_3(x_1, x_2, x_3); \end{aligned} \tag{5.47}$$

about the plane $x_2 = 0$:

$$\begin{aligned} V_1(x_1, -x_2, x_3) &= V_1(x_1, x_2, x_3), \\ V_2(x_1, -x_2, x_3) &= -V_2(x_1, x_2, x_3), \\ V_3(x_1, -x_2, x_3) &= V_3(x_1, x_2, x_3). \end{aligned} \tag{5.48}$$

The symmetry about the vertical axis $x_1 = x_2 = 0$ (5.9) is a composition of these two reflections.

Table 5.1 Centres of parity invariance of a plan form in a layer with stress-free boundaries and a vertical profile (5.46).

n	p	Centres of parity invariance
Odd	$\alpha_2 = 0$	$\left(\frac{L_1}{2}\left(l_1+\frac{1}{2}\right), \frac{L_2 l_2}{2}, \frac{\pi}{2}\right)$
Odd	Odd, or $\alpha_2 = 0$	$\left(\frac{L_1 l_1}{2}, \frac{L_2}{2}\left(l_2+\frac{1}{2}\right), \frac{\pi}{2}\right)$
Even	Any	$\left(\frac{L_1 l_1}{2}, \frac{L_2 l_2}{2}, \frac{\pi}{2}\right)$
Even	Even, or $\alpha_2 = 0$	$\left(\frac{L_1}{2}\left(l_1+\frac{1}{2}\right), \frac{L_2}{2}\left(l_2+\frac{1}{2}\right), \frac{\pi}{2}\right)$

l_1 and l_2 are arbitrary integers

In the case of square convective cells, i.e. for $L_1 = L_2$ and $\alpha_2 = 0$, the plan form has an independent reflection symmetry about the diagonal plane $x_1 = x_2$:

$$V_1(x_1, x_2, x_3) = V_2(x_2, x_1, x_3), \quad V_3(x_1, x_2, x_3) = V_3(x_2, x_1, x_3). \tag{5.49}$$

A plan form in a layer with stress-free boundaries and a vertical profile (5.46) is parity-invariant about the centres listed in Table 5.1. Parity invariance guarantees the absence of the α-effect in the leading order.

5.5.3 *Magnetic Eddy Diffusion Operator for Convective Plan Forms in a Layer*

Each of the reflection symmetries (5.47), (5.48) and (5.49) splits the domain of the operator of magnetic induction $\mathscr{L}$ into a direct sum of two invariant subspaces consisting of symmetric and antisymmetric vector fields for the respective symmetry. A vector field has the antisymmetry of reflection about the plane $x_1 = 0$, if it satisfies the modified relations (5.47), where the signs in the r.h.s. are reversed:

$$\begin{aligned} f_1(-x_1, x_2, x_3) &= f_1(x_1, x_2, x_3), \\ f_2(-x_1, x_2, x_3) &= -f_2(x_1, x_2, x_3), \\ f_3(-x_1, x_2, x_3) &= -f_3(x_1, x_2, x_3); \end{aligned}$$

the antisymmetries of reflection about the planes $x_2 = 0$ and $x_1 = x_2$ are defined similarly.

Consequently, solutions to the auxiliary problems of type I, $\mathbf{S}_k$, have the same symmetries/antisymmetries, as the vector field $\partial \mathbf{V}/\partial x_k$ (see (5.21)). Furthermore, solutions to the auxiliary problems of type II, $\mathbf{G}_{mk}$ have the same symmetries/antisymmetries, as the field $\partial^2 \mathbf{V}/\partial x_m \partial x_k$ (see (5.28)). In particular, $\mathbf{G}_{kk}$ possess both reflection symmetries (about the planes $x_1 = 0$ and $x_1 = 0$), and hence

$$\langle \mathbf{V} \times \mathbf{G}_{kk} \rangle_v = 0.$$

This implies that, restricted to the subspace of vector fields which are solenoidal in the slow variables and have a zero vertical component, the operator of magnetic eddy diffusion is diagonal:

$$\mathscr{E} = (\eta - \langle \mathbf{V} \times \mathbf{G}_{12} \rangle_3) \frac{\partial^2}{\partial X_1^2} + (\eta + \langle \mathbf{V} \times \mathbf{G}_{21} \rangle_3) \frac{\partial^2}{\partial X_2^2}.$$

We find therefore, that for a flow having the symmetries (5.47) and (5.48), the minimum magnetic eddy diffusivity

$$\eta_{\text{eddy}} \equiv \min_{|\mathbf{q}|=1} (-\lambda_2(\mathbf{q}))$$

can be expressed as

$$\eta_{\text{eddy}} = \eta + \min(-\langle \mathbf{V} \times \mathbf{G}_{12} \rangle_3, \langle \mathbf{V} \times \mathbf{G}_{21} \rangle_3). \quad (5.50)$$

Let us now consider the case of square convective cells, in which plan forms have the reflection symmetry (5.49) about the diagonal plane. It is simple to check that the mapping

$$\mathscr{S}(\mathbf{V}) = (V_2(x_2, x_1, x_3), V_1(x_2, x_1, x_3), V_3(x_2, x_1, x_3))$$

commutes with $\mathscr{L}$. (Fields, possessing the symmetry (5.49), are fixed points of $\mathscr{S}$). Evidently,

$$\mathscr{S}\left(\frac{\partial \mathbf{V}}{\partial x_1}\right) = \frac{\partial \mathbf{V}}{\partial x_2},$$

and hence $\mathbf{S}_2 = \mathscr{S}(\mathbf{S}_1)$. Consequently, the r.h.s. of the equations (5.28), defining solutions $\mathbf{G}_{12}$ and $\mathbf{G}_{21}$ to the auxiliary problems of type II, are also mapped into each other by the mapping $\mathscr{S}$. This implies

$$\mathbf{G}_{12} = \mathscr{S}(\mathbf{G}_{21}),$$

and hence

$$\eta - \langle \mathbf{V} \times \mathbf{G}_{12} \rangle_3 = \eta + \langle \mathbf{V} \times \mathbf{G}_{21} \rangle_3.$$

Therefore, in the presence of the reflection symmetry (5.49) about the diagonal plane, magnetic eddy diffusion is isotropic and it is described by a single scalar coefficient.

5.6 Numerical Investigation of Magnetic Eddy Diffusion

As we have shown in the previous section, for computation of the tensor of eddy correction of magnetic diffusion it suffices to solve two auxiliary problems of type I (to determine $\mathbf{S}_1$ and $\mathbf{S}_2$) and two auxiliary problems of type II (to determine $\mathbf{G}_{12}$ and $\mathbf{G}_{21}$). However, one can also apply the method described in Sect. 4.4, which requires to solve one auxiliary problem for the adjoint operator:

$$\mathscr{L}^* \mathbf{Z}_3 = \mathbf{V} \times \mathbf{e}_3$$

instead of computing $\mathbf{G}_{12}$ and $\mathbf{G}_{21}$. (The differences in the geometry of the fluid container or in the symmetries of the flow do not affect this method based on expression (4.51).)

Computations presented in this section were performed for plan forms (5.43)–(5.44) with the vertical profile

$$w(x_3) = \sum_{n=1}^{N} \beta_n \sin(nx_3), \tag{5.51}$$

where β_n are constants. The flows were normalised: $\langle |\mathbf{V}|^2 \rangle = 1$. Plots in Figs. 5.1–5.3 are constructed for plan forms with the vertical profile (5.46).

Since solutions to the auxiliary problems have two reflection symmetries (5.47) and (5.48) (see Sect. 5.5.2), and satisfy the periodicity (5.7) and boundary (5.3) conditions, they can be expanded in the following Fourier series:

$$\mathbf{S}_1 = \sum_{n_i \geq 0} \begin{bmatrix} s_{1,\mathbf{n},1} & \cos n_1K_1x_1 & \cos n_2K_2x_2 & \cos n_3x_3 \\ s_{1,\mathbf{n},2} & \sin n_1K_1x_1 & \sin n_2K_2x_2 & \cos n_3x_3 \\ s_{1,\mathbf{n},3} & \sin n_1K_1x_1 & \cos n_2K_2x_2 & \sin n_3x_3 \end{bmatrix}, \tag{5.52}$$

$$\mathbf{S}_2 = \sum_{n_i \geq 0} \begin{bmatrix} s_{2,\mathbf{n},1} & \sin n_1K_1x_1 & \sin n_2K_2x_2 & \cos n_3x_3 \\ s_{2,\mathbf{n},2} & \cos n_1K_1x_1 & \cos n_2K_2x_2 & \cos n_3x_3 \\ s_{2,\mathbf{n},3} & \cos n_1K_1x_1 & \sin n_2K_2x_2 & \sin n_3x_3 \end{bmatrix}, \tag{5.53}$$

$$\mathbf{G}_{mk} = \sum_{n_i \geq 0} \begin{bmatrix} \gamma_{mk,\mathbf{n},1} & \cos n_1K_1x_1 & \sin n_2K_2x_2 & \cos n_3x_3 \\ \gamma_{mk,\mathbf{n},2} & \sin n_1K_1x_1 & \cos n_2K_2x_2 & \cos n_3x_3 \\ \gamma_{mk,\mathbf{n},3} & \sin n_1K_1x_1 & \sin n_2K_2x_2 & \sin n_3x_3 \end{bmatrix} \tag{5.54}$$

(the last formula is applicable for $m \neq k$). Solenoidality conditions (5.23) and (5.29) now become

$$\begin{aligned} -s_{1,\mathbf{n},1}n_1K_1 + s_{1,\mathbf{n},2}n_2K_2 + s_{1,\mathbf{n},3}n_3 &= 0, \\ s_{2,\mathbf{n},1}n_1K_1 - s_{2,\mathbf{n},2}n_2K_2 + s_{2,\mathbf{n},3}n_3 &= 0, \\ -\gamma_{mk,\mathbf{n},1}n_1K_1 - \gamma_{mk,\mathbf{n},2}n_2K_2 + \gamma_{mk,\mathbf{n},3}n_3 + s_{k,\mathbf{n},m} &= 0. \end{aligned}$$

It is natural to solve auxiliary problems in the terms of coefficients of the series (5.52)–(5.54). Each of the symmetries discussed below in this paragraph halves the number of the unknown non-zero coefficients. If p is an odd number or $\alpha_2 = 0$ in the potential of the plan form (5.44), then the sum of wave numbers in horizontal directions is even in all harmonics comprising the flow. Consequently, the domain of the operator $\mathscr{L}$ splits into a direct sum of two invariant subspaces, categorised by the parity of the sum of wave numbers in horizontal directions, $n_1 + n_2$, of the harmonics spanning the subspaces. Thus, if p is odd or $\alpha_2 = 0$, then the coefficients of the series (5.52)–(5.54) vanish for wave vectors $\mathbf{n}$ with odd sums $n_1 + n_2$. Similarly, if the sum (5.51) involves sines for odd wave numbers n only, then the coefficients vanish for wave vectors $\mathbf{n}$ with the odd sum $n_1 + n_3$. For plan forms with the vertical profile (5.46) for $n > 1$, whose coefficients of the series (5.52)–(5.54) are zero for all n_3 not divisible by a certain integer n, the last symmetry is modified: the coefficients are non-zero only for even $n_1 + n_3/n$.

All computations were performed with the resolution of 64 trigonometric function in each direction. The energy spectrum of the solutions to auxiliary problems computed for this resolution decays by several (at least 4) orders of magnitude.

Figures 5.1 and 5.2 show plots of the minimum magnetic eddy diffusivity $\eta_{\rm eddy}$ (5.50) and the growth rates of the dominant short-scale magnetic modes with zero-mean horizontal components (of the same space periodicity as the convective plan form) for two families of parameter values for $\alpha_2 = 0$ (the associated eigenvalues of the magnetic induction operator are real). According to [177, 178], square

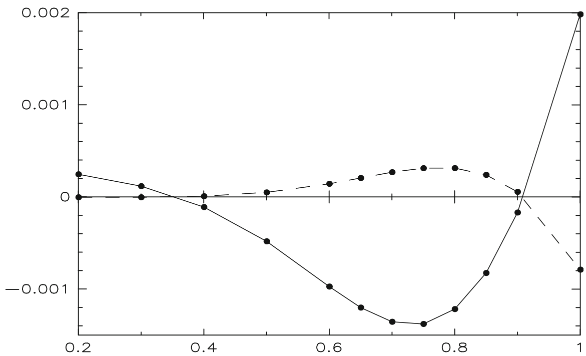

Fig. 5.1 The minimum magnetic eddy diffusivity $\eta_{\rm eddy}$ (*solid line*, *the vertical axis*) in plan forms for $n = 1$, $K_2 = 2$, $\alpha_2 = 0$ and the growth rates of the dominant short-scale magnetic modes with zero-mean horizontal components generated by these flows (*dashed line*, *the vertical axis*) for molecular diffusivity $\eta = 0.06$, as functions of K_1 (the horizontal axis). *Dots* show the computed values

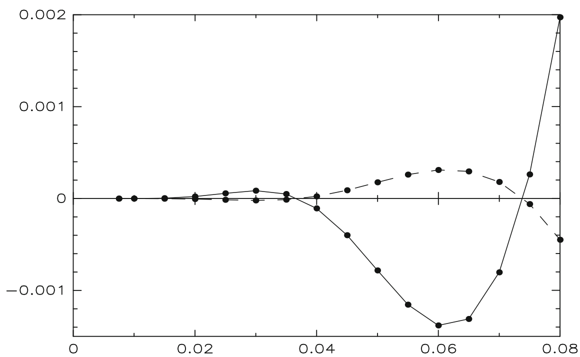

Fig. 5.2 The minimum magnetic eddy diffusivity $\eta_{\rm eddy}$ (*solid line*, *the vertical axis*) in plan forms for $n = 1$, $K_1 = 0.75$, $K_2 = 2$, $\alpha_2 = 0$ and the growth rates of the dominant short-scale magnetic modes with zero-mean horizontal components generated by these flows (*dashed line*, *the vertical axis*), as functions of molecular diffusivity η (the horizontal axis). *Dots* show the computed values

convective plan forms for $K_1 = K_2 = 1/2$ and $\alpha_2 = 0$ are not kinematic dynamos. Magnetic fields satisfying the boundary conditions (5.3) and periodic along diagonals of the convective cells with the period equal to a half of the diagonal length were considered in these papers (such flows constitute a subspace in the class of short-scale flows that we are considering). Figures 5.1 and 5.2 attest that for some sets of parameter values square cells are short-scale kinematic dynamos.

Figures 5.1 and 5.2 illustrate an unexpected phenomenon: the critical value of magnetic molecular diffusivity for the onset of short-scale magnetic field generation coincides with the critical value for the onset of large-scale field generation. In other words, the minimum magnetic eddy diffusivity vanishes for the same sets of parameter values, for which the growth rate of the dominant short-scale magnetic mode vanishes. Since growth rates of large-scale magnetic fields are small (the order of ε^2), the large-scale magnetic field instability is weak compared to the short-scale one.

The coincidence of the windows of kinematic generation of short- and large-scale magnetic field for $\alpha_2 = 0$ has been established numerically for various plan forms, whose vertical profiles (5.51) are sums of up to three sines with the same or different parity of the wave numbers n. The critical magnetic molecular diffusivities were determined to the accuracy of at least 10^{-8}. All modes found in these computations have common parity properties. The dominant short-scale magnetic modes possess the reflection antisymmetries about the planes $x_1 = 0$ and $x_2 = 0$ (the same ones that $\mathbf{G}_{12}$ and $\mathbf{G}_{21}$ have). Consequently, the short-scale neutral magnetic modes can be expanded in the Fourier series (5.54), in which the coefficients for wave vectors $\mathbf{n}$ with even sums $n_1 + n_2$ vanish (in contrast to the solutions to the auxiliary problems of type II). Furthermore, when wave numbers n in all terms constituting the vertical profile $w(x_3)$ are odd, terms in the series (5.54) representing the short-scale neutral mode do not vanish for wave vectors $\mathbf{n}$ with components of the parities (odd, even, even) or (even, odd, odd), while in such series for $\mathbf{G}_{12}$ and $\mathbf{G}_{21}$ terms do not vanish for wave vectors, the parity of whose all components is the same.

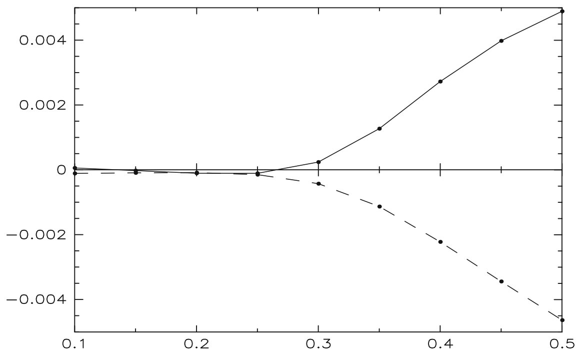

Fig. 5.3 The minimum magnetic eddy diffusivity η_{eddy} (*solid line*, *the vertical axis*) in plan forms for $n = 1$, $p = 2$, $K_1 = 1.5\sqrt{3}$ and the growth rates of the dominant short-scale magnetic modes with zero-mean horizontal components generated by these flows (*dashed line*, *the vertical axis*), as functions of the ratio α_2/α_1 (the horizontal axis). *Dots* show the computed values

Plots of the minimum magnetic eddy diffusivity and the growth rates of the dominant short-scale magnetic modes with zero-mean horizontal components for a family of plan forms for $\alpha_2 \neq 0$ are shown in Fig. 5.3 (the respective eigenvalues of the short-scale magnetic induction operator are real). The plan forms for the considered sets of parameter values fail to generate magnetic fields of the same periodicity as that of the flow, but there exists a window of the ratios α_2/α_1, for which $\eta_{\text{eddy}} < 0$. Thus, generation of large-scale magnetic fields by the respective plan forms from this family can be observed.

The plan forms for $p = 2,\ \alpha_2 = \alpha_1/2$ consist of hexagonal convective cells [6]. It was found in [334] that this flow can generate short-scale magnetic modes for a relatively small magnetic molecular diffusivity (for the conditions for the magnetic field at the upper boundary, different from those considered here—for the case of a dielectric above the layer of fluid). No short-scale generation by hexagonal cells for $K_1 = \sqrt{3/8},\ K_2 = \sqrt{1/8}$ and the perfect conductor conditions on the horizontal boundaries (5.3) was found in [178]. We have computed η_{eddy} for such hexagonal plan forms with vertical profiles (5.46) or (5.51) involving two sine functions for various ratios β_2/β_1, but we did not find any case of negative eddy diffusivity.

5.7 Conclusions

(1) We have constructed in this chapter complete formal asymptotic expansions of large-scale magnetic modes generated kinematically by flows, symmetric about the vertical axis, in a layer of fluid with perfectly electrically conducting horizontal boundaries, together with the associated eigenvalues of the magnetic induction operator. We have shown that the α-effect is insignificant in the leading order (i.e., it does not affect generation of large-scale magnetic field) and derived the operator of magnetic eddy diffusion (5.33). We have proved that in plane-parallel flows eddy correction of magnetic diffusion can only enhance molecular diffusion. If the flow has the symmetries of reflection about the vertical planes $x_1 = 0$ and $x_2 = 0$, then the magnetic eddy diffusion operator is diagonal. Magnetic eddy diffusion is isotropic, if in addition the flow in a square periodicity cell has the symmetry of reflection about the diagonal vertical plane $x_1 = x_2$

(2) These analytical results were applied to investigate the kinematic generation of large-scale magnetic field by plan forms of thermal convection in a horizontal layer of electrically conducting fluid without rotation. The minimum magnetic eddy diffusivity η_{eddy} was computed for three families of plan forms. Numerical results reveal that for $\alpha_2 = 0$ in the potential (5.44) of a plan form (and hence if the flow is invariant under swapping of the horizontal coordinate axes, together with an appropriate rescaling if periods along the two coordinate axes are different), the regions of parameter values of kinematic

generation of short-scale and large-scale magnetic fields coincide. No cases of negative magnetic eddy diffusivity for hexagonal cells were identified. A family of less symmetric plan forms for $\alpha_2 \neq 0$ is found, for which the minimum magnetic eddy diffusivity is negative, but no short-scale magnetic fields are generated. In such flows generation of large-scale magnetic fields can be observed, however, the intervals of parameter values for which $\eta_{\mathrm{eddy}} < 0$ are short, and the absolute values of the computed negative minimum magnetic eddy diffusivities are small.

(3) Let us also note a side result. Convective plan forms are poloidal flows (5.43), and hence their helicity is pointwise zero. We have thus demonstrated once again that short-scale or large-scale magnetic dynamos do not rely on helicity of the generating fluid flows.

Chapter 6
Linear Stability of Steady MHD States to Large-Scale Perturbations

It was shown in [82] that stability of three-dimensional space-periodic steady flows to large-scale perturbations is controlled by the so-called AKA-effect (the acronym means the anisotropic kinematic α-effect). Its absence is guaranteed only in a single wide class: parity-invariant flows. Stability of such flows is determined by the effect of eddy viscosity. The analysis of kinematic generation of large-scale magnetic fields (see Chaps. 3, 4 and 5) reveals a similar picture: If a generating flow does not possess special properties (e.g., a symmetry, like parity invariance), then the magnetic α-effect is predominantly responsible for generation of large-scale magnetic fields [307] (see Chap. 3). If the flow is parity-invariant, then the magnetic α-effect does not emerge, the short-scale fluid motions give rise to magnetic eddy diffusion, and large-scale field generation is possible when magnetic eddy diffusivity is negative [161, 335, 336]. The growth rates of instability modes, both kinematic and magnetic ones, for which the α-effect is responsible, are the order of the ratio of the large and small spatial scales, ε; the growth rates of the modes appearing due to eddy diffusion are the order of ε^2.

The analogy can be traced further: after elimination of the fluctuating components of the respective modes by averaging over small scales, both the large-scale magnetic field generation and the large-scale hydrodynamic instability due to the action of the α-effect are governed by first-order partial differential equations, that do not involve terms describing dissipation of energy. The spectra of the operators of the α-effect are symmetric about the imaginary axis; thus, generically hydrodynamic and magnetic systems possessing the α-effect are unstable. In order to distinguish between this situation and a physically more natural one, in which the α-effect competes with molecular diffusion, it was proposed in [307] to call γ-effect the former case of the isolated α-effect. (Molecular diffusion and the α-effect are acting simultaneously, if the amplitude of the fluctuating part of a large-scale flow increases as $\varepsilon^{-1/2}$ [306, 321–323], see Chaps. 10 and 11).

In view of this similarity of the stability properties of the hydrodynamic and magnetic systems to large-scale perturbations, it is natural to generalise the

V. Zheligovsky, *Large-Scale Perturbations of Magnetohydrodynamic Regimes*, Lecture Notes in Physics, 829, DOI: 10.1007/978-3-642-18170-2_6,

analysis and investigate a combined MHD system, in which perturbation modes possess both magnetic and hydrodynamic components. We do this in the present chapter following the paper [326]. We consider linear stability of three-dimensional space-periodic steady MHD states and construct a complete formal asymptotic expansion of the perturbation modes in power series in the scale ratio ε. We show that in a generic MHD state the evolution of large-scale perturbations is controlled in the leading order by the action of the combined MHD α-effect. Unlike in the previous chapters, we construct the asymptotic expansion of the modes for this general case (assuming that the operator of the MHD α-effect is elliptic). If the steady MHD state is parity-invariant, then the α-effect is absent, and the behaviour of the perturbation modes averaged over small scales is determined in the leading order by a second-order partial differential operator—the so-called operator of the anisotropic combined MHD eddy (turbulent) diffusion. We study numerically the distribution of the combined MHD eddy diffusivity in an ensemble of steady MHD states, synthesised from random-amplitude Fourier harmonics with an exponentially decaying energy spectrum.

6.1 Statement of the Linear Stability Problem

6.1.1 The Governing Equations

An MHD steady state $\mathbf{V}, \mathbf{H}, P$ satisfies the Navier–Stokes and magnetic induction equations, and the solenoidality conditions:

$$\nu\nabla^2\mathbf{V} + \mathbf{V} \times (\nabla \times \mathbf{V}) + (\nabla \times \mathbf{H}) \times \mathbf{H} - \nabla P + \mathbf{F} = 0, \qquad (6.1)$$

$$\eta\nabla^2\mathbf{H} + \nabla \times (\mathbf{V} \times \mathbf{H}) + \mathbf{J} = 0, \qquad (6.2)$$

$$\nabla \cdot \mathbf{V} = \nabla \cdot \mathbf{H} = 0. \qquad (6.3)$$

Here $\mathbf{V}(\mathbf{x})$ is the velocity of a conducting fluid flow, $\mathbf{H}(\mathbf{x})$ magnetic field, $P(\mathbf{x}) - \frac{1}{2}|\mathbf{V}|^2$ pressure, ν molecular viscosity, η molecular magnetic diffusivity, $\mathbf{F}$ an external body force. The term $\mathbf{J}$ is, strictly speaking, unphysical; it may reflect the presence of external currents in the volume occupied by the fluid; it is introduced into (6.2) in order to allow for the maximum possible generality. If $\mathbf{F} = \mathbf{J} = 0$, the only solution to the system of equations (6.1)–(6.3) is the stable steady state $\mathbf{V} = \mathbf{H} = 0$. The presence of the source terms $\mathbf{F}(\mathbf{x})$ and/or $\mathbf{J}(\mathbf{x})$ Sect. 6.2, not supposed to be both identically zero, breaks the spatial invariance of the system; we will note in the importance of this for our constructions. We assume, that the fields are $\mathbf{L}$-periodic, i.e. have period L_i in each Cartesian variable x_i. Investigation of linear stability of the MHD steady state $\mathbf{V}, \mathbf{H}, P$ gives rise to the eigenvalue problem

$$\mathscr{M}^v(\mathbf{v},\mathbf{h},p) \equiv \nu\nabla^2\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{V}) + \mathbf{V} \times (\nabla \times \mathbf{v}) \\ + (\nabla \times \mathbf{h}) \times \mathbf{H} + (\nabla \times \mathbf{H}) \times \mathbf{h} - \nabla p = \lambda\mathbf{v}, \tag{6.4}$$

$$\mathscr{M}^h(\mathbf{v},\mathbf{h}) \equiv \eta\nabla^2\mathbf{h} + \nabla \times (\mathbf{v} \times \mathbf{H} + \mathbf{V} \times \mathbf{h}) = \lambda\mathbf{h}, \tag{6.5}$$

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{h} = 0. \tag{6.6}$$

for the operator of linearisation $\mathscr{M} = (\mathscr{M}^v, \mathscr{M}^h)$ of the equations (6.1)–(6.3) describing the MHD steady state. (Note that the meaning of the labels v and h depends on their location: the superscripts v and h denote the flow and magnetic components of operators and vector fields, respectively, and the subscripts v and h denote the vertical and horizontal components of the averaged three-dimensional fields, see (5.14) and (5.15).)

6.1.2 Asymptotic Expansion of Large-Scale MHD Stability Modes

A large-scale stability mode $\mathbf{v},\mathbf{h},p$ is supposed to possess a large spatial scale of the order $|\mathbf{L}|/\varepsilon$, where $\varepsilon > 0$ is a small parameter. As in the previous chapters, $\mathbf{x}$ denotes the fast spatial variable, and $\mathbf{X} = \varepsilon\mathbf{x}$ the slow one. A solution to the eigenvalue problem (6.4)–(6.6) is sought in the form of power series

$$\mathbf{v} = \sum_{n=0}^{\infty} \mathbf{v}_n(\mathbf{X},\mathbf{x})\varepsilon^n, \quad \mathbf{h} = \sum_{n=0}^{\infty} \mathbf{h}_n(\mathbf{X},\mathbf{x})\varepsilon^n, \tag{6.7}$$

$$p = \sum_{n=0}^{\infty} p_n(\mathbf{X},\mathbf{x})\varepsilon^n, \tag{6.8}$$

$$\lambda = \sum_{n=0}^{\infty} \lambda_n\varepsilon^n. \tag{6.9}$$

6.1.3 The Hierarchy of Equations for the Large-Scale Modes

We start by considering the solenoidality conditions (6.6) for large-scale MHD perturbations. Substituting the series (6.7), equating to zero the terms at each order ε^n and extracting the mean and fluctuating parts (formally defined in Sect. 3.1.1), we find

$$\nabla_{\mathbf{X}} \cdot \langle \mathbf{v}_n \rangle = 0, \tag{6.10}$$

$$\nabla_{\mathbf{X}} \cdot \langle \mathbf{h}_n \rangle = 0, \tag{6.11}$$

$$\nabla_{\mathbf{x}} \cdot \{\mathbf{v}_n\} + \nabla_{\mathbf{X}} \cdot \{\mathbf{v}_{n-1}\} = 0, \tag{6.12}$$

$$\nabla_{\mathbf{x}} \cdot \{\mathbf{h}_n\} + \nabla_{\mathbf{X}} \cdot \{\mathbf{h}_{n-1}\} = 0 \tag{6.13}$$

for all $n \geq 0$. Differential operators with the indices $\mathbf{x}$ and $\mathbf{X}$ denote the operations with differentiation in the fast and slow spatial variables, respectively; any term with a negative index n is zero by definition.

We substitute now series (6.7)–(6.9) into the eigenvalue equations (6.4)–(6.5) and convert them into equalities for power series in ε. Considering the terms at each power of ε, we obtain a hierarchy of systems of equations

$$\mathscr{M}^v(\mathbf{v}_n, \mathbf{h}_n, p_n) + \nu\big(2(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\{\mathbf{v}_{n-1}\} + \nabla^2_{\mathbf{X}}\mathbf{v}_{n-2}\big)$$
$$+ \mathbf{V} \times (\nabla_{\mathbf{X}} \times \mathbf{v}_{n-1}) + (\nabla_{\mathbf{X}} \times \mathbf{h}_{n-1}) \times \mathbf{H} = \sum_{m=0}^{n} \lambda_{n-m}\mathbf{v}_m, \tag{6.14}$$

$$\mathscr{M}^h(\mathbf{v}_n, \mathbf{h}_n) + \eta\big(2(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\{\mathbf{h}_{n-1}\} + \nabla^2_{\mathbf{X}}\mathbf{h}_{n-2}\big)$$
$$+ \nabla_{\mathbf{X}} \times (\mathbf{V} \times \mathbf{h}_{n-1} + \mathbf{v}_{n-1} \times \mathbf{H}) = \sum_{m=0}^{n} \lambda_{n-m}\mathbf{h}_m. \tag{6.15}$$

Henceforth, we reserve the notation $\mathscr{M} = (\mathscr{M}^v, \mathscr{M}^h)$ for the operator of linearisation of the equations of magnetohydrodynamics (6.4)–(6.5), that involves differentiation in fast variables only.

6.2 Solvability of Auxiliary Problems

We intend to solve successively the systems (6.14)–(6.15) together with the conditions (6.10)–(6.13). As in the previous chapters, this gives rise to auxiliary problems; we consider their solvability in this section.

Let $\mathscr{L}' = (\mathscr{L}'^v, \mathscr{L}'^h)$ be the restriction of the operator of linearisation $\mathscr{M}$ to the subspace of solenoidal vector fields, $\mathbf{L}$-periodic in fast variables (acting into the same subspace), and $\mathscr{L} = (\mathscr{L}^v, \mathscr{L}^h)$ the further restriction of $\mathscr{L}$ to the subspace of zero-mean vector fields.

The adjoint operator to linearisation $\mathscr{L}'^* = ((\mathscr{L}'^*)^v, (\mathscr{L}'^*)^h)$ is derived performing integration by parts in the identity $\langle \mathscr{L}'^*(\mathbf{v}, \mathbf{h}) \cdot (\mathbf{v}', \mathbf{h}') \rangle \equiv \langle (\mathbf{v}, \mathbf{h}) \cdot \mathscr{L}'(\mathbf{v}', \mathbf{h}') \rangle$:

$$(\mathscr{L}'^*)^v(\mathbf{v}, \mathbf{h}) = \nu\nabla^2_{\mathbf{x}}\mathbf{v} - \nabla_{\mathbf{x}} \times (\mathbf{V} \times \mathbf{v}) + \mathscr{P}_{\rm sol}(\mathbf{H} \times (\nabla_{\mathbf{x}} \times \mathbf{h}) - \mathbf{v} \times (\nabla_{\mathbf{x}} \times \mathbf{V})),$$

$$(\mathscr{L}'^{*})^{h}(\mathbf{v},\mathbf{h}) = \eta\nabla_{\mathbf{x}}^{2}\mathbf{h} + \nabla_{\mathbf{x}} \times (\mathbf{H}\times\mathbf{v}) + \mathscr{P}_{\mathrm{sol}}(\mathbf{v}\times(\nabla_{\mathbf{x}}\times\mathbf{H}) - \mathbf{V}\times(\nabla_{\mathbf{x}}\times\mathbf{h})).$$

Here $\mathscr{P}_{\mathrm{sol}}$ is the projection onto the subspace of solenoidal vector fields,

$$\mathscr{P}_{\mathrm{sol}}\mathbf{A} \equiv \mathbf{A} - \nabla_{\mathbf{x}}a,$$

where

$$a = \nabla_{\mathbf{x}}^{-2}(\nabla_{\mathbf{x}}\cdot\mathbf{A})$$

is the $\mathbf{L}$-periodic solution to the Laplace equation $\nabla_{\mathbf{x}}^{2}a = \nabla_{\mathbf{x}}\cdot\mathbf{A}$. Evidently, any pair of constant vectors $(\mathbf{C}^{v},\mathbf{C}^{h})$ belongs to the kernel of $\mathscr{L}'$, and hence $\ker\mathscr{L}'$ is at least six-dimensional.

Solenoidality (6.3) and periodicity of $\mathbf{V}$ and $\mathbf{H}$ imply the identities

$$\langle\mathscr{M}^{v}(\mathbf{v},\mathbf{h},p)\rangle = \langle\mathbf{V}\nabla_{\mathbf{x}}\cdot\mathbf{v} - \mathbf{H}\nabla_{\mathbf{x}}\cdot\mathbf{h}\rangle, \quad \langle\mathscr{M}^{h}(\mathbf{v},\mathbf{h})\rangle = 0. \tag{6.16}$$

for any fields $\mathbf{v},\mathbf{h},p$, $\mathbf{L}$-periodic in fast variables. Consequently, a pair of short-scale solenoidal vector fields $\mathbf{v},\mathbf{h}$, whose spatial means do not vanish simultaneously, can be an eigenfunction or a generalised eigenfunction of $\mathscr{L}'^{*}$, only if it is associated with a zero eigenvalue. Using the projection expressed as an integral of the resolvent of $\mathscr{L}'$ over a closed contour, we find as in Chap. 3, that for any pair of constant three-dimensional vectors $\langle\mathbf{v}\rangle,\langle\mathbf{h}\rangle$ there exists a pair of solenoidal vector fields $\mathbf{v},\mathbf{h}\in\ker\mathscr{L}'$, such that their spatial means coincide with the constant fields $\langle\mathbf{v}\rangle,\langle\mathbf{h}\rangle$.

In what follows, we assume that the kernel of $\mathscr{L}'$ is six-dimensional, and hence the kernel of $\mathscr{L}'$ is trivial. This assumption is not satisfied, if the spatial invariance of the MHD system is not broken by the source terms $\mathbf{F}(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$; we will discuss this in detail in Chap. 9. Otherwise, if one of the molecular diffusivities ν and η is fixed, the assumption is not satisfied only for a countable set of values of the second diffusivity.

Consider the system of equations

$$\mathscr{M}^{v}(\mathbf{v},\mathbf{h},p) = \mathbf{f}^{v}, \quad \mathscr{M}^{h}(\mathbf{v},\mathbf{h}) = \mathbf{f}^{h}, \tag{6.17}$$

$$\nabla_{\mathbf{x}}\cdot\mathbf{v} = 0 \tag{6.18}$$

under the additional conditions

$$\nabla_{\mathbf{x}}\cdot\mathbf{f}^{v} = 0, \quad \nabla_{\mathbf{x}}\cdot\mathbf{f}^{h} = 0 \tag{6.19}$$

From the divergence of the second equation in the system (6.17) for the magnetic component $\mathbf{h}$ together with the solenoidality condition (6.19), we find by the usual arguments that $\mathbf{h}$ is solenoidal. Hence the identities (6.16) for the image of the mean part of the operator of linearisation and the solenoidality (6.18) imply that the conditions

$$\langle\mathbf{f}^{v}\rangle = \langle\mathbf{f}^{h}\rangle = 0 \tag{6.20}$$

are necessary for solvability of the system (6.17)–(6.19). Let us apply the operators $\nu^{-1}\nabla_{\mathbf{x}}^{-2}$ and $\eta^{-1}\nabla_{\mathbf{x}}^{-2}$ to the first and second equations in (6.17), respectively (this is permitted as long as (6.17) holds). Applying the Fredholm alternative theorem [154, 171] to the resultant problem and using our assumption that the kernel of $\mathscr{L}'$ is six-dimensional and hence the kernel of $\mathscr{L}$ is trivial, we conclude that the system (6.17)–(6.19) under consideration is solvable if and only if the conditions (6.20) are satisfied. Subtracting from the solution an appropriate vector field from the kernel of $\mathscr{L}'$, we find that the system (6.17)–(6.19) has a unique zero-mean solution $\mathbf{v}, \mathbf{h}$ as long as (6.20) is satisfied.

Let us now consider the system (6.17) under the condition

$$\nabla_{\mathbf{x}} \cdot \mathbf{v} = w(\mathbf{x}), \quad \langle w \rangle = 0 \tag{6.21}$$

in place of (6.20). Averaging the second equation in (6.17) over the fast spatial variables we find that the relation

$$\langle \mathbf{f}^h \rangle = 0 \tag{6.22}$$

remains a necessary condition for solvability of the system (6.17), (6.19). Furthermore, the second equation in (6.17) implies

$$\nabla_{\mathbf{x}} \cdot \mathbf{h} = \eta^{-1}\nabla_{\mathbf{x}}^{-2}(\nabla_{\mathbf{x}} \cdot \mathbf{f}^h).$$

Hence we find from (6.16) that the second necessary solvability condition for the system (6.17), (6.21) is

$$\langle \mathbf{V}w - \eta^{-1}\mathbf{H}\nabla_{\mathbf{x}}^{-2}(\nabla_{\mathbf{x}} \cdot \mathbf{f}^h)\rangle = \langle \mathbf{f}^v \rangle. \tag{6.23}$$

(If (6.23) is not satisfied, then there exists a solution, in which $\nabla_{\mathbf{x}}p$ is $\mathbf{L}$-periodic as required, but the function p is not space-periodic). If the solvability conditions (6.22) and (6.23) are satisfied, we split the r.h.s. of the first equation in (6.17) into a sum

$$\mathbf{f}^v = \langle \mathbf{f}^v \rangle + \mathscr{P}_{\mathrm{sol}}\mathbf{f}^v + \nabla_{\mathbf{x}}\phi, \quad \text{where} \quad \langle \mathscr{P}_{\mathrm{sol}}\mathbf{f}^v \rangle = \langle \nabla_{\mathbf{x}}\phi \rangle = 0.$$

and substitute

$$\mathbf{v} = \mathbf{v}' + \nabla_{\mathbf{x}}\nabla_{\mathbf{x}}^{-2}w, \quad \mathbf{h} = \mathbf{h}' + \eta^{-1}\nabla_{\mathbf{x}}\nabla_{\mathbf{x}}^{-2}(\nabla_{\mathbf{x}} \cdot \mathbf{f}^h), \quad p = p' - \phi.$$

This transforms equations (6.17), (6.21) into a system of equations in $\mathbf{v}'$, $\mathbf{h}'$ and p' of the form (6.17)–(6.20), for which existence of a unique zero-mean solution is established above.

6.3 The Combined MHD α-Effect

Each system (6.14)–(6.15) in the hierarchy will be solved under the condition (6.10)–(6.13) in two steps:

1°. Satisfying the solvability conditions (6.22) and (6.23), we find $\langle \mathbf{v}_{n-2} \rangle$, $\langle \mathbf{h}_{n-2} \rangle$ and λ_n.
2°. The resultant system is solved for $\mathbf{v}_n$ and $\mathbf{h}_n$.
(The presence or absence of the α-effect will be seen not to affect this plan.)

6.3.1 Solution of Order ε^0 Equations

Step 1° for $n = 0$. The Eqs. 6.14–6.15 for $n = 0$ reduce to

$$\mathscr{M}^v(\{\mathbf{v}_0\}, \{\mathbf{h}_0\}, \{p_0\}) + \langle \mathbf{v}_0 \rangle \times (\nabla_\mathbf{x} \times \mathbf{V}) + (\nabla_\mathbf{x} \times \mathbf{H}) \times \langle \mathbf{h}_0 \rangle = \lambda_0 \mathbf{v}_0, \tag{6.24}$$

$$\mathscr{M}^h(\{\mathbf{v}_0\}, \{\mathbf{h}_0\}) + (\langle \mathbf{h}_0 \rangle \cdot \nabla_\mathbf{x})\mathbf{V} - (\langle \mathbf{v}_0 \rangle \cdot \nabla_\mathbf{x})\mathbf{H} = \lambda_0 \mathbf{h}_0. \tag{6.25}$$

We average these equations over the fast spatial variables, using the expressions (6.16) for the mean part of the image of the operator of linearisation, as well as the solenoidality of $\{\mathbf{v}_0\}$ and $\{\mathbf{v}_0\}$ in the fast variables, stemming from (6.12)–(6.13) for $n = 0$. We obtain $0 = \lambda_0 \langle \mathbf{v}_0 \rangle = \lambda_0 \langle \mathbf{h}_0 \rangle$, and consequently assume $\lambda_0 = 0$. (The alternative possibility $\langle \mathbf{v}_0 \rangle = \langle \mathbf{h}_0 \rangle = 0$ is potentially interesting if λ_0 is an imaginary eigenvalue. Since oscillatory short-scale stability modes are non-generic, we do not consider further this possibility. The reader can construct the amplitude equation arising in this case following the approach of Sect. 3.8).

Step 2° for $n = 0$. By linearity, the fluctuating parts of equations (6.24)–(6.25) have solutions of the following structure:

$$\begin{pmatrix} \{\mathbf{v}_0\} \\ \{\mathbf{h}_0\} \\ \{p_0\} \end{pmatrix} = \sum_{k=1}^{3} \left(\begin{pmatrix} \mathbf{S}_k^{vv} \\ \mathbf{S}_k^{vh} \\ S_k^{vp} \end{pmatrix} \langle \mathbf{v}_0 \rangle_k + \begin{pmatrix} \mathbf{S}_k^{hv} \\ \mathbf{S}_k^{hh} \\ S_k^{hp} \end{pmatrix} \langle \mathbf{h}_0 \rangle_k \right), \tag{6.26}$$

Here $\mathbf{L}$-periodic zero-mean vector fields $\mathbf{S}(\mathbf{x}, t)$ are solutions to auxiliary problems of type I:

Auxiliary Problem I.1

$$\mathscr{M}^v(\mathbf{S}_k^{vv}, \mathbf{S}_k^{vh}, S_k^{vp}) = -\mathbf{e}_k \times (\nabla_\mathbf{x} \times \mathbf{V}), \tag{6.27}$$

$$\nabla_\mathbf{x} \cdot \mathbf{S}_k^{vv} = 0, \tag{6.28}$$

$$\mathscr{M}^h(\mathbf{S}_k^{vv}, \mathbf{S}_k^{vh}) = \frac{\partial \mathbf{H}}{\partial x_k}. \tag{6.29}$$

Auxiliary problem I.2

$$\mathscr{M}^v(\mathbf{S}_k^{hv}, \mathbf{S}_k^{hh}, S_k^{hp}) = \mathbf{e}_k \times (\nabla_\mathbf{x} \times \mathbf{H}), \tag{6.30}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{S}_k^{hv} = 0, \tag{6.31}$$

$$\mathscr{M}^h(\mathbf{S}_k^{hv}, \mathbf{S}_k^{hh}) = -\frac{\partial \mathbf{V}}{\partial x_k}. \tag{6.32}$$

The spatial means of the r.h.s. of equations (6.27)–(6.32) vanish due to periodicity of $\mathbf{V}$ and $\mathbf{H}$. This ensures, by the criterion (6.20) for solvability of the problem (6.17)–(6.19), that the problems (6.27)–(6.29) and (6.30)–(6.32) have unique zero-mean solutions. Conditions (6.28) and (6.31) guarantee solenoidality of $\{\mathbf{v}_0\}$ in the fast variables. Solenoidality of $\{\mathbf{h}_0\}$ stems from the relations

$$\nabla_{\mathbf{x}} \cdot \mathbf{S}_k^{vh} = \nabla_{\mathbf{x}} \cdot \mathbf{S}_k^{hh} = 0; \tag{6.33}$$

as shown in Sect. 6.2, they are consequences of solenoidality of the fields $\mathbf{V}, \mathbf{H}$.

The problems (6.27)–(6.29) and (6.30)–(6.32) can be expressed as

$$\mathscr{L}'(\mathbf{S}_k^{vv} + \mathbf{e}_k, \mathbf{S}_k^{vh}) = 0 \quad \text{and} \quad \mathscr{L}'(\mathbf{S}_k^{hv}, \mathbf{S}_k^{hh} + \mathbf{e}_k) = 0,$$

respectively. Their solvability is equivalent to existence of a pair of vector fields in the kernel of the operator of linearisation $\mathscr{L}'$, whose spatial means coincide with two arbitrary constant vectors (the fields solenoidal by the requirements for the domain of $\mathscr{L}'$)—see Sect. 6.2.

6.3.2 *The Solvability Condition for Order ε^1 Equations: The Operator of the Combined MHD α-Effect*

Step 1° for $n = 1$. Applying identities (6.16) for the mean part of the operator of linearisation, relations (6.12)–(6.13) for the divergence of the vector fields for $n = 1$, and expressions (6.26) for the fluctuating parts of the zero-order terms of the perturbation mode, we reduce the mean parts of the equations (6.14)–(6.15) for $n = 1$ to the MHD mean-field equations

$$\sum_{k=1}^{3} \left(\mathbf{A}_k^{vv} \nabla_{\mathbf{X}} \langle \mathbf{v}_0 \rangle_k + \mathbf{A}_k^{hv} \nabla_{\mathbf{X}} \langle \mathbf{h}_0 \rangle_k \right) - \nabla_{\mathbf{X}} p_0^* = \lambda_1 \langle \mathbf{v}_0 \rangle, \tag{6.34}$$

$$\nabla_{\mathbf{X}} \times \sum_{k=1}^{3} \left(\mathbf{A}_k^{vh} \langle \mathbf{v}_0 \rangle_k + \mathbf{A}_k^{hh} \langle \mathbf{h}_0 \rangle_k \right) = \lambda_1 \langle \mathbf{h}_0 \rangle. \tag{6.35}$$

We have denoted here

$$p_n^* \equiv \langle p_n \rangle - \left\langle \mathbf{V} \cdot \sum_{k=1}^{3} \left(\mathbf{S}_k^{vv} \langle \mathbf{v}_n \rangle_k + \mathbf{S}_k^{hv} \langle \mathbf{h}_n \rangle_k \right) + \mathbf{H} \cdot \sum_{k=1}^{3} \left(\mathbf{S}_k^{vh} \langle \mathbf{v}_n \rangle_k + \mathbf{S}_k^{hh} \langle \mathbf{h}_n \rangle_k \right) \right\rangle;$$

$\mathbf{A}_k^{\cdot v}$ are 3×3 matrices with the entries

$$(A_k^{vv})_j^m \equiv \langle -V_m((S_k^{vv})_j + \delta_k^j) - V_j((S_k^{vv})_m + \delta_k^m) + H_m(S_k^{vh})_j + H_j(S_k^{vh})_m \rangle, \quad (6.36)$$

$$(A_k^{hv})_j^m \equiv \langle -V_m(S_k^{hv})_j - V_j(S_k^{hv})_m + H_m((S_k^{hh})_j + \delta_k^j) + H_j((S_k^{hh})_m + \delta_k^m) \rangle \quad (6.37)$$

(δ_k^j is the Kronecker symbol), and $\mathbf{A}_k^{\cdot h}$ are three-dimensional vectors:

$$\mathbf{A}_k^{vh} \equiv \langle \mathbf{V} \times \mathbf{S}_k^{vh} - \mathbf{H} \times (\mathbf{S}_k^{vv} + \mathbf{e}_k) \rangle, \quad (6.38)$$

$$\mathbf{A}_k^{hh} \equiv \langle \mathbf{V} \times (\mathbf{S}_k^{hh} + \mathbf{e}_k) - \mathbf{H} \times \mathbf{S}_k^{hv} \rangle. \quad (6.39)$$

The first-order partial differential operator in the l.h.s. of (6.34)–(6.35), acting on the vector field $(\langle \mathbf{v}_0 \rangle, \langle \mathbf{h}_0 \rangle)$, is called the *operator of the combined MHD α-effect*. We denote it by $\mathscr{A}$. Solution of the subsequent ($n > 0$) systems of equations (6.14)–(6.15) is significantly different in the presence ($\mathscr{A} \neq 0$) and the absence ($\mathscr{A} = 0$) of the α-effect. We consider the case where the combined MHD α-effect is present in the next section, and the case of emergence of the combined MHD eddy diffusion in the absence of the α-effect in Sects. 6.5 and 6.7.

6.4 Complete Expansion of Large-Scale MHD Modes in the Presence of α-Effect

Step 1° for $n = 1$ (*continued in the presence of the α-effect*). As in the dynamo problems considered in the previous chapters, one can explore the evolution of mean-field perturbations in a bounded region in the space of the slow variables with suitable conditions at the boundary. However, in this chapter in both cases (in the presence and absence of the α-effect) we will only consider the linear stability modes defined in the entire space of the slow variables, demanding that the modes are globally bounded.

6.4.1 Solution of Order ε^1 Equations

Globally bounded eigenfunctions of the eigenvalue problem (6.34)–(6.35) are Fourier harmonics:

$$\langle \mathbf{v}_0 \rangle = \hat{\mathbf{v}} \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}, \quad \langle \mathbf{h}_0 \rangle = \hat{\mathbf{h}} \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}},$$

where $\mathbf{q}$ is an arbitrary wave vector, and vectors $\hat{\mathbf{v}}$ and $\hat{\mathbf{h}}$ satisfy the equations

$$-\frac{\mathrm{i}\mathbf{q}}{|\mathbf{q}|^2}\times\left(\mathbf{q}\times\sum_{k=1}^{3}\left(\hat{v}_k\mathbf{A}_k^{vv}+\hat{h}_k\mathbf{A}_k^{hv}\right)\mathbf{q}\right)=\lambda_1\hat{\mathbf{v}}, \tag{6.40}$$

$$\mathrm{i}\mathbf{q}\times\sum_{k=1}^{3}\left(\hat{v}_k\mathbf{A}_k^{vh}+\hat{h}_k\mathbf{A}_k^{hh}\right)=\lambda_1\hat{\mathbf{h}}. \tag{6.41}$$

The solenoidality conditions (6.10)–(6.11) for $n=0$ reduce to

$$\hat{\mathbf{v}}\cdot\mathbf{q}=\hat{\mathbf{h}}\cdot\mathbf{q}=0. \tag{6.42}$$

We denote $\mathbf{A}(\mathbf{q})$ the 6×6 matrix by which the vector $(\hat{\mathbf{v}},\hat{\mathbf{h}})$ in the l.h.s. of (6.40)–(6.41) is multiplied; it is the symbol of the differential operator of the combined MHD α-effect $\mathscr{A}$. It is elementary to verify that its spectrum is symmetric about the imaginary axis: if an eigenvalue λ is associated with a harmonic for a wave vector $\mathbf{q}$, then $-\lambda$ is an eigenvalue associated with a harmonic for the wave vector $-\mathbf{q}$. Thus, an alternative typical for the γ-effect arises: either for any wave vector the temporal behaviour of a perturbation mode consists of harmonic oscillations of a constant amplitude, or there exists a wave vector, for which the perturbations exponentially grow in time.

By solving the eigenvalue equations (6.34)–(6.35), we satisfy the solvability conditions for the problem (6.14)–(6.15) for $n=1$. It is convenient to solve it, applying the same substitution as used for the systems of equations for $n>1$, which we discuss in the next subsection.

6.4.2 *Solution of Order ε^n, $n>1$ Equations*

We denote

$$\mathbb{V}_{\mathbf{q}}=\left\{(\hat{\mathbf{f}}^v,\hat{\mathbf{f}}^h)\;\middle|\;\hat{\mathbf{f}}^v,\hat{\mathbf{f}}^h\in\mathbb{C}^3,\ \hat{\mathbf{f}}^v\cdot\mathbf{q}=\hat{\mathbf{f}}^h\cdot\mathbf{q}=0\right\}\subset\mathbb{C}^6,$$

$$\bar{\mathbb{H}}^s=\{\mathbf{f}(\mathbf{X})\in\mathbb{W}_2^s([0,\mathbf{L}])\mid\nabla\cdot\mathbf{f}=0,\ \langle\mathbf{f}\rangle=0\},$$

where $\mathbb{W}_2^s([0,\mathbf{L}])$ is the Sobolev space of vector-functions, L_i-periodic in the Cartesian variables x_i. In this subsection we assume that the operator of the MHD α-effect has the following ellipticity property: for any $\mathbf{q}\neq 0$ the matrix $\mathbf{A}(\mathbf{q})$ defines a linear operator, which is invertible in $\mathbb{V}_{\mathbf{q}}$. Then the image of the linear operator $\mathscr{A}:\bar{\mathbb{H}}^{s+1}\to\bar{\mathbb{H}}^s$ is closed, $\mathscr{A}$ has a discrete spectrum, and the resolvent $(\zeta\mathscr{I}-\mathscr{A})^{-1}$ is a compact operator for all ζ not belonging to the spectrum of $\mathscr{A}$.

Furthermore, for the sake of simplicity we assume that the eigenvalue λ_1 of the matrix $\mathbf{A}(\mathbf{q})$ has multiplicity one. (Similar constructions can be implemented if λ_1

is an eigenvalue of a higher multiplicity, but they are more technical). This assumption generically holds true, i.e., if for a steady state $\mathbf{V}, \mathbf{H}$ the multiplicity of the eigenvalue exceeds one, then for almost any small perturbation of the fields $\mathbf{F}$ and $\mathbf{J}$, determining the steady state, this eigenvalue splits into the respective number of close eigenvalues of multiplicity one.

To solve the systems of equations (6.14)–(6.15) for $n > 0$, it is convenient to make the substitution

$$\begin{pmatrix} \{\mathbf{v}_n\} \\ \{\mathbf{h}_n\} \\ \{p_n\} \end{pmatrix} = \begin{pmatrix} \{\mathbf{v}'_n\} \\ \{\mathbf{h}'_n\} \\ \{p'_n\} \end{pmatrix} + \sum_{k=1}^{3} \left(\begin{pmatrix} \mathbf{S}_k^{vv} \\ \mathbf{S}_k^{vh} \\ S_k^{vp} \end{pmatrix} \langle \mathbf{v}_n \rangle_k + \begin{pmatrix} \mathbf{S}_k^{hv} \\ \mathbf{S}_k^{hh} \\ S_k^{hp} \end{pmatrix} \langle \mathbf{h}_n \rangle_k \right). \tag{6.43}$$

After the substitution, the system takes the form

$$\begin{aligned} &\mathscr{M}^v(\mathbf{v}'_n, \mathbf{h}'_n, p'_n) + \nu\big(2(\nabla_\mathbf{x} \cdot \nabla_\mathbf{X})\{\mathbf{v}_{n-1}\} + \nabla^2_\mathbf{X}\mathbf{v}_{n-2}\big) \\ &\quad + \mathbf{V} \times (\nabla_\mathbf{X} \times \mathbf{v}_{n-1}) + (\nabla_\mathbf{X} \times \mathbf{h}_{n-1}) \times \mathbf{H} - \nabla_\mathbf{X} p_{n-1} = \sum_{m=0}^{n-1} \lambda_{n-m}\mathbf{v}_m, \end{aligned} \tag{6.44}$$

$$\begin{aligned} &\mathscr{M}^h(\mathbf{v}'_n, \mathbf{h}'_n) + \eta\big(2(\nabla_\mathbf{x} \cdot \nabla_\mathbf{X})\{\mathbf{h}_{n-1}\} + \nabla^2_\mathbf{X}\mathbf{h}_{n-2}\big) \\ &\quad + \nabla_\mathbf{X} \times (\mathbf{V} \times \mathbf{h}_{n-1} + \mathbf{v}_{n-1} \times \mathbf{H}) = \sum_{m=0}^{n-1} \lambda_{n-m}\mathbf{h}_m. \end{aligned} \tag{6.45}$$

We will solve the systems (6.44)–(6.45) successively, noting that solutions have the following structure:

$$\mathbf{v}_n(\mathbf{X}, \mathbf{x}) = \mathbf{w}_n(\mathbf{x})\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}, \quad \mathbf{h}_n(\mathbf{X}, \mathbf{x}) = \mathbf{g}_n(\mathbf{x})\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}, \quad p_n(\mathbf{X}, \mathbf{x}) = q_n(\mathbf{x})\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}. \tag{6.46}$$

Suppose that, for a given N, we have solved all the equations for $n < N$ and determined

- the fields $\mathbf{v}_n, \mathbf{h}_n, p_n$ for all $n < N - 1$,
- the fields $\mathbf{v}'_{N-1}, \mathbf{h}'_{N-1}, p'_{N-1}$,
- λ_n for all $n < N$,

and, in agreement with (6.46), the dependence of the obtained vector fields on the slow variables is via the factor $\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}$. We consider the system (6.44)–(6.45) for $n = N$.

Step 1° for $n = N$. Applying the substitution (6.43) for $n = N - 1$, we transform the equations (6.44)–(6.45) for $n = N$ averaged over the fast spatial variables into the relations

$$\begin{aligned} &\sum_{k=1}^{3} \left(\mathbf{A}_k^{vv}\nabla_\mathbf{X}\langle \mathbf{v}_{N-1}\rangle_k + \mathbf{A}_k^{hv}\nabla_\mathbf{X}\langle \mathbf{h}_{N-1}\rangle_k\right) - \nabla_\mathbf{X} p^*_{N-1} - \lambda_1\langle \mathbf{v}_{N-1}\rangle - \lambda_N\langle \mathbf{v}_0\rangle \\ &\quad = -\langle \mathbf{V} \times (\nabla_\mathbf{X} \times \mathbf{v}'_{N-1}) - \mathbf{V}\nabla_\mathbf{X} \cdot \mathbf{v}'_{N-1} + (\nabla_\mathbf{X} \times \mathbf{h}'_{N-1}) \times \mathbf{H} + \mathbf{H}\nabla_\mathbf{X} \cdot \mathbf{h}'_{N-1}\rangle \\ &\qquad - \nu\nabla^2_\mathbf{X}\langle \mathbf{v}_{N-2}\rangle + \sum_{m=1}^{N-2} \lambda_{N-m}\langle \mathbf{v}_m\rangle, \end{aligned} \tag{6.47}$$

$$\nabla_{\mathbf{X}} \times \sum_{k=1}^{3} \left(\mathbf{A}_k^{vh} \langle \mathbf{v}_{N-1} \rangle_k + \mathbf{A}_k^{hh} \langle \mathbf{h}_{N-1} \rangle_k \right) - \lambda_1 \langle \mathbf{h}_{N-1} \rangle - \lambda_N \langle \mathbf{h}_0 \rangle$$
$$= -\eta \nabla_{\mathbf{X}}^2 \langle \mathbf{h}_{N-2} \rangle - \nabla_{\mathbf{X}} \times \langle \mathbf{V} \times \mathbf{h}'_{N-1} + \mathbf{v}'_{N-1} \times \mathbf{H} \rangle + \sum_{m=1}^{N-2} \lambda_{N-m} \langle \mathbf{h}_m \rangle. \quad (6.48)$$

By our induction assumption, the r.h.s. of (6.47)–(6.48) are of the form $\hat{\mathbf{f}}_N^v e^{i\mathbf{q}\cdot\mathbf{X}}$ and $\hat{\mathbf{f}}_N^h e^{i\mathbf{q}\cdot\mathbf{X}}$, where constant vectors $\hat{\mathbf{f}}_N^v$ and $\hat{\mathbf{f}}_N^h$ are known, and $\hat{\mathbf{f}}_N^h \cdot \mathbf{q} = 0$. This implies that the unknown mean fields are Fourier harmonics in the slow variables for the wave vector $\mathbf{q}$ (cf. (6.46) for $n = N - 1$), and we find from (6.47)–(6.48)

$$(\mathbf{A}(\mathbf{q}) - \lambda_1) \begin{pmatrix} \langle \mathbf{w}_{N-1} \rangle \\ \langle \mathbf{g}_{N-1} \rangle \end{pmatrix} - \lambda_N \begin{pmatrix} \hat{\mathbf{v}} \\ \hat{\mathbf{h}} \end{pmatrix} = \begin{pmatrix} -|\mathbf{q}|^{-2} \mathbf{q} \times (\mathbf{q} \times \hat{\mathbf{f}}_N^v) \\ \hat{\mathbf{f}}_N^h \end{pmatrix}. \quad (6.49)$$

This problem is considered in $\mathbb{V}_{\mathbf{q}}$. We find λ_N from the invariant projection of (6.49) parallel to $(\hat{\mathbf{v}}, \hat{\mathbf{h}}) \in \mathbb{V}_{\mathbf{q}}$. By the assumption that λ_1 is an eigenvalue of multiplicity one, the matrix $\mathbf{A}(\mathbf{q}) - \lambda_1$ is invertible in the invariant subspace, complementary to $(\hat{\mathbf{v}}, \hat{\mathbf{h}})$ in $\mathbb{V}_{\mathbf{q}}$. We assume that the component of $(\langle \mathbf{w}_{N-1} \rangle, \langle \mathbf{g}_{N-1} \rangle) \in \mathbb{V}_{\mathbf{q}}$, parallel to $(\hat{\mathbf{v}}, \hat{\mathbf{h}})$ vanishes (this is the normalisation condition, stemming from the possibility of multiplication of the eigenfunction by an arbitrary series in ε). Hence, (6.49) yields the value of $(\langle \mathbf{w}_{N-1} \rangle, \langle \mathbf{g}_{N-1} \rangle)$. Finally, we find p'_{N-1} considering the divergence of (6.47), and vector fields $\{\mathbf{v}_{N-1}\}$, $\{\mathbf{h}_{N-1}\}$ and $\{p_{N-1}\}$ are determined by the substitution (6.43) for $n = N - 1$.

Step 2° for $n = N$. Relations (6.47)–(6.48) ensure the solvability of the system of equations (6.44)–(6.45) for $n = N$, considered under the conditions

$$\nabla_{\mathbf{x}} \cdot \mathbf{v}'_N = -\nabla_{\mathbf{X}} \cdot \{\mathbf{v}_{N-1}\}, \quad \nabla_{\mathbf{x}} \cdot \mathbf{h}'_N = -\nabla_{\mathbf{X}} \cdot \{\mathbf{h}_{N-1}\}$$

(stemming from the identities (6.12)–(6.13), for $n = N$, for the divergence of the terms of the series (6.7), and from solenoidality of solutions to the auxiliary problems of type I, see (6.28), (6.31) and (6.33)). These conditions are used to extract the potential part of the unknown vector fields. The fluctuating part of equations (6.44)–(6.45) for $n = N$ can be solved in $\mathbf{v}'_n, \mathbf{h}'_n$ and p'_n following the procedure for solution of the problem (6.17), (6.21) discussed in Sect. 6.2. Evidently, the solution complies with (6.46), since the dependence of the r.h.s. in (6.17), on the slow variables consists of the proportionality to $e^{i\mathbf{q}\cdot\mathbf{X}}$.

Thus, we have constructed a complete formal asymptotic expansion of large-scale modes of perturbations and the associated eigenvalues, when the operator of the α-effect, $\mathscr{A}$, has the ellipticity property. It can be proved that under this assumption every eigenvalue of $\mathscr{A}$ gives rise to a branch of eigenvalues of the original operator of linearisation $\mathscr{L}$ (see the eigenvalue problem (6.4)–(6.6)), and the asymptotic series (6.7)–(6.9) converge for sufficiently small ε. Such a proof relies on the perturbation theory for linear operators [152] and follows the proof [307] for kinematic magnetic dynamos.

6.5 MHD Eddy Diffusion in Steady Parity-Invariant MHD States

We consider in this section an MHD steady state, in which the fields $\mathbf{V}, \mathbf{H}$ are parity-invariant; without any loss of generality we assume that the centre of symmetry coincides with the origin of the coordinate system:

$$\mathbf{V}(\mathbf{x}) = -\mathbf{V}(-\mathbf{x}), \quad \mathbf{H}(\mathbf{x}) = -\mathbf{H}(-\mathbf{x}). \tag{6.50}$$

6.5.1 Solution of Order ε^1 Equations in the Absence of α-Effect

Step 1° for $n = 1$ (*continued in the absence of the* α*-effect*). If the MHD steady state is parity-invariant, the domain of the operator of linearisation, $\mathscr{M}$, splits into two invariant subspaces: parity-invariant ($\mathbf{f}(\mathbf{x}) = -\mathbf{f}(-\mathbf{x})$) and parity-antiinvariant ($\mathbf{f}(\mathbf{x}) = \mathbf{f}(-\mathbf{x})$) fields. Consequently, parity-antiinvariance of the r.h.s. in the auxiliary problems of type I implies that vector, $\mathbf{S}_k^{vv}, \mathbf{S}_k^{vh}, \mathbf{S}_k^{hv}, \mathbf{S}_k^{hh}$, and scalar, S_k^{vp}, S_k^{hp}, fields are parity-antiinvariant. (Parity invariance of scalar fields is defined, as usual, by the relation $f(\mathbf{x}) = f(-\mathbf{x})$, and parity antiinvariance by $f(\mathbf{x}) = -f(-\mathbf{x})$). Thus, we find from expressions (6.36)–(6.39) for the tensor of the combined α-effect, that $\mathscr{A} = 0$ and $\mathbf{A}(\mathbf{q})$ is a zero matrix. The eigenvalue equations (6.34)–(6.35) averaged over the fast spatial variables become $-\nabla_{\mathbf{X}}\langle p_0\rangle = \lambda_1\langle \mathbf{v}_0\rangle$ and $0 = \lambda_1\langle \mathbf{h}_0\rangle$, whereby we assume $\lambda_1 = 0$ and $\langle p_0\rangle = 0$.

Step 2° for $n = 1$. By virtue of expressions (6.26) for solutions of the order ε^0 equations and by linearity, the fluctuating parts of the equations (6.44)–(6.45), obtained from (6.14)–(6.15) for $n = 1$ by the substitution (6.43), have solutions of the form

$$\begin{pmatrix} \mathbf{v}'_1 \\ \mathbf{h}'_1 \\ p'_1 \end{pmatrix} = \sum_{k=1}^{3}\sum_{m=1}^{3}\left(\begin{pmatrix} \mathbf{G}_{mk}^{vv} \\ \mathbf{G}_{mk}^{vh} \\ G_{mk}^{vp} \end{pmatrix}\frac{\partial\langle \mathbf{v}_0\rangle_k}{\partial X_m} + \begin{pmatrix} \mathbf{G}_{mk}^{hv} \\ \mathbf{G}_{mk}^{hh} \\ G_{mk}^{hp} \end{pmatrix}\frac{\partial\langle \mathbf{h}_0\rangle_k}{\partial X_m}\right), \tag{6.51}$$

where vector fields $\mathbf{G}(\mathbf{x}, t)$ are $\mathbf{L}$-periodic zero-mean solutions to auxiliary problems of type II:

Auxiliary problem II.1

$$\mathscr{M}^v(\mathbf{G}_{mk}^{vv}, \mathbf{G}_{mk}^{vh}, G_{mk}^{vp}) = -2\nu\frac{\partial \mathbf{S}_k^{vv}}{\partial x_m} - V_k\mathbf{e}_m + V_m\mathbf{e}_k - (\mathbf{V}\cdot\mathbf{S}_k^{vv})\mathbf{e}_m + V_m\mathbf{S}_k^{vv} + (\mathbf{H}\cdot\mathbf{S}_k^{vh})\mathbf{e}_m - H_m\mathbf{S}_k^{vh} + \mathbf{e}_m S_k^{vp}, \tag{6.52}$$

$$\nabla_{\mathbf{x}}\cdot\mathbf{G}_{mk}^{vv} = -(S_k^{vv})_m, \tag{6.53}$$

$$\mathscr{M}^h(\mathbf{G}^{vv}_{mk}, \mathbf{G}^{vh}_{mk}) = -2\eta\frac{\partial \mathbf{S}^{vh}_k}{\partial x_m} - \mathbf{V}(S^{vh}_k)_m + V_m\mathbf{S}^{vh}_k + \mathbf{H}(S^{vv}_k)_m - H_m(\mathbf{S}^{vv}_k + \mathbf{e}_k); \tag{6.54}$$

Auxiliary problem II.2

$$\mathscr{M}^v(\mathbf{G}^{hv}_{mk}, \mathbf{G}^{hh}_{mk}, G^{hp}_{mk}) = -2\nu\frac{\partial \mathbf{S}^{hv}_k}{\partial x_m} + H_k\mathbf{e}_m - H_m\mathbf{e}_k - (\mathbf{V}\cdot\mathbf{S}^{hv}_k)\mathbf{e}_m + V_m\mathbf{S}^{hv}_k + (\mathbf{H}\cdot\mathbf{S}^{hh}_k)\mathbf{e}_m - H_m\mathbf{S}^{hh}_k + \mathbf{e}_m S^{hp}_k, \tag{6.55}$$

$$\nabla_{\mathbf{x}}\cdot\mathbf{G}^{hv}_{mk} = -(S^{hv}_k)_m, \tag{6.56}$$

$$\mathscr{M}^h(\mathbf{G}^{hv}_{mk}, \mathbf{G}^{hh}_{mk}) = -2\eta\frac{\partial \mathbf{S}^{hh}_k}{\partial x_m} - \mathbf{V}(S^{hh}_k)_m + V_m(\mathbf{S}^{hh}_k + \mathbf{e}_k) + \mathbf{H}(S^{hv}_k)_m - H_m\mathbf{S}^{hv}_k. \tag{6.57}$$

Relations (6.53) and (6.56) ensure that (6.12) is satisfied for $n = 1$. Taking the divergence of equations (6.54) and (6.57), and subtracting from the results the mth components of equations (6.29) and (6.32), respectively, we find by the usual arguments that

$$\nabla_{\mathbf{x}}\cdot\mathbf{G}^{vh}_{mk} = -(S^{vh}_k)_m, \quad \nabla_{\mathbf{x}}\cdot\mathbf{G}^{hh}_{mk} = -(S^{hh}_k)_m.$$

This guarantees that relation (6.13) for the divergence of the terms of expansions (6.7) holds true for $n = 1$. Due to parity-invariance of the MHD steady state $\mathbf{V}, \mathbf{H}$ (6.50) and parity-antiinvariance of solutions to the auxiliary problems of type I, the r.h.s. of (6.52), (6.54), (6.55) and (6.57) are parity-invariant. Hence, vector, $\mathbf{G}^{vv}_{mk}, \mathbf{G}^{vh}_{mk}, \mathbf{G}^{hv}_{mk}, \mathbf{G}^{hh}_{mk}$, and scalar, G^{vp}_{mk}, G^{hp}_{mk}, fields constituting solutions of auxiliary problems of type II are parity-invariant.

6.5.2 *The Solvability Condition for Order ε^2 Equations: The Operator of the Combined MHD Eddy Diffusion*

Step 1° for $n = 2$. By virtue of expressions (6.16) for the mean part of the operator of linearisation, the relations (6.12)–(6.13) for $n = 1$ specifying the divergence of the terms of the series (6.7), and the substitution (6.51), the mean parts of the Eqs. 6.44–6.45 for $n = 2$ can be rendered as

$$\nu\nabla^2_{\mathbf{X}}\langle\mathbf{v}_0\rangle + \sum_{k=1}^{3}\sum_{m=1}^{3}\sum_{j=1}^{3}\left(\mathbf{D}^{vv}_{jmk}\frac{\partial^2\langle\mathbf{v}_0\rangle_k}{\partial X_m\partial X_j} + \mathbf{D}^{hv}_{jmk}\frac{\partial^2\langle\mathbf{h}_0\rangle_k}{\partial X_m\partial X_j}\right) - \nabla_{\mathbf{X}}p^{\diamond}_1 = \lambda_2\langle\mathbf{v}_0\rangle, \tag{6.58}$$

$$\eta\nabla^2_{\mathbf{X}}\langle\mathbf{h}_0\rangle+\nabla_{\mathbf{X}}\times\sum_{k=1}^{3}\sum_{m=1}^{3}\left(\mathbf{D}^{vh}_{mk}\frac{\partial\langle\mathbf{v}_0\rangle_k}{\partial X_m}+\mathbf{D}^{hh}_{mk}\frac{\partial\langle\mathbf{h}_0\rangle_k}{\partial X_m}\right)=\lambda_2\langle\mathbf{h}_0\rangle. \tag{6.59}$$

In these MHD mean-field equations, we have denoted

$$\begin{aligned}p^{\diamond}_n=\langle p_n\rangle-\sum_{k=1}^{3}\sum_{m=1}^{3}\Bigg(&\langle\mathbf{V}\cdot\mathbf{G}^{vv}_{mk}-\mathbf{H}\cdot\mathbf{G}^{vh}_{mk}\rangle\frac{\partial\langle\mathbf{v}_{N-1}\rangle_k}{\partial X_m}\\&+\langle\mathbf{V}\cdot\mathbf{G}^{hv}_{mk}-\mathbf{H}\cdot\mathbf{G}^{hh}_{mk}\rangle\frac{\partial\langle\mathbf{h}_{N-1}\rangle_k}{\partial X_m}\Bigg),\end{aligned} \tag{6.60}$$

$$\mathbf{D}^{vv}_{jmk}=\langle -V_j\mathbf{G}^{vv}_{mk}-\mathbf{V}(G^{vv}_{mk})_j+H_j\mathbf{G}^{vh}_{mk}+\mathbf{H}(G^{vh}_{mk})_j\rangle, \tag{6.61}$$

$$\mathbf{D}^{hv}_{jmk}=\langle -V_j\mathbf{G}^{hv}_{mk}-\mathbf{V}(G^{hv}_{mk})_j+H_j\mathbf{G}^{hh}_{mk}+\mathbf{H}(G^{hh}_{mk})_j\rangle, \tag{6.62}$$

$$\mathbf{D}^{vh}_{mk}=\langle\mathbf{V}\times\mathbf{G}^{vh}_{mk}-\mathbf{H}\times\mathbf{G}^{vv}_{mk}\rangle, \tag{6.63}$$

$$\mathbf{D}^{hh}_{mk}=\langle\mathbf{V}\times\mathbf{G}^{hh}_{mk}-\mathbf{H}\times\mathbf{G}^{hv}_{mk}\rangle. \tag{6.64}$$

The second-order partial differential operator defined by the l.h.s. of (6.58)–(6.59) is called the *operator of the combined hydrodynamic and magnetic eddy diffusion*; in general, it is anisotropic.

Eigenfunctions of the problem (6.58)–(6.59), that are globally bounded in the entire space, are Fourier harmonics: $\langle\mathbf{v}_0\rangle=\hat{\mathbf{v}}e^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}$ and $\langle\mathbf{h}_0\rangle=\hat{\mathbf{h}}e^{\mathrm{i}\mathbf{q}\cdot\mathbf{X}}$. Here $\mathbf{q}$ is a constant wave vector, and three-dimensional vectors $\hat{\mathbf{v}}$ and $\hat{\mathbf{h}}$ satisfy the orthogonality condition (6.42) and equations

$$-\nu|\mathbf{q}|^2\hat{\mathbf{v}}+\mathbf{q}\times\left(\mathbf{q}\times\sum_{k=1}^{3}\sum_{m=1}^{3}\sum_{j=1}^{3}(\mathbf{D}^{vv}_{jmk}\hat{v}_k+\mathbf{D}^{hv}_{jmk}\hat{h}_k)\frac{q_mq_j}{|\mathbf{q}|^2}\right)=\lambda_2\hat{\mathbf{v}}, \tag{6.65}$$

$$-\eta|\mathbf{q}|^2\hat{\mathbf{h}}-\mathbf{q}\times\sum_{k=1}^{3}\sum_{m=1}^{3}q_m(\mathbf{D}^{vh}_{mk}\hat{v}_k+\mathbf{D}^{hh}_{mk}\hat{h}_k)=\lambda_2\hat{\mathbf{h}}. \tag{6.66}$$

6.6 Complete Expansion of Large-Scale MHD Stability Modes in the Absence of the α-Effect

For the sake of simplicity, we assume in this section that the eigenvalue λ_2, that we found solving the problem (6.65)–(6.66), (6.42), has multiplicity one (generically this holds true).

Step 1° for $n = 2$. To solve the Eqs. 6.44–6.45 for $n > 1$, we employ new substitutions

$$\begin{pmatrix} \mathbf{v}'_n \\ \mathbf{h}'_n \\ p'_n \end{pmatrix} = \begin{pmatrix} \mathbf{v}''_n \\ \mathbf{h}''_n \\ p''_n \end{pmatrix} + \sum_{k=1}^{3}\sum_{m=1}^{3}\left(\begin{pmatrix} \mathbf{G}^{vv}_{mk} \\ \mathbf{G}^{vh}_{mk} \\ G^{vp}_{mk} \end{pmatrix} \frac{\partial\langle \mathbf{v}_{n-1}\rangle_k}{\partial X_m} + \begin{pmatrix} \mathbf{G}^{hv}_{mk} \\ \mathbf{G}^{hh}_{mk} \\ G^{hp}_{mk} \end{pmatrix} \frac{\partial\langle \mathbf{h}_{n-1}\rangle_k}{\partial X_m} \right). \quad (6.67)$$

Substituting the flow components of (6.43) (where the index is changed: $n \to n-1$) and of (6.67) into the identity (6.12) for divergence of the terms of the series (6.7), and employing relations (6.53) and (6.56) for the divergence of solutions to auxiliary problems of type II, we obtain

$$\nabla_{\mathbf{x}} \cdot \mathbf{v}''_n + \nabla_{\mathbf{X}} \cdot \mathbf{v}'_{n-1} = 0; \quad (6.68)$$

similarly, we find

$$\nabla_{\mathbf{x}} \cdot \mathbf{h}''_n + \nabla_{\mathbf{X}} \cdot \mathbf{h}'_{n-1} = 0. \quad (6.69)$$

In the new variables, Eqs. 6.44–6.45 reduce to

$$\mathscr{M}^v(\mathbf{v}''_n, \mathbf{h}''_n, p''_n) + \nu\big(2(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\mathbf{v}'_{n-1} + \nabla^2_{\mathbf{X}}\mathbf{v}_{n-2}\big) + \mathbf{V} \times (\nabla_{\mathbf{X}} \times \mathbf{v}'_{n-1})$$
$$+ (\nabla_{\mathbf{X}} \times \mathbf{h}'_{n-1}) \times \mathbf{H} - \nabla_{\mathbf{X}}(\langle p_{n-1}\rangle + p'_{n-1}) = \sum_{m=0}^{n-2} \lambda_{n-m}\mathbf{v}_m, \quad (6.70)$$

$$\mathscr{M}^h(\mathbf{v}''_n, \mathbf{h}''_n) + \eta(2(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\mathbf{h}'_{n-1} + \nabla^2_{\mathbf{X}}\mathbf{h}_{n-2})$$
$$+ \nabla_{\mathbf{X}} \times (\mathbf{V} \times \mathbf{h}'_{n-1} + \mathbf{v}'_{n-1} \times \mathbf{H}) = \sum_{m=0}^{n-2} \lambda_{n-m}\mathbf{h}_m. \quad (6.71)$$

Our goal is to solve successively this hierarchy of equations and find all terms in the asymptotic expansions (6.7)–(6.9).

Suppose that for some N all equations for $n < N$ are solved yielding the following information:

- the unknown fields $\mathbf{v}_n, \mathbf{h}_n, p_n$ for all $n < N-2$,
- $\langle p_{N-2}\rangle$,
- the fields $\mathbf{v}''_n, \mathbf{h}''_n, p''_n$ for all $n < N$,
- λ_n for all $n < N$,

and the dependence of all unknown fields on the slow variables consists of their proportionality to $e^{i\mathbf{q}\cdot\mathbf{X}}$ in agreement with (6.46).

Step 1° for $n = N$. By virtue of expressions (6.16) for the mean part of the operator of linearisation, identities (6.68)–(6.69) for the divergence of the unknown fields, and the relations (6.67) for $n = N-1$, the mean parts of the Eqs. 6.70–6.71 for $n = N$ reduce to

$$\nu\nabla^2_{\mathbf{X}}\langle\mathbf{v}_{N-2}\rangle + \sum_{k=1}^{3}\sum_{m=1}^{3}\sum_{j=1}^{3}\left(\mathbf{D}^{vv}_{jmk}\frac{\partial^2\langle\mathbf{v}_{N-2}\rangle_k}{\partial X_m\partial X_j} + \mathbf{D}^{hv}_{jmk}\frac{\partial^2\langle\mathbf{h}_{N-2}\rangle_k}{\partial X_m\partial X_j}\right)$$
$$- \nabla_{\mathbf{X}} p^{\diamond}_{N-1} - \lambda_2\langle\mathbf{v}_{N-2}\rangle - \lambda_N\langle\mathbf{v}_0\rangle = \langle -\mathbf{V}\times(\nabla_{\mathbf{X}}\times\mathbf{v}''_{N-1}) + \mathbf{V}\nabla_{\mathbf{X}}\cdot\mathbf{v}''_{N-1}$$
$$- (\nabla_{\mathbf{X}}\times\mathbf{h}''_{N-1})\times\mathbf{H} - \mathbf{H}\nabla_{\mathbf{X}}\cdot\mathbf{h}''_{N-1}\rangle + \sum_{m=1}^{N-3}\lambda_{N-m}\langle\mathbf{v}_m\rangle, \tag{6.72}$$

$$\eta\nabla^2_{\mathbf{X}}\langle\mathbf{h}_{N-2}\rangle + \nabla_{\mathbf{X}}\times\sum_{k=1}^{3}\sum_{m=1}^{3}\left(\mathbf{D}^{vh}_{mk}\frac{\partial\langle\mathbf{v}_{N-2}\rangle_k}{\partial X_m} + \mathbf{D}^{hh}_{mk}\frac{\partial\langle\mathbf{h}_{N-2}\rangle_k}{\partial X_m}\right) - \lambda_2\langle\mathbf{h}_{N-2}\rangle$$
$$- \lambda_N\langle\mathbf{h}_0\rangle = -\nabla_{\mathbf{X}}\times\langle\mathbf{V}\times\mathbf{h}''_{N-1} + \mathbf{v}''_{N-1}\times\mathbf{H}\rangle + \sum_{m=1}^{N-3}\lambda_{N-m}\langle\mathbf{h}_m\rangle. \tag{6.73}$$

By the induction assumption, the r.h.s. in (6.72)–(6.73) take the form $\hat{\mathbf{f}}^v_N e^{i\mathbf{q}\cdot\mathbf{X}}$ and $\hat{\mathbf{f}}^h_N e^{i\mathbf{q}\cdot\mathbf{X}}$, where constant vectors $\hat{\mathbf{f}}^v_N$ and $\hat{\mathbf{f}}^h_N$ are known, and $\hat{\mathbf{f}}^h_N\cdot\mathbf{q}=0$. Denote the 6×6 matrix in the l.h.s. of (6.65)–(6.66) by $\mathbf{D}(\mathbf{q})$. Substituting the dependencies (6.46) on the slow variables for $n=N-2$ and cancelling out the factor $e^{i\mathbf{q}\cdot\mathbf{X}}$, we transform (6.72)–(6.73) into an equivalent system

$$(\mathbf{D}(\mathbf{q})-\lambda_2)\begin{pmatrix}\langle\mathbf{w}_{N-2}\rangle\\ \langle\mathbf{g}_{N-2}\rangle\end{pmatrix} - \lambda_N\begin{pmatrix}\hat{\mathbf{v}}\\ \hat{\mathbf{h}}\end{pmatrix} = \begin{pmatrix}-|\mathbf{q}|^{-2}\mathbf{q}\times(\mathbf{q}\times\hat{\mathbf{f}}^v_N)\\ \hat{\mathbf{f}}^h_N\end{pmatrix},$$

which we solve in $\mathbb{V}_{\mathbf{q}}$ (similarly to how the problem (6.49) was solved, see Sect. 6.4.2): We first find λ_N and the vector $(\langle\mathbf{w}_{N-2}\rangle, \langle\mathbf{g}_{N-2}\rangle)$ in the invariant subspace of $\mathbb{V}_{\mathbf{q}}$, complementary to the subspace spanned by $(\hat{\mathbf{v}}, \hat{\mathbf{h}})$. Next, from the divergence of the hydrodynamic equation (6.72) we find $p^{\diamond}_{N-1}$ and hence $\langle p_{N-1}\rangle$ (see (6.60)), and from expressions (6.43) for $n=N-2$ the fields $\{\mathbf{v}_{N-2}\}$, $\{\mathbf{h}_{N-2}\}$ and $\{p_{N-2}\}$. Thus, $\mathbf{v}_{N-2}, \mathbf{h}_{N-2}$ and p_{N-2} are now completely determined; from the expressions (6.67) for $n=N-1$ we also find $\mathbf{v}'_{N-1}, \mathbf{h}'_{N-1}$ and p'_{N-1}.

Step 2° for $n=N$. The fluctuating parts of the Eqs. 6.70–6.71 for $n=N$ considered together with the conditions (6.68)–(6.69), specifying the divergencies of the unknown fields, constitute a problem of the kind (6.17), (6.21) We can therefore find the zero-mean fields $\mathbf{v}''_N$ and $\mathbf{h}''_N$ together with p''_N from these equations, following the procedure for solution of the problem (6.17), (6.21) discussed in Sect. 6.2.

We have therefore constructed a complete formal asymptotic expansion of large-scale linear stability modes and the associated eigenvalues for a parity-invariant MHD steady state, in which the α-effect is absent. If the operator of the anisotropic combined MHD eddy diffusion, defined by the l.h.s. of Eqs. 6.58–6.59, is elliptic, then one can use the approach of [306] to prove that each eigenvalue of this operator gives rise to a branch of eigenvalues of the original operator of linearisation $\mathscr{L}'$ (see the eigenvalue problem (6.4)–(6.6)).

A simple further analysis of symmetries in the algebraic constructions presented above reveals that (i) in the expansions (6.7) and (6.8), the order ε^n terms are parity-antiinvariant for all even n, and parity-invariant for all odd n (and therefore $\langle \mathbf{v}_n \rangle = \langle \mathbf{h}_n \rangle = 0$ for all odd n, and $\langle p_n \rangle = 0$ for all even n); (ii) $\lambda_n = 0$ for all odd n, i.e., the eigenvalues λ in the problem (6.4)–(6.6) are expanded in power series in ε^2.

6.7 How Rare is Negative MHD Eddy Diffusivity?

We have shown in the previous sections that, in order to study the linear stability of a parity-invariant MHD steady state $\mathbf{V}, \mathbf{H}$ to large-scale perturbations, it suffices to solve numerically the auxiliary problems of types I ((6.27)–(6.29) and (6.30)–(6.32)) and II ((6.52)–(6.54) and (6.55)–(6.57)). (The use of a more efficient strategy of computations following the approach discussed in Sect. 4.4, is also possible, see Sect. 7.5 in the next chapter; then the number of the auxiliary problems to be solved decreases twice from 24 to 12). Afterwards, we compute the entries of the tensor of combined eddy correction of magnetic diffusion and kinematic viscosity, $\mathbf{D}$ (6.61)– (6.64), and determine the minimum MHD eddy diffusivity

$$\eta_{\text{eddy}} = \min_{|\mathbf{q}|=1} (-\text{Re}\lambda_2(\mathbf{q})).$$

We have studied, similarly to [113, 336], how often the phenomenon of negative combined MHD eddy diffusivity is observed in model MHD states. Steady fields $\mathbf{V}, \mathbf{H}$ were synthesised in a cube of periodicity side $L_i = 2\pi$ as random-amplitude Fourier series projected into the subspace of solenoidal vector fields. (Such fields are solutions to the system (6.1)–(6.3) under an appropriate choice of the source terms $\mathbf{F}(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$. The behaviour of a turbulent MHD state at the scale lengths, that are not maximum, is determined by interaction with other scales present in the system, resulting in the direct and inverse energy cascades. The physical sense that can be attributed to the source terms in our simulations consists of modelling the processes of energy exchange within the hierarchy of spatial scales). The energy spectrum of the synthetic fields falls off exponentially by 6 orders of magnitude from the 1st to the 10th spherical shell in the space of wave vectors. The series are cut off at wave vectors of length 10, and the r.m.s. amplitude of both fields (the flow and magnetic field) is one. The auxiliary problems were solved by the iterative methods [231]. Solutions were sought as Fourier series with the resolution of 64^3 harmonics, which provides a sufficient accuracy (the energy spectrum falls off by at least 10 orders of magnitude). We have checked that for the three considered values of molecular viscosity and magnetic diffusivity (they were assigned equal values) each MHD steady state from our ensemble of 25 pairs of synthetic fields $\mathbf{V}, \mathbf{H}$ is stable to short-scale perturbations of the same spatial periodicity.

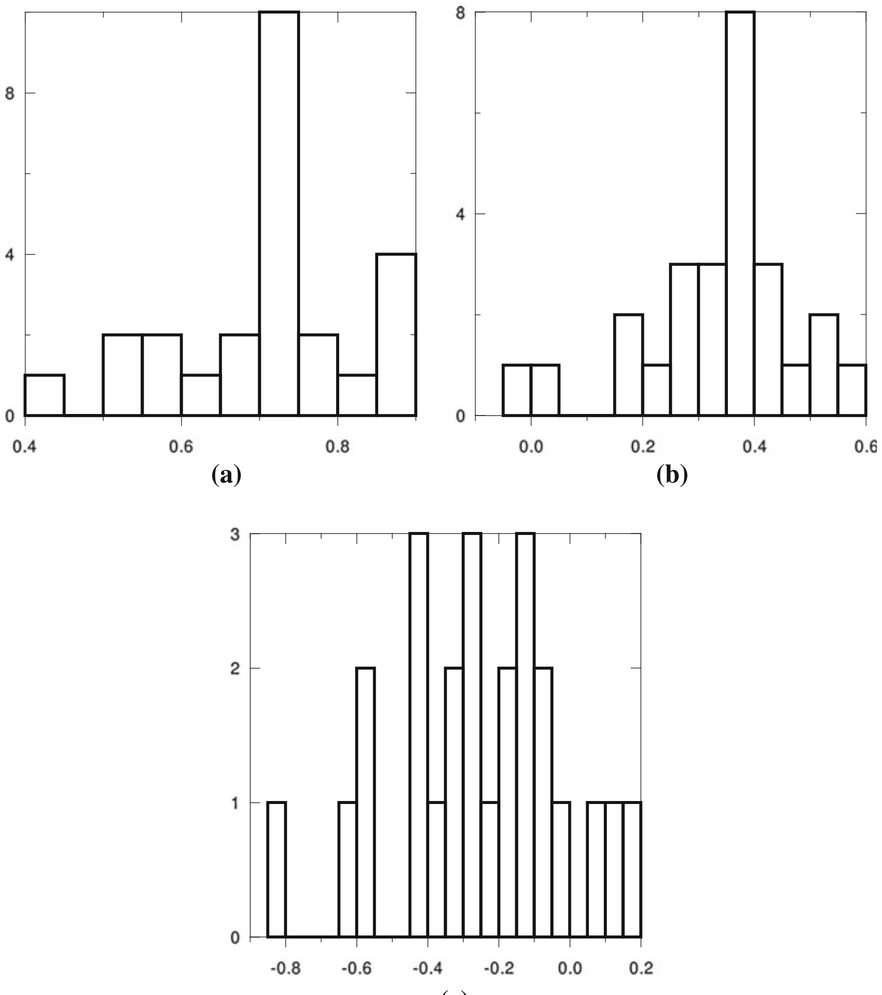

Fig. 6.1 Histograms of the minimum combined MHD eddy diffusivity in an ensemble of 25 synthetic MHD steady states for $\nu = \eta = 1$ (**a**), $\nu = \eta = 3/4$ (**b**), $\nu = \eta = 1/2$ (**c**)

Histograms of the obtained values of the minimum MHD eddy diffusivity are shown in Fig. 6.1. Negative combined MHD eddy diffusivity is observed already for $\nu = \eta = 3/4$, which corresponds to the short-scale kinematic and magnetic Reynolds numbers as low as 4/3 (assuming that the characteristic spatial scale is 1, the order of the periodicity box size). It is instructive to compare this with the results found in Chap. 3 [336], according to which negative magnetic eddy diffusivity in the kinematic dynamos occurs for significantly lower molecular magnetic diffusivities: the critical values of η for the onset of negative magnetic eddy diffusivity lie in the interval $0.2 < \eta < 0.3$.

6.8 Conclusions

1. We have constructed in this chapter complete asymptotic expansions of large-scale modes of MHD perturbations of short-scale space-periodic MHD steady states in power series in the scale ratio, together with the associated eigenvalues of the operator of linearisation. We have shown that generically the combined magnetic and hydrodynamic α-effect arises, described by a first-order partial (pseudo)differential operator in the slow variables (see the l.h.s. of the system of equations (6.34)–(6.35)). Its spectrum is symmetric about the imaginary axis, and hence generically steady MHD states are unstable to large-scale perturbations. If the α-effect does not occur—for instance, if the MHD state is parity-invariant—then the evolution of the perturbation averaged over short scales is controlled by a second-order partial differential operator in the slow variables (see the l.h.s. of equations (6.58)–(6.59)), which describes the so-called anisotropic combined magnetic and hydrodynamic eddy diffusion. Thus, we have shown that large-scale MHD perturbations, which have both a hydrodynamic and a magnetic component, exhibit a combination of the same physical phenomena, as perturbations of particular kinds in which one of these components vanishes identically (see [82], and Chaps. 3 and 4).
2. We have computed combined MHD eddy diffusivities for three equal values of molecular viscosity and magnetic diffusivity in an ensemble of 25 synthetic MHD steady states modelling short-scale turbulence, which were synthesised form random-amplitude Fourier harmonics with an exponential energy spectrum fall off. The minimum combined MHD eddy diffusivity $\eta_{\rm eddy}$ turns out to be negative for $\nu = \eta = 3/4$ and 1/2 in 4 and 92% cases, respectively (for $\nu = \eta = 1$, we have found $0 < \eta_{\rm eddy} < 1$ in all instances, i.e., combined eddy correction of magnetic diffusion and kinematic viscosity always decreases the combined MHD eddy diffusivity). Thus, our simulations demonstrate that MHD eddy diffusivity can be negative for a significantly higher molecular magnetic diffusivity than in kinematic dynamos, when a more general class of perturbations involving both hydrodynamic and magnetic components is considered.

Chapter 7
Weakly Nonlinear Stability of MHD Regimes to Large-Scale Perturbations

In Chap. 6 we have considered linear stability of parity-invariant MHD steady states to large-scale perturbations. Equations, governing stability modes, were obtained from the equations of magnetohydrodynamics by neglecting the terms, quadratic in perturbations. This procedure is legitimate, when development of perturbations is inspected at an initial stage and their amplitudes are small; when the amplitudes are increasing, the accuracy of description by the linearised equations at large time intervals, evidently, deteriorates. It is therefore desirable to derive a system of nonlinear mean-field equations for the perturbations averaged over small scales, in which the large-scale dynamics of the mean perturbations is uncoupled from the evolution of their short-scale component, and which might be used for the study of the evolution of the perturbations at subsequent stages, when nonlinearity sets in.

In this chapter we consider the so-called weakly nonlinear stability of solutions to equations of magnetohydrodynamics, i.e., we study equations, which *precisely* describe the evolution of perturbations. Taking into account the nonlinear terms, one can, evidently, construct more precise asymptotic expansions, in which the effects arising due to nonlinearity of the MHD system are not ignored (and hence, for instance, a chaotic behaviour is not ruled out). The material of this chapter, the exposition in which follows [328], is a direct extension of the results discussed in the previous chapter. We will show that while the amplitude of a large-scale perturbation does not exceed the scale ratio ε, the asymptotic multiscale methods used to carry out the analysis in the linear stability problem can be also applied for investigation of the weakly nonlinear stage of development of the perturbation. Surprisingly, the resultant mean-field equations turn out to be nonlinear, only if the α-effect is not present in the MHD perturbations in the leading order—e.g., if the MHD regime is parity-invariant. In the latter case the mean-field equations for perturbations generalise the Navier–Stokes and magnetic induction equations: additional terms emerge in them—not only the operator of anisotropic combined MHD eddy diffusion, which we encountered in the linear stability problem

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considered in Chap. 6, but also new quadratic terms analogous to the terms describing advection, which can be interpreted as a *nonlinear (quadratic) anisotropic operator of combined eddy correction of advection of fluid and magnetic field*.

We were exploring in Chap. 6 stability of steady MHD states. In this chapter, we will inspect a more general case, allowing time dependence of the MHD regimes, whose stability is examined. This gives rise to a new difficulty: the non-uniqueness of the choice of initial conditions for solutions of the auxiliary problems. When an eigenvalue problem for large-scale perturbation modes for a steady MHD state (like in Chaps. 3 and 6) or a Floquet problem for a time-periodic regime (like in Chap. 4) is solved, it is natural to demand that solutions to the auxiliary problems are, respectively, steady or time-periodic with the same temporal period; this usually guarantees uniqueness of the solutions. The same approach can be applied for quasi-periodic MHD regimes (i.e., regimes which have more than one incommensurate basic frequency). If the time dependence of an MHD regime is of a general type, a prerequisite for our constructions is the correctness of averaging over the fast variables. We will show that despite the non-uniqueness of solutions to the auxiliary problems, the coefficients of the operator of the combined MHD α-effect, the operator of combined eddy correction of magnetic diffusion and kinematic viscosity and the quadratic operator of combined eddy correction of advection can be uniquely determined, if the MHD regime, whose weakly nonlinear stability to large-scale perturbations is studied, is stable to short-scale perturbations (this requirement seems natural, because the growth rates of large-scale instability modes are at most the order of the spatial scale ratio ε, and hence the large-scale instability is too weak for detection in experiments when a short-scale instability develops in the background, but see Sect. 7.2).

Spatial periodicity of the MHD regime, whose stability is studied, is also inessential (although it is very convenient for computation of solutions to auxiliary problems and evaluation of the eddy coefficients), and we do not prescribe any periodicity in this chapter. Instead, we impose the requirements that all auxiliary problems emerging in the course of derivation of the mean-field equations have solutions (under the natural conditions), that are globally bounded together with a sufficient number of derivatives, and all the mean quantities that we will consider are well-defined.

7.1 Statement of the Problem

The short-scale MHD regime $\mathbf{V}(\mathbf{x},t)$, $\mathbf{H}(\mathbf{x},t)$, $P(\mathbf{x},t)$, whose weakly nonlinear stability is considered, satisfies the equations

$$\frac{\partial \mathbf{V}}{\partial t} = \nu\nabla^2\mathbf{V} + \mathbf{V}\times(\nabla\times\mathbf{V}) + (\nabla\times\mathbf{H})\times\mathbf{H} - \nabla P + \mathbf{F}, \tag{7.1}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \eta \nabla^2 \mathbf{H} + \nabla \times (\mathbf{V} \times \mathbf{H}) + \mathbf{J}, \tag{7.2}$$

$$\nabla \cdot \mathbf{V} = \nabla \cdot \mathbf{H} = 0. \tag{7.3}$$

Here all the variables have the same sense as in the equations for an MHD steady state, the linear stability problem for which was considered in the previous chapter (see Sect. 6.1.1). In this chapter we do not assume that the fields $\mathbf{V}, \mathbf{H}, P$ are steady or periodic in space or in time. The spatial invariance of the MHD state $\mathbf{V}, \mathbf{H}, P$, as well as the temporal invariance of the regime, if $\mathbf{V}, \mathbf{H}, P$ are unsteady, is supposed to be broken by the source terms $\mathbf{F}$ and $\mathbf{J}$. In particular, $\mathbf{F}$ and $\mathbf{J}$ are supposed not to be both identically zero. (For $\mathbf{F} = \mathbf{J} = 0$ any solution to the system (7.1)–(7.3) tends to zero when $t \to \infty$ because of dissipation of energy, and thus the stability problem in the sourceless case is not of interest). In Sect. 7.4 we will also assume that the fields $\mathbf{V}, \mathbf{H}, P$ are parity-invariant.

7.1.1 The Governing Equations

The operator of linearisation of the system of equations (7.1)–(7.2) in the vicinity of the MHD regime under investigation is $\mathscr{M} = (\mathscr{M}^v, \mathscr{M}^h)$, where

$$\mathscr{M}^v(\mathbf{v}, \mathbf{h}, p) \equiv -\frac{\partial \mathbf{v}}{\partial t} + \nu \nabla^2_{\mathbf{x}} \mathbf{v} + \mathbf{V} \times (\nabla_{\mathbf{x}} \times \mathbf{v}) + \mathbf{v} \times (\nabla_{\mathbf{x}} \times \mathbf{V})$$
$$+ (\nabla_{\mathbf{x}} \times \mathbf{H}) \times \mathbf{h} + (\nabla_{\mathbf{x}} \times \mathbf{h}) \times \mathbf{H} - \nabla_{\mathbf{x}} p,$$
$$\mathscr{M}^h(\mathbf{v}, \mathbf{h}) \equiv -\frac{\partial \mathbf{h}}{\partial t} + \eta \nabla^2_{\mathbf{x}} \mathbf{h} + \nabla_{\mathbf{x}} \times (\mathbf{V} \times \mathbf{h} + \mathbf{v} \times \mathbf{H}).$$

Let us consider a perturbation of an amplitude of the order ε. The perturbed state, $\mathbf{V} + \varepsilon \mathbf{v}, \mathbf{H} + \varepsilon \mathbf{h}, P + \varepsilon p$ also satisfies Eqs. 7.1–7.3, and hence the profiles of the perturbation, $\mathbf{v}, \mathbf{h}, p$, (which we will call henceforth just a "perturbation") satisfy

$$\mathscr{M}^v(\mathbf{v}, \mathbf{h}, p) + \varepsilon(\mathbf{v} \times (\nabla \times \mathbf{v}) + (\nabla \times \mathbf{h}) \times \mathbf{h}) = 0, \tag{7.4}$$

$$\mathscr{M}^h(\mathbf{v}, \mathbf{h}) + \varepsilon \nabla \times (\mathbf{v} \times \mathbf{h}) = 0, \tag{7.5}$$

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{h} = 0. \tag{7.6}$$

These equations are obtained as a difference of Eqs. 7.1–7.3 for the perturbed and unperturbed states. Investigation of weakly nonlinear stability of the regime $\mathbf{V}, \mathbf{H}, P$, amounts to the study of the behaviour of solutions to this system of equations.

The perturbations are supposed to depend on the fast and slow variables. As usual, by $\mathbf{x}$ we denote the fast spatial variable and t the fast time; the respective slow variables are defined as $\mathbf{X} = \varepsilon\mathbf{x}$ and $\boldsymbol{T} = \varepsilon^s t$. Here the exponent s depends on the properties of the problem under consideration: the presence of the α-effect implies $s = 1$, and in its absence and, therefore, in the presence of the MHD eddy diffusion $s = 2$.

The fields $\mathbf{V}, \mathbf{H}, P$, constituting the MHD regime whose stability we investigate, are supposed to be smooth and globally bounded, so that all spatio-temporal mean quantities that we will consider in the course of derivation of the mean-field equations were well-defined. Averaging over the fast variables is meant here:

$$\langle\!\langle \mathbf{f}(\mathbf{X}, T, \mathbf{x}, t) \rangle\!\rangle \equiv \lim_{\tilde{t}\to\infty} \lim_{\ell\to\infty} \frac{1}{\tilde{t}\ell^3} \int_0^{\tilde{t}} \int_{-\ell/2}^{\ell/2} \int_{-\ell/2}^{\ell/2} \int_{-\ell/2}^{\ell/2} \mathbf{f}(\mathbf{X}, T, \mathbf{x}, t) \mathrm{d}\mathbf{x}\, \mathrm{d}t$$

denotes the spatio-temporal mean,[1] and

$$\{\!\{\mathbf{f}\}\!\} \equiv \mathbf{f} - \langle\!\langle \mathbf{f} \rangle\!\rangle$$

the fluctuating part of a scalar or a vector field $\mathbf{f}$. Clearly, these conditions are satisfied, if the MHD regime whose stability is examined is smooth, space-periodic, and steady or periodic in time, or, more generally, if it is quasi-periodic in space and time and the energy spectrum falls off sufficiently fast. (A function F is quasi-periodic in a scalar variable x, if $F(x) = f(f_1 x, \ldots, f_m x)$ and f is 2π-periodic in each variable, all ratios of the *basic frequencies* f_{n_1}/f_{n_2} being irrational for $n_1 \neq n_2$). We will also use the notation $\langle\!\langle \mathbf{f} \rangle\!\rangle_k$ for the $k-$th component of the mean $\langle\!\langle \mathbf{f} \rangle\!\rangle$, and

$$\langle \mathbf{f} \rangle \equiv \lim_{\ell\to\infty} \frac{1}{\ell^3} \int_{-\ell/2}^{\ell/2} \int_{-\ell/2}^{\ell/2} \int_{-\ell/2}^{\ell/2} \mathbf{f}(\mathbf{X}, T, \mathbf{x}, t) \mathrm{d}\mathbf{x}, \qquad \{\mathbf{f}\} \equiv \mathbf{f} - \langle \mathbf{f} \rangle$$

for the spatial (over the fast variables) mean and fluctuating part of $\mathbf{f}$, respectively.

[1] The condition of global boundedness is convenient, because the mean of a derivative of a globally bounded field in any fast variable vanishes, for instance

$$\left\langle\!\!\left\langle \frac{\partial \mathbf{f}}{\partial x_1} \right\rangle\!\!\right\rangle = \lim_{\tilde{t}\to\infty} \lim_{\ell\to\infty} \frac{1}{\tilde{t}\ell^3} \int_0^{\tilde{t}} \int_{-\ell/2}^{\ell/2} \int_{-\ell/2}^{\ell/2} \int_{-\ell/2}^{\ell/2} \frac{\partial \mathbf{f}}{\partial x_1} \mathrm{d}\mathbf{x}\, \mathrm{d}t$$
$$= \lim_{\tilde{t}\to\infty} \lim_{\ell\to\infty} \frac{1}{\tilde{t}\ell^2} \int_0^{\tilde{t}} \int_{-\ell/2}^{\ell/2} \int_{-\ell/2}^{\ell/2} \frac{\mathbf{f}(\ell/2, x_2, x_3, t) - \mathbf{f}(-\ell/2, x_2, x_3, t)}{\ell} \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}t = 0.$$

7.1.2 Asymptotic Expansion of Weakly Nonlinear Large-Scale Perturbations

A solution to the problem (7.4)–(7.6) is sought in the form of power series

$$\mathbf{v} = \sum_{n=0}^{\infty} \mathbf{v}_n(\mathbf{X}, T, \mathbf{x}, t)\varepsilon^n, \qquad \mathbf{h} = \sum_{n=0}^{\infty} \mathbf{h}_n(\mathbf{X}, T, \mathbf{x}, t)\varepsilon^n, \tag{7.7}$$

$$p = \sum_{n=0}^{\infty} p_n(\mathbf{X}, T, \mathbf{x}, t)\varepsilon^n. \tag{7.8}$$

We will consider therefore only perturbations[2] that are compatible with (7.7) at $t = 0$.

7.1.3 The Hierarchy of Equations for the Perturbations

The solenoidality conditions (7.6) for fields constituting a large-scale perturbation are treated as in Sect. 6.1.3: Substituting the series (7.7) into (7.6) and extracting the mean and fluctuating parts at each order ε^n, we find

$$\nabla_{\mathbf{X}} \cdot \langle\!\langle \mathbf{v}_n \rangle\!\rangle = \nabla_{\mathbf{X}} \cdot \langle\!\langle \mathbf{h}_n \rangle\!\rangle = 0, \tag{7.9}$$

$$\nabla_{\mathbf{x}} \cdot \{\!\{\mathbf{v}_n\}\!\} + \nabla_{\mathbf{X}} \cdot \{\!\{\mathbf{v}_{n-1}\}\!\} = \nabla_{\mathbf{x}} \cdot \{\!\{\mathbf{h}_n\}\!\} + \nabla_{\mathbf{X}} \cdot \{\!\{\mathbf{h}_{n-1}\}\!\} = 0 \tag{7.10}$$

for all $n \geq 0$ (all quantities with an index $n < 0$ are assumed, by definition, to vanish).

Upon substitution of the series (7.7) and (7.8) into Eqs. 7.4 and 7.5 governing the evolution of the perturbation, we transform them into equalities of power series in ε. Equating the terms of the series at each order ε^n, we obtain a hierarchy of systems of equations, considered together with the conditions for the divergencies (7.9) and (7.10). As in the linear problem, in principle we can successively solve the systems at any order and thus find all terms in the series (7.7) and (7.8). However, since the appearing equations are bulky and non-illuminating, we will restrict ourselves to derivation of the *mean-field equations*, i.e., a closed system of equations in the slow variables for the averaged leading terms of the expansions of perturbations (7.7) and (7.8). These equations are the solvability conditions for the equations in the fast variables at orders ε or ε^2.

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[2] Clearly, not every family of vector fields depending on ε can be expressed as power series (7.7). However, this constraint is not too restrictive, because it is natural to apply our stability analysis for the study of the weakly nonlinear evolution of large-scale MHD perturbation modes and their superpositions, and such modes admit asymptotic expansions in power series (7.7), see Chap. 6.

7.2 Solvability of Auxiliary Problems

Suppose the fields $\mathbf{v}, \mathbf{h}, p$ and their derivatives are smooth and globally bounded—we will only seek solutions to auxiliary problems from this class. Since the fields $\mathbf{V}$ and $\mathbf{H}$ are solenoidal (7.3) and also belong to this class, the following identities hold true:

$$\langle \mathscr{M}^v(\mathbf{v}, \mathbf{h}, p) \rangle = -\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{V}\nabla_\mathbf{x} \cdot \mathbf{v} - \mathbf{H}\nabla_\mathbf{x} \cdot \mathbf{h} \rangle; \tag{7.11}$$

$$\left\langle \mathscr{M}^h(\mathbf{v}, \mathbf{h}) \right\rangle = -\frac{\partial \langle \mathbf{h} \rangle}{\partial t} \tag{7.12}$$

$$\Rightarrow \quad \langle\!\langle \mathscr{M}^v(\mathbf{v}, \mathbf{h}, p) \rangle\!\rangle = \langle\!\langle \mathbf{V}\nabla_\mathbf{x} \cdot \mathbf{v} - \mathbf{H}\nabla_\mathbf{x} \cdot \mathbf{h} \rangle\!\rangle; \qquad \left\langle\!\left\langle \mathscr{M}^h(\mathbf{v}, \mathbf{h}) \right\rangle\!\right\rangle = 0. \tag{7.13}$$

Relations (7.13) imply that the conditions

$$\langle\!\langle \mathbf{f}^v(\mathbf{x}, t) \rangle\!\rangle = \left\langle\!\left\langle \mathbf{f}^h(\mathbf{x}, t) \right\rangle\!\right\rangle = 0 \tag{7.14}$$

are necessary for existence of solutions to the system of equations

$$\mathscr{M}^v(\mathbf{v}, \mathbf{h}, p) = \mathbf{f}^v, \quad \mathscr{M}^h(\mathbf{v}, \mathbf{h}) = \mathbf{f}^h, \tag{7.15}$$

$$\nabla_\mathbf{x} \cdot \mathbf{v} = \nabla_\mathbf{x} \cdot \mathbf{h} = 0 \tag{7.16}$$

from the class under consideration. By virtue of the expressions (7.11)–(7.12) for spatial means of the image of the operator of linearisation, the spatial mean of solutions of the homogeneous system

$$\mathscr{M}^v(\mathbf{v}, \mathbf{h}, p) = 0, \quad \mathscr{M}^h(\mathbf{v}, \mathbf{h}) = 0, \tag{7.17}$$

$$\nabla_\mathbf{x} \cdot \mathbf{v} = \nabla_\mathbf{x} \cdot \mathbf{h} = 0 \tag{7.18}$$

from this class of fields does not depend on time. The divergence of the second equation in 7.17 implies that $\nabla_\mathbf{x} \cdot \mathbf{h}$ is conserved in time, and hence the solenoidality condition for $\mathbf{h}$ is satisfied if $\mathbf{h}$ is solenoidal at $t = 0$.

We assume henceforth that for any pair of smooth solenoidal zero-mean (7.14) fields $\mathbf{f}^v(\mathbf{x}, t), \mathbf{f}^h(\mathbf{x}, t)$, globally bounded together with the derivatives, the problem (7.15)–(7.16) has a solution $\mathbf{v}, \mathbf{h}, p$, globally bounded together with the derivatives, at least for some initial conditions. This solvability condition is satisfied, e.g., for generic space-periodic MHD regimes $\mathbf{V}, \mathbf{H}, P$ that are steady or time-periodic: If zero is not in the spectrum of the operator of linearisation $\mathscr{M}$ restricted to the subspace of solenoidal zero-mean vector fields, having the periodicities of $\mathbf{V}$ and $\mathbf{H}$, then there exist steady or time-periodic, respectively, solutions of the same periodicity. For space-periodic MHD steady states, we have shown this in Sect. 6.2; for MHD regimes, periodic in space and time, a demonstration can be easily obtained by a modification of the arguments presented in Sect. 4.1.3 for the operator of magnetic induction.

Note that temporally or spatially invariant MHD regimes are not generic. If the system (7.1)–(7.3) describes an MHD regime, which is translation-invariant in time (the sources **F** and **J** are time-independent), but the fields **V** and **H** are unsteady, then differentiating (7.1) and (7.2) in time we find, that the kernel of $\mathscr{M}$ includes the neutral short-scale mode $(\partial\mathbf{V}/\partial t, \partial\mathbf{H}/\partial t)$ with a vanishing spatio-temporal mean. (Evidently, differentiation does not violate solenoidality, periodicity or quasi-periodicity). Alternatively, let the system of equations (7.1)–(7.3) describe an MHD regime, translation-invariant in space (the sources **F** and **J** depend only on time), in which **V** and **H**, P are non-uniform in space. Then differentiation of (7.1) and (7.2) in $x_k, k = 1, 2, 3$, shows that $(\partial\mathbf{V}/\partial x_k, \partial\mathbf{H}/\partial x_k) \in \ker\mathscr{M}$; this neutral mode has a vanishing spatial mean. Note that we do not regard solutions to (7.17)–(7.18) as neutral modes, if they decay exponentially in time. (Existence of MHD regimes, translation-invariant in space and possessing neutral modes—for instance, of time-periodic regimes, non-uniform in space, although arising under a time-periodic uniform forcing—seems plausible because of non-uniqueness of MHD attractors for relatively small viscosities and magnetic diffusivities).

A solution to the problem (7.15)–(7.16) can be constructed as a solution to the system of parabolic partial differential equations. However, since the region occupied by fluid is non-compact, this solution is not guaranteed to be globally bounded. Quasi-periodicity in space or time with just two basic frequencies is already problematic: it is impossible to generalise directly the arguments, put forward for periodic solutions. Suppose, for instance, that the steady state $\mathbf{V}, \mathbf{H}, P$ is quasi-periodic in space, and a solution of the same quasi-periodicity to the problem (7.15)–(7.16) is sought. Existence of a space-periodic solution was shown in Sect. 6.2 by application of the Fredholm alternative theorem. To use it, the inverse Laplacian $\nabla_{\mathbf{x}}^{-2}$ is applied to Eq. 7.15. For vector fields of a fixed space periodicity, $\nabla_{\mathbf{x}}^{-2}$ is a compact operator, and its application enables us to reformulate the problem (7.15)–(7.16) in the terms of an operator of the suitable structure. By contrast, the inverse Laplacian can be applied to quasi-periodic vector fields of a zero spatial mean, but it is not bounded, let alone compact. We illustrate this by an example of quasi-periodic functions of a scalar variable x with the basic frequencies f_1 and f_2, for which

$$\nabla_{\mathbf{x}}^{-2} \mathrm{e}^{\mathrm{i}x(k_1 f_1 + k_2 f_2)} = -\mathrm{e}^{\mathrm{i}x(k_1 f_1 + k_2 f_2)} / (k_1 f_1 + k_2 f_2)^2. \tag{7.19}$$

The irrational number f_1/f_2 can be approximated to any accuracy by rational numbers, and hence the denominator in the r.h.s. of (7.19) approaches zero indefinitely,[3] and hence the inverse Laplacian is unbounded in the class of quasi-

[3] Here is a rigorous proof: Denote $\mu = \inf |p_1(f_1/f_2) - p_2|$, where the infimum is over all integer numbers p_1 and p_2. Evidently, $0 \le \mu < 1/2$. Assuming $\mu > 0$, choose integer $p_1 > 0$ and $p_2 \ge 0$ such that $\alpha \equiv |p_1(f_1/f_2) - p_2|$ satisfies $0 \le \alpha - \mu \ll \mu$. Denote by K the integer part of α^{-1}. Evidently, at least one of the two positive numbers: $1 - |Kp_1(f_1/f_2) - Kp_2|$ and $|(K+1)p_1(f_1/f_2) - (K+1)p_2| - 1$ does not exceed $\alpha/2 < \mu$. This, however, contradicts with the definition of μ; therefore, $\mu = 0$, as required.

periodic functions. Thus, existence of solutions must be verified in each application of our theory. We just postulate that the problems that we encounter below are solvable; otherwise, our algebraic constructions cannot be implemented directly, this indicating that new physical effects, not considered here, are present in the large-scale MHD system under consideration.

An MHD regime $\mathbf{V}, \mathbf{H}, P$ is called *linearly stable to short-scale perturbations*, if any solution to the problem (7.17)–(7.18) for globally bounded zero-mean initial conditions exponentially decays in (the fast) time together with the derivatives. (We consider only zero-mean solutions, since by virtue of (7.11)–(7.12) the spatial mean of any solution is conserved in time). For physical applications, it is natural to study large-scale stability of such MHD regimes (because large-scale instabilities develop in the slow time, whereas short-scale ones in the fast time). However, there exists a class of MHD regimes, which are not linearly stable to short-scale perturbations as individual trajectories in the phase space, but nevertheless can be observed, being dense in stable geometric objects: these trajectories belong to chaotic attractors. We want our theory to be applicable to such trajectories, and hence do not demand that the MHD regimes that we consider are linearly stable to short-scale perturbations (with the exception of Sect. 7.4.2).

Thus, the condition (7.14) does not guarantee existence of a solution, globally bounded together with the derivatives, to the problem (7.15)–(7.16) for *all* initial conditions in the considered class. For instance, suppose the MHD state whose stability is examined is steady, and a steady solution to the problem (7.15)–(7.16) exists. Then any evolutionary solution is a sum of the steady solution and a solution to the homogeneous problem (7.17)–(7.18). If the MHD steady state $\mathbf{V}, \mathbf{H}, P$ is not linearly stable to short-scale perturbations, then a solution to the homogeneous problem can experience an unbounded growth in time—this will happen, if an unstable mode is present in the expansion of the initial condition for the homogeneous problem in the basis of stability modes. We ban such indefinite growth; it does not occur for correctly chosen initial conditions, such that the respective initial condition for the homogeneous problem belongs to the subspace of decaying eigenmodes of the operator of linearisation $\mathscr{M}$.

Finally, we assume that any solution to the system of homogeneous equations (7.17)–(7.18), that does not grow exponentially in the fast time and has a zero spatio-temporal mean, exponentially decays in fast time. If the MHD regime $\mathbf{V}, \mathbf{H}, P$ is space-periodic and steady or time-periodic, this condition implies that the kernel of the operator of linearisation is six-dimensional (as we assumed in the previous chapter), and hence that under the condition (7.14) there exists a steady or time-periodic, respectively, space-periodic solution to the problem (7.15)–(7.16) (provided the r.h.s. belongs to the same class).

We will follow essentially the same two-step plan for the solution of the hierarchy of equations, as in the linear problem:

1°. Verify the solvability conditions (7.14).
2°. Solve the resultant system for $\mathbf{v}_n, \mathbf{h}_n$ and p_n.

7.3 Large-Scale Instability in the Presence of the MHD α-Effect

For any choice of the exponent $s > 0$ in the definition of the slow time, we find from the leading (order ε^0) terms of Eqs. 7.4 and 7.5:

$$\mathscr{M}^v(\{\!\{\mathbf{v}_0\}\!\}, \{\!\{\mathbf{h}_0\}\!\}, \{\!\{p_0\}\!\}) = -\langle\!\langle \mathbf{v}_0 \rangle\!\rangle \times (\nabla_\mathbf{x} \times \mathbf{V}) - (\nabla_\mathbf{x} \times \mathbf{H}) \times \langle\!\langle \mathbf{h}_0 \rangle\!\rangle, \quad (7.20)$$

$$\mathscr{M}^h(\{\!\{\mathbf{v}_0\}\!\}, \{\!\{\mathbf{h}_0\}\!\}) = -(\langle\!\langle \mathbf{h}_0 \rangle\!\rangle \cdot \nabla_\mathbf{x})\mathbf{V} + (\{\!\{\mathbf{v}_0\}\!\} \cdot \nabla_\mathbf{x})\mathbf{H}. \quad (7.21)$$

These equations are supplemented by the solenoidality conditions (see (7.10) for $n = 0$):

$$\nabla_\mathbf{x} \cdot \{\!\{\mathbf{v}_0\}\!\} = \nabla_\mathbf{x} \cdot \{\!\{\mathbf{h}_0\}\!\} = 0. \quad (7.22)$$

7.3.1 *Solution of Order ε^0 Equations*

Step 1° for $n = 0$. The spatial means of the r.h.s. of Eqs. 7.20–7.22 vanish due to the global boundedness of the fields **V** and **H**, and thus the solvability condition (7.14) is satisfied.

Step 2° for $n = 0$. Since differentiation only in the fast variables is performed in $\mathscr{M}$, and the spatio-temporal means $\langle\!\langle \mathbf{v}_0 \rangle\!\rangle$ and $\langle\!\langle \mathbf{h}_0 \rangle\!\rangle$ depend only on the slow variables, by linearity the system (7.20)–(7.22) has a solution of the following structure:

$$\begin{pmatrix} \{\!\{\mathbf{v}_0\}\!\} \\ \{\!\{\mathbf{h}_0\}\!\} \\ \{\!\{p_0\}\!\} \end{pmatrix} = \begin{pmatrix} \xi_0^v \\ \xi_0^h \\ \xi_0^p \end{pmatrix} + \sum_{k=1}^{3} \left(\begin{pmatrix} \mathbf{S}_k^{vv} \\ \mathbf{S}_k^{vh} \\ S_k^{vp} \end{pmatrix} \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k + \begin{pmatrix} \mathbf{S}_k^{hv} \\ \mathbf{S}_k^{hh} \\ S_k^{hp} \end{pmatrix} \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k \right), \quad (7.23)$$

where zero-mean vector fields $\mathbf{S}(\mathbf{x}, t)$ are solutions to auxiliary problems of type I:

Auxiliary problem I.1

$$\mathscr{M}^v(\mathbf{S}_k^{vv}, \mathbf{S}_k^{vh}, S_k^{vp}) = -\mathbf{e}_k \times (\nabla_\mathbf{x} \times \mathbf{V}), \quad (7.24)$$

$$\nabla_\mathbf{x} \cdot \mathbf{S}_k^{vv} = 0, \quad (7.25)$$

$$\mathscr{M}^h(\mathbf{S}_k^{vv}, \mathbf{S}_k^{vh}) = \frac{\partial \mathbf{H}}{\partial x_k}, \quad (7.26)$$

$$\nabla_\mathbf{x} \cdot \mathbf{S}_k^{vh} = 0. \quad (7.27)$$

Auxiliary problem I.2

$$\mathscr{M}^v(\mathbf{S}_k^{hv}, \mathbf{S}_k^{hh}, S_k^{hp}) = \mathbf{e}_k \times (\nabla_\mathbf{x} \times \mathbf{H}), \quad (7.28)$$

$$\nabla_\mathbf{x} \cdot \mathbf{S}_k^{hv} = 0, \quad (7.29)$$

$$\mathscr{M}^h(\mathbf{S}_k^{hv}, \mathbf{S}_k^{hh}) = -\frac{\partial \mathbf{V}}{\partial x_k}, \tag{7.30}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{S}_k^{hh} = 0. \tag{7.31}$$

The field $\xi_0(\mathbf{X}, T, \mathbf{x}, t)$ is supposed to decay exponentially in time; it satisfies

$$\mathscr{M}^v(\xi_0^v, \xi_0^h, \xi_0^p) = \mathscr{M}^h(\xi_0^v, \xi_0^h) = 0, \quad \nabla_{\mathbf{x}} \cdot \xi_0^v = \nabla_{\mathbf{x}} \cdot \xi_0^h = 0, \quad \langle \xi_0 \rangle = 0. \tag{7.32}$$

The spatial means of the r.h.s. of Eqs. 7.24, 7.26, 7.28, and 7.30 vanish due to the global boundedness of the fields $\mathbf{V}$ and $\mathbf{H}$. Consequently, the solvability condition (7.14) is satisfied for the auxiliary problems (7.24)–(7.27) and (7.28)–(7.31), and hence, by our assumption, globally bounded with derivatives solutions to the auxiliary problems exist. Furthermore, by virtue of expressions (7.11)–(7.12) for the spatial mean of the image of the operator of linearisation, $\langle \mathbf{S} \rangle$ and $\langle \xi_0 \rangle$ are independent of fast time, and hence the condition $\langle\!\langle \mathbf{S}_k^v \rangle\!\rangle = \langle\!\langle \mathbf{S}_k^h \rangle\!\rangle = 0$ implies that $\langle \mathbf{S} \rangle$ vanish at any $t \leq 0$; similarly, $\langle \xi_0 \rangle$ is zero at any $t \leq 0$. Solenoidality conditions (7.25), (7.27), (7.29) and (7.31) together with the solenoidality of ξ_0 guarantee that the leading terms of the expansion of the perturbation are solenoidal (7.22). Solenoidality conditions for the magnetic components—for ξ_0^h in the problem (7.32), and (7.27), (7.31) in the auxiliary problems I.1 and I.2—are satisfied for $t > 0$, if they hold true at $t = 0$. Thus any smooth solenoidal fields, globally bounded with the derivatives, can serve as initial conditions for the problems (7.42)–(7.27), (7.28), (7.31) and (7.32), provided their spatial means vanish, the solution remains globally bounded in space and time, and ξ_0 and its derivatives exponentially decay in time. If the MHD regime, whose stability is examined, is a steady state, and/or it is periodic or quasi-periodic in space and/or time, it is natural to demand that $\mathbf{S}$ have the same properties.

Averaging the relations

$$\mathbf{v}_0 = \{\!\{\mathbf{v}_0\}\!\} + \langle\!\langle \mathbf{v}_0 \rangle\!\rangle, \qquad \mathbf{h}_0 = \{\!\{\mathbf{h}_0\}\!\} + \langle\!\langle \mathbf{h}_0 \rangle\!\rangle$$

over the fast spatial variables at $t = 0$, we find

$$\langle\!\langle \mathbf{v}_0 \rangle\!\rangle|_{T=0} = \langle \mathbf{v}_0 \rangle|_{t=0}, \quad \langle\!\langle \mathbf{h}_0 \rangle\!\rangle|_{T=0} = \langle \mathbf{h}_0 \rangle|_{t=0},$$

and hence

$$\{\!\{\mathbf{v}_0\}\!\}|_{t=0} = \{\mathbf{v}_0\}|_{t=0}, \quad \{\!\{\mathbf{h}_0\}\!\}|_{t=0} = \{\mathbf{h}_0\}|_{t=0}.$$

Initial conditions for the problem (7.32) can be found therefore from the expressions (7.23) for the fluctuating parts of the leading terms in the expansion of perturbations (7.7), applied at $t = 0$ (assuming $\mathbf{v}_0, \mathbf{h}_0$ and $\mathbf{S}$ are specified at $t = 0$). If the MHD regime $\mathbf{V}, \mathbf{H}, P$ is stable to short-scale perturbations, then a change in the initial conditions for $\mathbf{S}$ in the class of acceptable initial conditions is compensated by the respective change in the initial conditions for ξ_0. As we will see, this non-uniqueness does not affect eddy terms in the mean-field equations, since the resultant variations of $\mathbf{S}$ and ξ_0 decay in time exponentially (being solutions to the problem (7.17) and (7.18) with zero-mean initial conditions belonging, by construction, to the stable subspace of the operator of linearisation $\mathscr{M}$).

We finish this subsection commenting on the assumption that any solution to the system of homogeneous equations (7.17)–(7.18), that does not grow exponentially in the fast time and has a zero spatio-temporal mean, must exponentially decay in the fast time (see the end of Sect. 7.2). Suppose, for the sake of simplicity, that the MHD regime is steady. Then, the assumption is equivalent to the condition that the spectrum of linearisation $\mathscr{M}$ does not involve imaginary eigenvalues associated with zero-mean short-scale modes (in particular, the kernel of $\mathscr{M}$ is six-dimensional, composed of non-zero-mean neutral stability modes). This is the generic case. Let now the operator $\mathscr{M}$ possess a non-zero imaginary eigenvalue (for simplicity, of multiplicity one). Then the associated short-scale oscillatory stability mode, $\mathbf{S}$, gives rise to a large-scale linear stability mode, for which an amplitude equation can be constructed following the approach of Sect. 3.8. Because of nonlinearity of equations for perturbations, interaction of short-scale steady and oscillatory stability modes takes place, and therefore in the framework of investigation of weakly nonlinear stability to large-scale perturbations the oscillatory mode (as well as the one associated with the complex-conjugate eigenvalue) must be included into the analysis together with all steady neutral modes. In other words, two new terms

$$c\begin{pmatrix}\mathbf{S}^v\\ \mathbf{S}^h\\ S^p\end{pmatrix} + \bar{c}\begin{pmatrix}\bar{\mathbf{S}}^v\\ \bar{\mathbf{S}}^h\\ \bar{S}^p\end{pmatrix}$$

must be added into the expression (7.23) for the leading term in the expansion of perturbations, and then the mean-free equations that we construct below are supplemented by equations for the real and imaginary parts of the complex-valued amplitude $c(\mathbf{X})$. The consequences of existence of a zero-mean steady short-scale stability mode are quite similar.

Furthermore, existence of an imaginary eigenvalue in the spectrum of linearisation $\mathscr{M}$ is generically accompanied by occurrence of a Hopf bifurcation (when the eigenvalue is non-zero), or saddle-node or pitchfork bifurcations (when it is zero), in which a branch of short-scale MHD regimes emerges. Large-scale stability of MHD regimes constituting this branch close to the point of bifurcation can be examined following a similar approach. We do not present here the asymptotic analysis of large-scale perturbations of such MHD regimes; instead, we present the mean-field and amplitude equations for leading terms of expansions of the perturbations in Appendix, since the derivations for three-dimensional MHD regimes in entire space and for convective hydromagnetic regimes in a plane layer are very similar.

7.3.2 The Solvability Condition for Order ε^1 Equations: Linear Equations for Weakly Nonlinear Perturbations

Step 1° for $n = 1$. Let the slow time be defined as $T = \varepsilon t$. The equations for perturbations, (7.4) and (7.5), yield at order ε the system

$$\begin{aligned}
&\mathscr{M}^v(\{\!\{\mathbf{v}_1\}\!\},\{\!\{\mathbf{h}_1\}\!\},\{\!\{p_1\}\!\}) - \frac{\partial \mathbf{v}_0}{\partial T} + \langle\!\langle \mathbf{v}_1 \rangle\!\rangle \times (\nabla_\mathbf{x} \times \mathbf{V}) + (\nabla_\mathbf{x} \times \mathbf{H}) \times \langle\!\langle \mathbf{h}_1 \rangle\!\rangle \\
&\quad + 2\nu(\nabla_\mathbf{x} \cdot \nabla_\mathbf{X})\{\!\{\mathbf{v}_0\}\!\} + \mathbf{V} \times (\nabla_\mathbf{X} \times \mathbf{v}_0) + (\nabla_\mathbf{X} \times \mathbf{h}_0) \times \mathbf{H} \\
&\quad + \mathbf{v}_0 \times (\nabla_\mathbf{x} \times \{\!\{\mathbf{v}_0\}\!\}) + (\nabla_\mathbf{x} \times \{\!\{\mathbf{h}_0\}\!\}) \times \mathbf{h}_0 - \nabla_\mathbf{X} p_0 = 0,
\end{aligned} \tag{7.33}$$

$$\begin{aligned}
&\mathscr{M}^h(\{\!\{\mathbf{v}_1\}\!\},\{\!\{\mathbf{h}_1\}\!\}) - \frac{\partial \mathbf{h}_0}{\partial T} + (\langle\!\langle \mathbf{h}_1 \rangle\!\rangle \cdot \nabla_\mathbf{x})\mathbf{V} - (\langle\!\langle \mathbf{v}_1 \rangle\!\rangle \cdot \nabla_\mathbf{x})\mathbf{H} + 2\eta(\nabla_\mathbf{x} \cdot \nabla_\mathbf{X})\{\!\{\mathbf{h}_0\}\!\} \\
&\quad + \nabla_\mathbf{X} \times (\mathbf{v}_0 \times \mathbf{H} + \mathbf{V} \times \mathbf{h}_0) + \nabla_\mathbf{x} \times (\mathbf{v}_0 \times \mathbf{h}_0) = 0.
\end{aligned} \tag{7.34}$$

Using expressions (7.13) for the spatio-temporal mean of the image of the operator of linearisation, expressions (7.23) for the leading terms in the expansion of large-scale perturbations, and solenoidality of the leading terms (7.10) for $n = 0$, we transform the mean parts of Eqs. 7.33–7.34 to

$$-\frac{\partial \langle\!\langle \mathbf{v}_0 \rangle\!\rangle}{\partial T} + \sum_{k=1}^{3} \left(\mathbf{A}_k^{vv} \nabla_\mathbf{X} \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k + \mathbf{A}_k^{hv} \nabla_\mathbf{X} \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k \right) - \nabla_\mathbf{X} p_0^* = 0, \tag{7.35}$$

$$-\frac{\partial \langle\!\langle \mathbf{h}_0 \rangle\!\rangle}{\partial T} + \nabla_\mathbf{X} \times \sum_{k=1}^{3} \left(\mathbf{A}_k^{vh} \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k + \mathbf{A}_k^{hh} \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k \right) = 0, \tag{7.36}$$

where we have denoted

$$p_0^* \equiv \langle\!\langle p_0 \rangle\!\rangle - \left\langle\!\!\left\langle \mathbf{V} \cdot \sum_{k=1}^{3} \left(\mathbf{S}_k^{vv} \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k + \mathbf{S}_k^{hv} \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k \right) + \mathbf{H} \cdot \sum_{k=1}^{3} \left(\mathbf{S}_k^{vh} \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k + \mathbf{S}_k^{hh} \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k \right) \right\rangle\!\!\right\rangle,$$

$\mathbf{A}_k^{\cdot v}$ are 3×3 matrices with the elements

$$\begin{aligned}
(A_k^{vv})_j^m &\equiv \left\langle\!\!\left\langle -V_m((S_k^{vv})_j + \delta_k^j) - V_j((S_k^{vv})_m + \delta_k^m) + H_m(S_k^{vh})_j + H_j(S_k^{vh})_m \right\rangle\!\!\right\rangle, \\
(A_k^{hv})_j^m &\equiv \left\langle\!\!\left\langle -V_m(S_k^{hv})_j - V_j(S_k^{hv})_m + H_m((S_k^{hh})_j + \delta_k^j) + H_j((S_k^{hh})_m + \delta_k^m) \right\rangle\!\!\right\rangle,
\end{aligned}$$

and $\mathbf{A}_k^{\cdot h}$ three-dimensional vectors

$$\begin{aligned}
\mathbf{A}_k^{vh} &\equiv \langle\!\langle \mathbf{V} \times \mathbf{S}_k^{vh} - \mathbf{H} \times (\mathbf{S}_k^{vv} + \mathbf{e}_k) \rangle\!\rangle, \\
\mathbf{A}_k^{hh} &\equiv \langle\!\langle \mathbf{V} \times (\mathbf{S}_k^{hh} + \mathbf{e}_k) - \mathbf{H} \times \mathbf{S}_k^{hv} \rangle\!\rangle.
\end{aligned}$$

(The terms in (7.33)–(7.34) involving the factors $\partial \xi_0 / \partial X_m$ do not contribute to the spatio-temporal means, since they decay exponentially in time). The first-order partial differential operator in the slow variables in the l.h.s. of (7.35) (7.36), acting on the vector field $(\langle \mathbf{v}_0 \rangle, \langle \mathbf{h}_0 \rangle)$, is called the *operator of the combined MHD α-effect*. If the MHD state, whose stability is examined, is steady, then this operator coincides with the operator of the MHD α-effect, that was derived in Sect. 6.3.2.

If the operator of the α-effect is non-zero (i.e., the MHD α-effect is present in the leading order), then Eqs. 7.35 and 7.36 constitute the desirable system of mean-field equations for an MHD perturbation, supplemented by the solenoidality conditions (7.9) for $n = 0$. The nonlinear terms disappear in (7.35)–(7.36) after spatial averaging due to solenoidality of $\{\!\{\mathbf{v}_0\}\!\}$ and $\{\!\{\mathbf{h}_0\}\!\}$ in the fast variables; nonlinearity is recovered only in the higher-order averaged equations. Thus, the case of the presence of the MHD α-effect in the leading order is completely analogous to the one encountered in the linear stability problem that we explored in Sect. 6.4; we do not consider it further here.

7.4 Equations for Weakly Nonlinear Perturbations in the Absence of the MHD α-Effect

The remainder of this chapter is devoted to the study of weakly nonlinear stability to large-scale perturbations of a parity-invariant MHD regime:

$$\mathbf{V}(\mathbf{x},t) = -\mathbf{V}(-\mathbf{x},t), \qquad \mathbf{H}(\mathbf{x},t) = -\mathbf{H}(-\mathbf{x},t). \tag{7.37}$$

For such $\mathbf{V}$ and $\mathbf{H}$, the operator $\mathscr{M}$ does not break parity invariance and antiinvariance, and thus vector fields $\mathbf{S}_k^{vv}, \mathbf{S}_k^{vh}, \mathbf{S}_k^{hv}, \mathbf{S}_k^{hh}$ and scalar fields S_k^{vp}, S_k^{hp} are parity-antiinvariant solutions to auxiliary problems of type I (provided they are parity-antiinvariant at $t = 0$; a parity-invariant component in the initial conditions for $\mathbf{S}$ gives rise to a parity-invariant component of the solution, which solves the system of homogeneous equations (7.17)–(7.18) and hence, by our assumption, decays exponentially—thus, it can be included into ξ_0). Consequently, all entries of the α-tensor are zero, and the mean-field equations (7.35)–(7.36) reduce to

$$-\frac{\partial\langle\!\langle \mathbf{v}_0\rangle\!\rangle}{\partial T} - \nabla_{\mathbf{X}}\langle\!\langle p_0\rangle\!\rangle = 0, \qquad -\frac{\partial\langle\!\langle \mathbf{h}_0\rangle\!\rangle}{\partial T} = 0,$$

i.e. $\langle\!\langle \mathbf{v}_0\rangle\!\rangle$ and $\langle\!\langle \mathbf{h}_0\rangle\!\rangle$ are time-independent. This indicates that an incorrect scaling of the slow time is chosen. We assume in this section, that the slow time is defined as $T = \varepsilon^2 t$: this scaling is compatible with the one of the eigenvalues of the operator of linearisation in the absence of the α-effect (see Sect. 6.5).

7.4.1 Solution of Order ε^1 Equations

Step 1° for $n = 1$ (*continued in the absence of the α-effect*). The equations, obtained from the equations for perturbations (7.4) and (7.5) at order ε, now take the form of (7.33)–(7.34), where the partial derivatives in the slow time T are discarded. The spatio-temporal mean part of these equations becomes $\nabla_{\mathbf{X}}\langle\!\langle p_0\rangle\!\rangle = 0$, i.e. $\langle\!\langle p_0\rangle\!\rangle = 0$. Hence, the solvability condition (7.14) is satisfied.

Step 2° for $n = 1$. By linearity, the equations obtained at order 1 have solutions of the following structure:

$$\begin{pmatrix} \{\!\{\mathbf{v}_1\}\!\} \\ \{\!\{\mathbf{h}_1\}\!\} \\ \{\!\{p_1\}\!\} \end{pmatrix} = \begin{pmatrix} \xi_1^v \\ \xi_1^h \\ \xi_1^p \end{pmatrix} + \sum_{k=1}^{3} \left(\begin{pmatrix} \mathbf{S}_k^{vv} \\ \mathbf{S}_k^{vh} \\ S_k^{vp} \end{pmatrix} \langle\!\langle \mathbf{v}_1 \rangle\!\rangle_k + \begin{pmatrix} \mathbf{S}_k^{hv} \\ \mathbf{S}_k^{hh} \\ S_k^{hp} \end{pmatrix} \langle\!\langle \mathbf{h}_1 \rangle\!\rangle_k \right.$$

$$+ \sum_{m=1}^{3} \left(\begin{pmatrix} \mathbf{G}_{mk}^{vv} \\ \mathbf{G}_{mk}^{vh} \\ G_{mk}^{vp} \end{pmatrix} \frac{\partial \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k}{\partial X_m} + \begin{pmatrix} \mathbf{G}_{mk}^{hv} \\ \mathbf{G}_{mk}^{hh} \\ G_{mk}^{hp} \end{pmatrix} \frac{\partial \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k}{\partial X_m} \right.$$

$$\left. \left. + \begin{pmatrix} \mathbf{Q}_{mk}^{vvv} \\ \mathbf{Q}_{mk}^{vvh} \\ Q_{mk}^{vvp} \end{pmatrix} \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_m + \begin{pmatrix} \mathbf{Q}_{mk}^{vhv} \\ \mathbf{Q}_{mk}^{vhh} \\ Q_{mk}^{vhp} \end{pmatrix} \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_m + \begin{pmatrix} \mathbf{Q}_{mk}^{hhv} \\ \mathbf{Q}_{mk}^{hhh} \\ Q_{mk}^{hhp} \end{pmatrix} \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_m \right) \right), \tag{7.38}$$

where vector fields $\mathbf{G}(\mathbf{x}, t)$ are zero-mean solutions to auxiliary problems of type II:

Auxiliary problem II.1

$$\mathcal{M}^v(\mathbf{G}_{mk}^{vv}, \mathbf{G}_{mk}^{vh}, G_{mk}^{vp}) = -2\nu \frac{\partial \mathbf{S}_k^{vv}}{\partial x_m} - V_k \mathbf{e}_m + V_m \mathbf{e}_k - (\mathbf{V} \cdot \mathbf{S}_k^{vv}) \mathbf{e}_m$$
$$+ V_m \mathbf{S}_k^{vv} + (\mathbf{H} \cdot \mathbf{S}_k^{vh}) \mathbf{e}_m - H_m \mathbf{S}_k^{vh} + \mathbf{e}_m S_k^{vp}, \tag{7.39}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{G}_{mk}^{vv} = -(S_k^{vv})_m, \tag{7.40}$$

$$\mathcal{M}^h(\mathbf{G}_{mk}^{vv}, \mathbf{G}_{mk}^{vh}) = -2\eta \frac{\partial \mathbf{S}_k^{vh}}{\partial x_m} - \mathbf{V}(S_k^{vh})_m + V_m \mathbf{S}_k^{vh} + \mathbf{H}(S_k^{vv})_m - H_m(\mathbf{S}_k^{vv} + \mathbf{e}_k), \tag{7.41}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{G}_{mk}^{vh} = -(S_k^{vh})_m. \tag{7.42}$$

Auxiliary problem II.2

$$\mathcal{M}^v(\mathbf{G}_{mk}^{hv}, \mathbf{G}_{mk}^{hh}, G_{mk}^{hp}) = -2\nu \frac{\partial \mathbf{S}_k^{hv}}{\partial x_m} + H_k \mathbf{e}_m - H_m \mathbf{e}_k - (\mathbf{V} \cdot \mathbf{S}_k^{hv}) \mathbf{e}_m$$
$$+ V_m \mathbf{S}_k^{hv} + (\mathbf{H} \cdot \mathbf{S}_k^{hh}) \mathbf{e}_m - H_m \mathbf{S}_k^{hh} + \mathbf{e}_m S_k^{hp}, \tag{7.43}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{G}_{mk}^{hv} = -(S_k^{hv})_m, \tag{7.44}$$

$$\mathcal{M}^h(\mathbf{G}_{mk}^{hv}, \mathbf{G}_{mk}^{hh}) = -2\eta \frac{\partial \mathbf{S}_k^{hh}}{\partial x_m} - \mathbf{V}(S_k^{hh})_m + V_m(\mathbf{S}_k^{hh} + \mathbf{e}_k) + \mathbf{H}(S_k^{hv})_m - H_m \mathbf{S}_k^{hv}, \tag{7.45}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{G}_{mk}^{hh} = -(S_k^{hh})_m. \tag{7.46}$$

Vector fields $\mathbf{Q}(\mathbf{x},t)$ are zero-mean solutions to auxiliary problems of type III:
Auxiliary problem III.1

$$\mathscr{M}^v(\mathbf{Q}^{vvv}_{mk},\mathbf{Q}^{vvh}_{mk},Q^{vvp}_{mk}) = -(\mathbf{S}^{vv}_k+\mathbf{e}_k)\times(\nabla_\mathbf{x}\times\mathbf{S}^{vv}_m)-(\nabla_\mathbf{x}\times\mathbf{S}^{vh}_k)\times\mathbf{S}^{vh}_m, \quad (7.47)$$

$$\mathscr{M}^h(\mathbf{Q}^{vvv}_{mk},\mathbf{Q}^{vvh}_{mk}) = -\nabla_\mathbf{x}\times((\mathbf{S}^{vv}_k+\mathbf{e}_k)\times\mathbf{S}^{vh}_m). \quad (7.48)$$

Auxiliary problem III.2

$$\mathscr{M}^v(\mathbf{Q}^{vhv}_{mk},\mathbf{Q}^{vhh}_{mk},Q^{vhp}_{mk}) = -(\mathbf{S}^{vv}_k+\mathbf{e}_k)\times(\nabla_\mathbf{x}\times\mathbf{S}^{hv}_m)-\mathbf{S}^{hv}_m\times(\nabla_\mathbf{x}\times\mathbf{S}^{vv}_k)$$
$$-(\nabla_\mathbf{x}\times\mathbf{S}^{vh}_k)\times(\mathbf{S}^{hh}_m+\mathbf{e}_m)-(\nabla_\mathbf{x}\times\mathbf{S}^{hh}_m)\times\mathbf{S}^{vh}_k, \quad (7.49)$$

$$\mathscr{M}^h(\mathbf{Q}^{vhv}_{mk},\mathbf{Q}^{vhh}_{mk}) = -\nabla_\mathbf{x}\times((\mathbf{S}^{vv}_k+\mathbf{e}_k)\times(\mathbf{S}^{hh}_m+\mathbf{e}_m)+\mathbf{S}^{hv}_m\times\mathbf{S}^{vh}_k). \quad (7.50)$$

Auxiliary problem III.3

$$\mathscr{M}^v(\mathbf{Q}^{hhv}_{mk},\mathbf{Q}^{hhh}_{mk},Q^{hhp}_{mk}) = -\mathbf{S}^{hv}_k\times(\nabla_\mathbf{x}\times\mathbf{S}^{hv}_m)-(\nabla_\mathbf{x}\times\mathbf{S}^{hh}_k)\times(\mathbf{S}^{hh}_m+\mathbf{e}_m), \quad (7.51)$$

$$\mathscr{M}^h(\mathbf{Q}^{hhv}_{mk},\mathbf{Q}^{hhh}_{mk}) = -\nabla_\mathbf{x}\times(\mathbf{S}^{hv}_k\times(\mathbf{S}^{hh}_m+\mathbf{e}_m)). \quad (7.52)$$

All auxiliary problems of type III.3 are supplemented by the solenoidality conditions

$$\nabla_\mathbf{x}\cdot\mathbf{Q}^{\cdots}_{mk}=0. \quad (7.53)$$

An exponentially decaying in fast time field $\xi_1(\mathbf{X},T,\mathbf{x},t)$ satisfies the equations

$$\mathscr{M}^v(\xi^v_1,\xi^h_1,\xi^p_1) = -2\nu(\nabla_\mathbf{x}\cdot\nabla_\mathbf{X})\xi^v_0-\mathbf{V}\times(\nabla_\mathbf{X}\times\xi^v_0)$$
$$-(\nabla_\mathbf{X}\times\xi^h_0)\times\mathbf{H}-\xi^v_0\times(\nabla_\mathbf{x}\times\mathbf{v}_0)-(\nabla_\mathbf{x}\times\mathbf{h}_0)\times\xi^h_0$$
$$-(\mathbf{v}_0-\xi^v_0)\times(\nabla_\mathbf{x}\times\xi^v_0)(\nabla_\mathbf{x}\times\xi^h_0)\times(\mathbf{h}_0-\xi^h_0)+\nabla_\mathbf{X}\xi^p_0, \quad (7.54)$$

$$\nabla_\mathbf{x}\cdot\xi^v_1+\nabla_\mathbf{X}\cdot\xi^v_0=0, \quad (7.55)$$

$$\mathscr{M}^h(\xi^v_1,\xi^h_1) = -2\eta(\nabla_\mathbf{x}\cdot\nabla_\mathbf{X})\xi^h_0-(\mathbf{H}\cdot\nabla_\mathbf{X})\xi^v_0+(\mathbf{V}\cdot\nabla_\mathbf{X})\xi^h_0-\mathbf{V}\nabla_\mathbf{X}\cdot\xi^h_0+\mathbf{H}\nabla_\mathbf{X}\cdot\xi^v_0,$$
$$-(\xi^h_0\cdot\nabla_\mathbf{x})\mathbf{v}_0+(\xi^v_0\cdot\nabla_\mathbf{x})\mathbf{h}_0-((\mathbf{h}_0-\xi^h_0)\cdot\nabla_\mathbf{x})\xi^v_0+((\mathbf{v}_0-\xi^v_0)\cdot\nabla_\mathbf{x})\xi^h_0, \quad (7.56)$$

$$\nabla_\mathbf{x}\cdot\xi^h_1+\nabla_\mathbf{X}\cdot\xi^h_0=0. \quad (7.57)$$

The r.h.s. of Eqs. 7.39–7.52 have zero spatial means, because the MHD regime $\mathbf{V},\mathbf{H},P$ is parity-invariant, and the solutions to the auxiliary problems of type I are parity-antiinvariant. For the same reason, the relations (7.40), (7.42), (7.44) and (7.46), specifying the divergencies of solutions to auxiliary problems of type II, imply $\langle\!\langle\mathbf{V}\nabla_\mathbf{x}\cdot\mathbf{G}^{\cdot v}_{mk}\rangle\!\rangle=\langle\!\langle\mathbf{H}\nabla_\mathbf{x}\cdot\mathbf{G}^{\cdot h}_{mk}\rangle\!\rangle=0$. Also, $\langle\!\langle\mathbf{V}\nabla_\mathbf{x}\cdot\xi^v_1\rangle\!\rangle=\langle\!\langle\mathbf{H}\nabla_\mathbf{x}\cdot\xi^h_1\rangle\!\rangle=0$ in view of relations (7.55) and (7.57) and because the divergencies of ξ^v_0 and ξ^h_0

exponentially decay in fast time. Hence, the solvability conditions (7.14) are satisfied for each of the six problems, stated by Eqs. 7.39–7.57.

The expressions (7.40), (7.42), (7.44), (7.46), (7.53), (7.55) and (7.57) for the divergencies of solutions to the auxiliary problems and ξ_1 guarantee that the relations (7.10) for $n=1$ for the divergencies of $\mathbf{v}_1$ and $\mathbf{h}_1$ are satisfied. Taking the divergencies of equations (7.41) and (7.45), and combining the resultant equalities with, respectively, Eqs. 7.26 and 7.30 in the statements of the auxiliary problems of type I, we find

$$\left(-\frac{\partial}{\partial t}+\eta\nabla^2_{\mathbf{x}}\right)\left(\nabla_{\mathbf{x}}\cdot\mathbf{G}^{\cdot h}_{mk}+(S^{\cdot h}_{k})_m\right)=0;$$

hence, Eqs. 7.42 and 7.46 are satisfied at any $t>0$, if they are satisfied at $t=0$. From the divergencies of equations (7.48), (7.50) and (7.52) we obtain that, similarly, the fields $\mathbf{Q}$ are solenoidal at any $t>0$, provided they are solenoidal at $t=0$. From the divergence of equation (7.56) in the fast variables we find, employing the identity $\mathscr{M}^h(\xi^v_0,\xi^h_0)=0$ (see (7.32)),

$$\begin{aligned}\left(-\frac{\partial}{\partial t}+\eta\nabla^2_{\mathbf{x}}\right)(\nabla_{\mathbf{x}}\cdot\xi^h_1)&=-\nabla_{\mathbf{x}}\cdot\left(\nabla_{\mathbf{X}}\times(\xi^v_0\times\mathbf{H}+\mathbf{V}\times\xi^h_0)\right)\\&=\nabla_{\mathbf{X}}\cdot\left(\nabla_{\mathbf{x}}\times(\xi^v_0\times\mathbf{H}+\mathbf{V}\times\xi^h_0)\right)\\&=-\left(-\frac{\partial}{\partial t}+\eta\nabla^2_{\mathbf{x}}\right)(\nabla_{\mathbf{X}}\cdot\xi^h_0),\end{aligned}$$

whereby relation (7.57) holds true for any $t>0$, if and only if it holds true at $t=0$.

Consequently, any pair of vector fields can serve as initial conditions for auxiliary problems of types II and III, if they are globally bounded together with their derivatives and have zero spatio-temporal means, their divergencies satisfy relations (7.40), (7.42), (7.44), (7.46) and (7.53), and the resultant solution is globally bounded in space and time. It is natural to choose parity-invariant initial conditions: since $\mathscr{M}$ preserves parity invariance and antiinvariance, the solutions $\mathbf{G}$ and $\mathbf{Q}$ are parity-invariant, provided they possess this symmetry at $t=0$. Zero-mean parity-antiinvariant components in the initial conditions for $\mathbf{G}$ and $\mathbf{Q}$ give rise to parity-invariant components, which solve the system of homogeneous equations (7.17)–(7.18) and hence, by our assumption, decay exponentially—thus, we include them into ξ_1. If the MHD regime, whose stability is examined, is steady, periodic or quasi-periodic in space and/or time, it is convenient to demand that $\mathbf{G}$ and $\mathbf{Q}$ have the same property. (If the MHD state is steady, then the auxiliary problems of types I and II are identical to the ones considered in the previous chapter).

Initial conditions for ξ_1 can be determined from the initial conditions for $\mathbf{v}_1,\mathbf{h}_1$ and solutions to the auxiliary problems. Integrating the evolutionary equation (7.54) in the fast time, considering the spatio-temporal mean of the result and using expressions (7.11)–(7.12) for the spatial mean of the image of the operator of linearisation $\mathscr{M}$ and expressions (7.55) and (7.57) for the divergencies of the components of ξ_1, we find from the condition $\langle\!\langle \xi_1^v \rangle\!\rangle = 0$ that

$$\langle \xi_1^v \rangle|_{t=0} = \left\langle\!\!\left\langle \int_0^t \left(\mathbf{V}\nabla_{\mathbf{X}} \cdot \xi_0^v - \mathbf{H}\nabla_{\mathbf{X}} \cdot \xi_0^h \right.\right.\right.$$

$$\left.\left.\left. - \mathbf{V} \times (\nabla_{\mathbf{X}} \times \xi_0^v) - (\nabla_{\mathbf{X}} \times \xi_0^h) \times \mathbf{H} + \nabla_{\mathbf{X}} \xi_0^p \right) \mathrm{d}t \right\rangle\!\!\right\rangle.$$

Similarly, (7.56) and the condition $\langle\!\langle \xi_1^h \rangle\!\rangle = 0$ yield

$$\langle \xi_1^h \rangle|_{t=0} = -\left\langle\!\!\left\langle \int_0^t \nabla_{\mathbf{X}} \times (\mathbf{V} \times \xi_0^h + \xi_0^v \times \mathbf{H}) \mathrm{d}t \right\rangle\!\!\right\rangle.$$

Any pair of globally bounded smooth fields satisfying relations (7.55) and (7.57) for the divergencies, whose spatial means are equal to the computed above values, can serve as initial conditions for ξ_1, if the respective solution decays exponentially in the fast time. Since $\langle \mathbf{S} \rangle = 0$ and the fields $\mathbf{G}$ and $\mathbf{Q}$ are parity-invariant, the spatial mean of every sum in k in the hydrodynamic and magnetic components of (7.38) vanishes, and hence

$$\langle \mathbf{v}_1 \rangle = \langle\!\langle \mathbf{v}_1 \rangle\!\rangle + \langle \xi_1^v \rangle, \qquad \langle \mathbf{h}_1 \rangle = \langle\!\langle \mathbf{h}_1 \rangle\!\rangle + \langle \xi_1^h \rangle.$$

For given $\mathbf{v}_1|_{t=0}$ and $\mathbf{h}_1|_{t=0}$, we determine at $t = 0$ from these equations $\langle\!\langle \mathbf{v}_1 \rangle\!\rangle|_{T=0}$ and $\langle\!\langle \mathbf{h}_1 \rangle\!\rangle|_{T=0}$; subsequently, from the flow and magnetic components of (7.38) (where $\{\!\{ \mathbf{v}_1 \}\!\} = \mathbf{v}_1 - \langle\!\langle \mathbf{v}_1 \rangle\!\rangle$ and $\{\!\{ \mathbf{h}_1 \}\!\} = \mathbf{h}_1 - \langle\!\langle \mathbf{h}_1 \rangle\!\rangle$) we find initial conditions for the problem (7.54)–(7.57). A change in initial conditions for $\mathbf{G}$ and $\mathbf{Q}$ in the class of fields under consideration is compensated by a respective change in initial conditions for ξ_1, but the respective changes in $\mathbf{G}$ and $\mathbf{Q}$ exponentially decay in fast time (by construction, being zero-mean solutions to the problem (7.17)–(7.18)—recall that an exponential growth of any solution to auxiliary problems is forbidden, and we have assumed that any non-growing zero-mean solution to the homogeneous equations (7.17)–(7.18) exponentially decay in time).

7.4.2 Bounds for Linear Problems with the Operator of Linearisation

We assume in this subsection that the MHD regime $\mathbf{V}, \mathbf{H}, P$ is periodic in the fast spatial variables and stable to short-scale perturbations. Stability is understood as an exponential decay in the fast time of the norms $|\cdot|_m$, for all $m > 0$, of any solution to the problem (7.17)–(7.18) with a zero spatial mean, where $|\cdot|_m$ is the norm in the Sobolev space $\mathbb{W}_2^m([0, \mathbf{L}])$, $\mathbf{L}$ denoting, as before, the vector of spatial periods of the MHD regime. The fields $\mathbf{V}, \mathbf{H}, P$ are supposed to be infinitely differentiable. We will address the question of the global boundedness (in time t) of $\mathbf{L}$-periodic in $\mathbf{x}$ solutions to the problem

$$\mathscr{M}\xi = \mathbf{f}, \quad \xi = (\xi^v, \xi^h), \quad \nabla_{\mathbf{x}} \cdot \xi^v = d^v, \quad \nabla_{\mathbf{x}} \cdot \xi^h = d^h, \tag{7.58}$$

applying Duhamel's principle [232]. Furthermore, we will show that the solutions decay exponentially in time t, if $\mathbf{f}, d^v$ and d^h have this property. (Slow variables play the rôle of passive parameters in this subsection, and we do not indicate the possible dependence of the data on the slow variables).

We reduce (7.58) to a problem with solenoidal unknown fields,

$$\mathscr{M}\tilde{\xi} = \tilde{f}, \quad \tilde{\xi} = (\tilde{\xi}^v, \tilde{\xi}^h), \quad \nabla_{\mathbf{x}} \cdot \tilde{\xi}^v = \nabla_{\mathbf{x}} \cdot \tilde{\xi}^h = 0 \tag{7.59}$$

by substitutions

$$\xi^v = \tilde{\xi}^v + \nabla_{\mathbf{x}} \xi^v, \quad \xi^h = \tilde{\xi}^h + \nabla_{\mathbf{x}} \xi^h, \tag{7.60}$$

where ξ is comprised of $\mathbf{L}$-periodic solutions to the Laplace equations

$$\nabla^2_{\mathbf{x}} \xi^v + d^v = 0, \quad \nabla^2_{\mathbf{x}} \xi^h + d^h = 0,$$

whose spatial means are zero. Evidently, the bounds

$$|\xi^v|_{m+2} \le C_{m,\mathbf{L}} |d^v|_m, \quad |\xi^h|_{m+2} \le C_{m,\mathbf{L}} |d^h|_m, \tag{7.61}$$

hold true, and thus

$$|\tilde{\mathbf{f}}|_m \le |\mathbf{f}|_m + C'_{m,\mathbf{L}}(|d^v|_m + |d^h|_m),$$

where the constants $C_{m,\mathbf{L}}$ and $C'_{m,\mathbf{L}}$ depend on m and the size of the parallelepiped of periodicity, and $C'_{m,\mathbf{L}}$ depends on the fields $\mathbf{V}$ and $\mathbf{H}$. (space periodicity of the regime $\mathbf{V}, \mathbf{H}, P$ and solutions to the problems (7.58) and (7.59) is necessary in this subsection to justify the inequalities (7.61)).

Linear stability of the regime $\mathbf{V}, \mathbf{H}, P$ implies that, for any globally bounded solenoidal zero-mean initial conditions, a solution ξ to the equations

$$\mathscr{M}\xi = 0, \quad \xi = (\xi^v, \xi^h), \quad \nabla_{\mathbf{x}} \cdot \xi^v = \nabla_{\mathbf{x}} \cdot \xi^h = 0 \tag{7.62}$$

satisfies for all $t_1 \ge t_2 \ge 0$ and m the inequality

$$|\xi(\mathbf{x}, t_1)|_m \le C_m \mathrm{e}^{-\alpha(t_1 - t_2)} |\xi(\mathbf{x}, t_2)|_m \tag{7.63}$$

for some constant C_m and $\alpha > 0$, independent of t_1, t_2 and ξ.

Let us demonstrate that a solution to the problem (7.58) is bounded, if $\langle \mathbf{f} \rangle = 0$, and $\mathbf{f}, d^v$ and d^h are globally bounded. We split a solution to (7.59) into a sum $\tilde{\xi} = \xi_I + \xi_{II}$, where ξ_I is a solution to the homogeneous problem (7.62) for zero-mean initial conditions $\xi_I|_{t=0} = \tilde{\xi}(\mathbf{x}, 0)$, and ξ_{II} a solution to the problem

$$\mathscr{M}\xi_{II} = \tilde{\mathbf{f}}(\mathbf{x}, t), \quad \nabla_{\mathbf{x}} \cdot \xi^v_{II} = \nabla_{\mathbf{x}} \cdot \xi^h_{II} = 0, \quad \xi_{II}|_{t=0} = 0. \tag{7.64}$$

By virtue of (7.63),

$$|\xi_I(\mathbf{x},t)|_m \le C_m \mathrm{e}^{-\alpha t} |\tilde{\xi}(\mathbf{x},0)|_m. \tag{7.65}$$

By Duhamel's principle,

$$\xi_{II} = \int_0^t \xi'(\mathbf{x},t,\tau)\mathrm{d}\tau, \tag{7.66}$$

where $\xi'(\mathbf{x},t,\tau)$ is a solution to the problem (7.62) for $t > \tau$ with the initial conditions $\xi'(\mathbf{x},t,\tau)|_{t=\tau} = \tilde{\mathbf{f}}(\mathbf{x},\tau)$. The inequality (7.63) implies

$$|\xi'(\mathbf{x},t,\tau)|_m \le C_m \mathrm{e}^{-\alpha(t-\tau)} |\tilde{\mathbf{f}}(\mathbf{x},\tau)|_m, \tag{7.67}$$

whereby

$$\begin{aligned} |\xi_{II}(\mathbf{x},t)|_m &\le \frac{C_m}{\alpha} \max_{\tau \le t} |\tilde{\mathbf{f}}(\mathbf{x},\tau)|_m \\ &\le \frac{C_m}{\alpha} \max_{\tau \le t} \Big(|\mathbf{f}(\mathbf{x},\tau)|_m + C'_{m,\mathbf{L}} (|d^v(\mathbf{x},\tau)|_m + |d^h(\mathbf{x},\tau)|_m) \Big). \end{aligned} \tag{7.68}$$

The inequalities (7.65) and (7.68) show that linear stability of the regime $\mathbf{V}, \mathbf{H}, P$ implies the boundedness in time of solutions to the auxiliary problems and their derivatives, for any zero-mean initial conditions satisfying the respective conditions for the divergence. (By virtue of the Sobolev embedding theorem [1, 170, 182, 299] this also implies the boundedness of the solutions in the norms of spaces of continuous functions $\mathbb{C}^m([0,\mathbf{L}])$).

We will show now that solutions to the problem (7.58), whose spatio-temporal mean is zero, decay exponentially in time t, if $\mathbf{f}, d^v$ and d^h do so. This establishes that the transients ξ_1 solving the problem (7.54)–(7.57) exponentially decay in time, as well as modifications in the fields $\mathbf{G}$ and $\mathbf{Q}$ caused by permissible changes in the initial conditions for $\mathbf{S}$.

As above, we make the substitutions (7.60) reducing (7.58) to a problem (7.59) with solenoidal unknown fields. Averaging the resultant equation (7.59) in the fast spatial variables and taking into account expressions (7.11) and (7.12) for the spatial means of the image of the operator of linearisation, we obtain

$$-\frac{\partial\langle\tilde{\xi}\rangle}{\partial t} = \langle\tilde{\mathbf{f}}\rangle \quad \Rightarrow \quad -\langle\tilde{\xi}\rangle|_{t=t_1} + \langle\tilde{\xi}\rangle|_{t=0} = \int_0^{t_1} \langle\tilde{\mathbf{f}}\rangle \mathrm{d}t.$$

Together with the condition $\langle\!\langle\tilde{\xi}\rangle\!\rangle = 0$ this implies

$$\langle\tilde{\xi}\rangle|_{t=0} = \lim_{\tilde{t}\to\infty} \frac{1}{\tilde{t}} \int_0^{\tilde{t}} \int_0^{t_1} \langle\tilde{\mathbf{f}}\rangle \mathrm{d}t\, \mathrm{d}t_1,$$

and hence

$$\langle\tilde{\xi}\rangle|_{t=t_2} = \lim_{\tilde{t}\to\infty}\frac{1}{\tilde{t}}\int_0^{\tilde{t}}\int_{t_2}^{t_1}\langle\tilde{\mathbf{f}}\rangle \mathrm{d}t\,\mathrm{d}t_1 = \lim_{\tilde{t}\to\infty}\frac{1}{\tilde{t}}\int_{t_2}^{\tilde{t}}\int_{t_2}^{t_1}\langle\tilde{\mathbf{f}}\rangle \mathrm{d}t\,\mathrm{d}t_1$$

$$\Rightarrow \qquad |\langle\tilde{\xi}\rangle|_{t=t_2}| \le \int_{t_2}^{\infty}|\langle\tilde{\mathbf{f}}\rangle|\mathrm{d}t.$$

Due to the exponential decay of $\mathbf{f}, d^v$ and d^h,

$$|\langle\tilde{\mathbf{f}}(\mathbf{x},t)\rangle| \le C_{\tilde{\mathbf{f}}}\mathrm{e}^{-\alpha t},$$

where $C_{\tilde{\mathbf{f}}}$ is a constant (without any loss of generality we assume that the exponent in this bound is the same as in (7.63). We find therefore

$$|\langle\tilde{\xi}\,(\mathbf{x},t_2)\rangle| \le (C_{\tilde{\mathbf{f}}}/\alpha)\mathrm{e}^{-\alpha t_2}$$

for any $t_2 \ge 0$.

It remains to derive an analogous inequality for the fluctuating part of the solution to the problem (7.59). We substitute $\tilde{\xi} = \{\tilde{\xi}\} + \langle\tilde{\xi}\rangle$ and again denote by $\tilde{\mathbf{f}}$ the r.h.s. of the equation in $\{\tilde{\xi}\}$:

$$\mathscr{M}\{\tilde{\xi}\} = \tilde{\mathbf{f}}, \quad \langle\tilde{\mathbf{f}}\rangle = 0.$$

$\{\tilde{\xi}\}$ has solenoidal components $\{\tilde{\xi}^v\}$ and $\{\tilde{\xi}^h\}$. We split $|\{\tilde{\xi}\}|_m$ further into a sum $\{\tilde{\xi}\} = \xi_I + \xi_{II}$, where ξ_I is a solution to the homogeneous problem (7.62) with the initial conditions $\xi_I|_{t=0} = \{\tilde{\xi}\,(\mathbf{x},0)\}$, and ξ_{II} is a solution to (7.64). Combining the bound

$$|\tilde{\mathbf{f}}(\mathbf{x},\tau)|_m \le C_{m,\tilde{\mathbf{f}}}\mathrm{e}^{-\alpha\tau},$$

where $C_{m,\tilde{\mathbf{f}}}$ is a time-independent constant, and the inequality (7.67) for solutions to the problem (7.62) satisfying the initial conditions $\xi'(\mathbf{x},t,\tau)|_{t=\tau} = \tilde{\mathbf{f}}(\mathbf{x},\tau)$, we obtain

$$|\xi'(\mathbf{x},t,\tau)|_m \le C_m\mathrm{e}^{-\alpha(t-\tau)}C_{m,\tilde{\mathbf{f}}}\mathrm{e}^{-\alpha\tau} = C_m C_{m,\tilde{\mathbf{f}}}\mathrm{e}^{-\alpha t}.$$

Thus, for any $\alpha' < \alpha$ the integral (7.66) admits the bound

$$|\xi_{II}(\mathbf{x},t)|_m \le C_m C_{m,\tilde{\mathbf{f}}}t\,\mathrm{e}^{-\alpha t} \le C'\mathrm{e}^{-\alpha' t}.$$

This inequality together with the inequality (7.65) demonstrates that ξ_1 decays exponentially, as desired.

7.4.3 Mean-Field Equations for Large-Scale Perturbations

At order ε^2, the Eqs. 7.4 and 7.5 for the evolution of a perturbation yield

$$\begin{aligned}
&\mathscr{M}^v(\{\!\{\mathbf{v}_2\}\!\},\{\!\{\mathbf{h}_2\}\!\},\{\!\{p_2\}\!\})-\frac{\partial\mathbf{v}_0}{\partial T}+\nu(2(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\{\!\{\mathbf{v}_1\}\!\}+\nabla^2_{\mathbf{X}}\mathbf{v}_0)+\langle\!\langle\mathbf{v}_2\rangle\!\rangle\times(\nabla_{\mathbf{x}}\times\mathbf{V})\\
&\quad+(\nabla_{\mathbf{x}}\times\mathbf{H})\times\langle\!\langle\mathbf{h}_2\rangle\!\rangle+\mathbf{V}\times(\nabla_{\mathbf{X}}\times\mathbf{v}_1)+(\nabla_{\mathbf{X}}\times\mathbf{h}_1)\times\mathbf{H}\\
&\quad+\mathbf{v}_0\times(\nabla_{\mathbf{x}}\times\{\!\{\mathbf{v}_1\}\!\})+\mathbf{v}_1\times(\nabla_{\mathbf{x}}\times\{\!\{\mathbf{v}_0\}\!\})+\mathbf{v}_0\times(\nabla_{\mathbf{X}}\times\mathbf{v}_0)\\
&\quad+(\nabla_{\mathbf{x}}\times\{\!\{\mathbf{h}_0\}\!\})\times\mathbf{h}_1+(\nabla_{\mathbf{x}}\times\{\!\{\mathbf{h}_1\}\!\})\times\mathbf{h}_0+(\nabla_{\mathbf{X}}\times\mathbf{h}_0)\times\mathbf{h}_0-\nabla_{\mathbf{X}}p_1=0,\\
&\mathscr{M}^h(\{\!\{\mathbf{v}_2\}\!\},\{\!\{\mathbf{h}_2\}\!\})-\frac{\partial\mathbf{h}_0}{\partial T}+\eta\big(2(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\{\!\{\mathbf{h}_0\}\!\}+\nabla^2_{\mathbf{X}}\mathbf{h}_0\big)+(\langle\!\langle\mathbf{h}_2\rangle\!\rangle\cdot\nabla_{\mathbf{x}})\mathbf{V}\\
&\quad-(\langle\!\langle\mathbf{v}_2\rangle\!\rangle\cdot\nabla_{\mathbf{x}})\mathbf{H}+\nabla_{\mathbf{X}}\times(\mathbf{v}_1\times\mathbf{H}+\mathbf{V}\times\mathbf{h}_1+\mathbf{v}_0\times\mathbf{h}_0)\\
&\quad+\nabla_{\mathbf{x}}\times(\mathbf{v}_1\times\mathbf{h}_0+\mathbf{v}_0\times\mathbf{h}_1)=0.
\end{aligned}$$

Step $2°$ for $n=2$. The spatio-temporal means of these equations can be obtained using expressions (7.13) for the spatio-temporal mean of the image of the operator of linearisation, solenoidality of the averaged terms of the expansion of the perturbation (7.7) in the slow variables (see (7.9) for $n=0$), and expressions for divergencies in the fast variables (see (7.10) for $n=1$, 2):

$$\begin{aligned}
&-\frac{\partial}{\partial T}\langle\!\langle\mathbf{v}_0\rangle\!\rangle+\nu\nabla^2_{\mathbf{X}}\langle\!\langle\mathbf{v}_0\rangle\!\rangle\\
&-\sum_{j=1}^{3}\frac{\partial}{\partial X_j}\Big\langle\!\!\Big\langle V_j\{\!\{\mathbf{v}_1\}\!\}+\mathbf{V}\Big\{\!\!\Big\{(v_1)_j\Big\}\!\!\Big\}+(v_0)_j\mathbf{v}_0-H_j\{\!\{\mathbf{h}_1\}\!\}-\mathbf{H}\Big\{\!\!\Big\{(h_1)_j\Big\}\!\!\Big\}-(h_0)_j\mathbf{h}_0\Big\rangle\!\!\Big\rangle\\
&-\nabla_{\mathbf{X}}p^{\diamond}=0
\end{aligned}$$

(here we have denoted $p^{\diamond}(\mathbf{X},T)=\Big\langle\!\!\Big\langle p_1-(|\mathbf{v}_0|^2-|\mathbf{h}_0|^2)/2-\mathbf{V}\cdot\mathbf{v}_1+\mathbf{H}\cdot\mathbf{h}_1\Big\rangle\!\!\Big\rangle$),

$$-\frac{\partial}{\partial T}\langle\!\langle\mathbf{h}_0\rangle\!\rangle+\eta\nabla^2_{\mathbf{X}}\langle\!\langle\mathbf{h}_0\rangle\!\rangle+\nabla_{\mathbf{X}}\times\langle\!\langle\{\!\{\mathbf{v}_1\}\!\}\times\mathbf{H}+\mathbf{V}\times\{\!\{\mathbf{h}_1\}\!\}+\mathbf{v}_0\times\mathbf{h}_0\rangle\!\rangle=0.$$

Since the operator of the α-effect is zero and the transients $\boldsymbol{\xi}_0$ and $\boldsymbol{\xi}_1$ exponentially decay, and by virtue of the expressions (7.23) and (7.38) for the first two terms in the power series expansion of the perturbation (7.7), the two equations reduce to the following ones:

The mean-field equation for the mean perturbation of the flow

$$\begin{aligned}
&-\frac{\partial}{\partial T}\langle\!\langle\mathbf{v}_0\rangle\!\rangle+\nu\nabla^2_{\mathbf{X}}\langle\!\langle\mathbf{v}_0\rangle\!\rangle+\sum_{j=1}^{3}\sum_{m=1}^{3}\sum_{k=1}^{3}\Bigg(\mathbf{D}^{vv}_{mkj}\frac{\partial^2\langle\!\langle\mathbf{v}_0\rangle\!\rangle_k}{\partial X_j\partial X_m}+\mathbf{D}^{hv}_{mkj}\frac{\partial^2\langle\!\langle\mathbf{h}_0\rangle\!\rangle_k}{\partial X_j\partial X_m}\\
&+\frac{\partial}{\partial X_j}\Big(\mathbf{A}^{vvv}_{mkj}\langle\!\langle\mathbf{v}_0\rangle\!\rangle_k\langle\!\langle\mathbf{v}_0\rangle\!\rangle_m+\mathbf{A}^{vhv}_{mkj}\langle\!\langle\mathbf{v}_0\rangle\!\rangle_k\langle\!\langle\mathbf{h}_0\rangle\!\rangle_m+\mathbf{A}^{hhv}_{mkj}\langle\!\langle\mathbf{h}_0\rangle\!\rangle_k\langle\!\langle\mathbf{h}_0\rangle\!\rangle_m\Big)\Bigg)\\
&+(\langle\!\langle\mathbf{v}_0\rangle\!\rangle\cdot\nabla_{\mathbf{X}})\langle\!\langle\mathbf{v}_0\rangle\!\rangle-(\langle\!\langle\mathbf{h}_0\rangle\!\rangle\cdot\nabla_{\mathbf{X}})\langle\!\langle\mathbf{h}_0\rangle\!\rangle-\nabla_{\mathbf{X}}p^{\diamond}=0.
\end{aligned}\tag{7.69}$$

The mean-field equation for the mean perturbation of magnetic field

$$-\frac{\partial}{\partial T}\langle\!\langle \mathbf{h}_0\rangle\!\rangle+\eta\nabla^2_{\mathbf{X}}\langle\!\langle \mathbf{h}_0\rangle\!\rangle+\nabla_{\mathbf{X}}\times\Bigg(\langle\!\langle \mathbf{v}_0\rangle\!\rangle\times\langle\!\langle \mathbf{h}_0\rangle\!\rangle+\sum_{m=1}^{3}\sum_{k=1}^{3}\Bigg(\mathbf{D}^{vh}_{mk}\frac{\partial\langle\!\langle \mathbf{v}_0\rangle\!\rangle_k}{\partial X_m}+\mathbf{D}^{hh}_{mk}\frac{\partial\langle\!\langle \mathbf{h}_0\rangle\!\rangle_k}{\partial X_m}$$
$$+\mathbf{A}^{vvh}_{mk}\langle\!\langle \mathbf{v}_0\rangle\!\rangle_k\langle\!\langle \mathbf{v}_0\rangle\!\rangle_m+\mathbf{A}^{vhh}_{mk}\langle\!\langle \mathbf{v}_0\rangle\!\rangle_k\langle\!\langle \mathbf{h}_0\rangle\!\rangle_m+\mathbf{A}^{hhh}_{mk}\langle\!\langle \mathbf{h}_0\rangle\!\rangle_k\langle\!\langle \mathbf{h}_0\rangle\!\rangle_m\Bigg)\Bigg)=0. \quad (7.70)$$

In these equations $\mathbf{D}$ denote the coefficients of the so-called *operator of combined eddy correction of magnetic diffusion and kinematic viscosity*:

$$\mathrm{D}^{vv}_{mkj}=\left\langle\!\!\left\langle -V_j\mathbf{G}^{vv}_{mk}-\mathbf{V}(G^{vv}_{mk})_j+H_j\mathbf{G}^{vh}_{mk}+\mathbf{H}(G^{vh}_{mk})_j\right\rangle\!\!\right\rangle, \quad (7.71)$$

$$\mathrm{D}^{hv}_{mkj}=\left\langle\!\!\left\langle -V_j\mathbf{G}^{hv}_{mk}-\mathbf{V}(G^{hv}_{mk})_j+H_j\mathbf{G}^{hh}_{mk}+\mathbf{H}(G^{hh}_{mk})_j\right\rangle\!\!\right\rangle, \quad (7.72)$$

$$\mathbf{D}^{vh}_{mk}=\langle\!\langle \mathbf{V}\times\mathbf{G}^{vh}_{mk}-\mathbf{H}\times\mathbf{G}^{vv}_{mk}\rangle\!\rangle, \quad (7.73)$$

$$\mathbf{D}^{hh}_{mk}=\langle\!\langle \mathbf{V}\times\mathbf{G}^{hh}_{mk}-\mathbf{H}\times\mathbf{G}^{hv}_{mk}\rangle\!\rangle. \quad (7.74)$$

The quantities (7.71)–(7.74) are identical to the entries of the tensor of combined eddy correction of magnetic diffusion and kinematic viscosity in the linear stability problem, see Sect. 6.5.2. $\mathbf{A}$ denote the coefficients of the quadratic *operator of combined eddy correction of advection*:

$$\mathrm{A}^{vvv}_{mkj}=\left\langle\!\!\left\langle -V_j\mathbf{Q}^{vvv}_{mk}-\mathbf{V}(Q^{vvv}_{mk})_j+H_j\mathbf{Q}^{vvh}_{mk}+\mathbf{H}(Q^{vvh}_{mk})_j+(S^{vv}_k)_j\mathbf{S}^{vv}_m-(S^{vh}_k)_j\mathbf{S}^{vh}_m\right\rangle\!\!\right\rangle, \quad (7.75)$$

$$\mathrm{A}^{vhv}_{mkj}=\Big\langle\!\!\Big\langle -V_j\mathbf{Q}^{vhv}_{mk}-\mathbf{V}(Q^{vhv}_{mk})_j+H_j\mathbf{Q}^{vhh}_{mk}+\mathbf{H}(Q^{vhh}_{mk})_j$$
$$+(S^{vv}_k)_j\mathbf{S}^{hv}_m+(S^{hv}_m)_j\mathbf{S}^{vv}_k-(S^{vh}_k)_j\mathbf{S}^{vh}_m-(S^{hh}_m)_j\mathbf{S}^{vh}_k\Big\rangle\!\!\Big\rangle, \quad (7.76)$$

$$\mathrm{A}^{hhv}_{mkj}=\left\langle\!\!\left\langle -V_j\mathbf{Q}^{hhv}_{mk}-\mathbf{V}(Q^{hhv}_{mk})_j+H_j\mathbf{Q}^{hhh}_{mk}+\mathbf{H}(Q^{hhh}_{mk})_j+(S^{hv}_k)_j\mathbf{S}^{hv}_m-(S^{hh}_k)_j\mathbf{S}^{hh}_m\right\rangle\!\!\right\rangle, \quad (7.77)$$

$$\mathbf{A}^{vvh}_{mk}=\langle\!\langle \mathbf{V}\times\mathbf{Q}^{vvh}_{mk}-\mathbf{H}\times\mathbf{Q}^{vvv}_{mk}+\mathbf{S}^{vv}_k\times\mathbf{S}^{vh}_m\rangle\!\rangle, \quad (7.78)$$

$$\mathbf{A}^{vhh}_{mk}=\langle\!\langle \mathbf{V}\times\mathbf{Q}^{vhh}_{mk}-\mathbf{H}\times\mathbf{Q}^{vhv}_{mk}+\mathbf{S}^{vv}_k\times\mathbf{S}^{hh}_m+\mathbf{S}^{hv}_m\times\mathbf{S}^{vh}_k\rangle\!\rangle, \quad (7.79)$$

$$\mathbf{A}^{hhh}_{mk}=\langle\!\langle \mathbf{V}\times\mathbf{Q}^{hhh}_{mk}-\mathbf{H}\times\mathbf{Q}^{hhv}_{mk}+\mathbf{S}^{hv}_k\times\mathbf{S}^{hh}_m\rangle\!\rangle. \quad (7.80)$$

In the review [197], the mean-field equations (7.69)–(7.70) are termed "order parameter equations for slowly modulated patterns". They involve additional "eddy" terms and in this sense generalise the Navier–Stokes (7.1) and magnetic induction (7.2) equations. Like in the linear stability problem, the operator of the combined MHD eddy diffusion emerges in them in place of the Laplace operator describing molecular diffusion. The equations cease to be parabolic, if the phenomenon of negative MHD eddy diffusivity occurs in the regime $\mathbf{V}, \mathbf{H}, P$.

7.5 Computation of Coefficients of the Combined MHD Eddy Operators

For computation of the eddy coefficients $\mathbf{D}$ and $\mathbf{A}$ in the mean-field equations (7.69)–(7.70) using the formulae (7.71)–(7.80), it suffices to solve 45 auxiliary problems (6, 18 and 21 problems of types I–III, respectively; formally, Eqs. 7.47–7.53 state 27 problems of type III, however, the fields $\mathbf{Q}^{vv,\cdot}_{mk}$ and $\mathbf{Q}^{hh,\cdot}_{mk}$ enter into the coefficients in Eqs. 7.69 and 7.70 not individually, but as sums $\mathbf{Q}^{vv,\cdot}_{mk} + \mathbf{Q}^{vv,\cdot}_{km}$ and $\mathbf{Q}^{hh,\cdot}_{mk} + \mathbf{Q}^{hh,\cdot}_{km}$, and hence the numbers of the auxiliary problems III.1 and III.3 can be decreased by 3, if these problems are restated for such sums for $m \neq k$). This number can be decreased three times, considering (following [327], see Sect. 4.4) auxiliary problems for the adjoint operator:

Auxiliary problem 1 for the adjoint operator

$$(\mathscr{M}^*)^v(\mathbf{Z}^{vv}_{jn}, \mathbf{Z}^{vh}_{jn}) = -\mathscr{P}_{\text{sol}}(V_j\mathbf{e}_n + V_n\mathbf{e}_j), \tag{7.81}$$

$$(\mathscr{M}^*)^h(\mathbf{Z}^{vv}_{jn}, \mathbf{Z}^{vh}_{jn}) = \mathscr{P}_{\text{sol}}(H_j\mathbf{e}_n + H_n\mathbf{e}_j), \tag{7.82}$$

$$\nabla \cdot \mathbf{Z}^{vv}_{jn} = \nabla \cdot \mathbf{Z}^{vh}_{jn} = 0. \tag{7.83}$$

Auxiliary problem 2 for the adjoint operator

$$(\mathscr{M}^*)^v(\mathbf{Z}^{hv}_{jn}, \mathbf{Z}^{hh}_{jn}) = \mathscr{P}_{\text{sol}}(\mathbf{H} \times \mathbf{e}_n), \tag{7.84}$$

$$(\mathscr{M}^*)^h(\mathbf{Z}^{hv}_{jn}, \mathbf{Z}^{hh}_{jn}) = -\mathscr{P}_{\text{sol}}(\mathbf{V} \times \mathbf{e}_n), \tag{7.85}$$

$$\nabla \cdot \mathbf{Z}^{hv}_{jn} = \nabla \cdot \mathbf{Z}^{hh}_{jn} = 0. \tag{7.86}$$

Here $\mathscr{M}^* = ((\mathscr{M}^*)^v, (\mathscr{M}^*)^h)$ is the operator, adjoint to linearisation $\mathscr{M}$, acting in the space of pairs of three-dimensional solenoidal vector fields bounded together with the derivatives,

$$(\mathscr{M}^*)^v(\mathbf{v}, \mathbf{h}) = \frac{\partial \mathbf{v}}{\partial t} + \nu\nabla^2\mathbf{v} - \nabla \times (\mathbf{V} \times \mathbf{v}) + \mathscr{P}_{\text{sol}}(\mathbf{H} \times (\nabla \times \mathbf{h}) - \mathbf{v} \times (\nabla \times \mathbf{V})),$$
$$(\mathscr{M}^*)^h(\mathbf{v}, \mathbf{h}) = \frac{\partial \mathbf{h}}{\partial t} + \eta\nabla^2\mathbf{h} + \nabla \times (\mathbf{H} \times \mathbf{v}) + \mathscr{P}_{\text{sol}}(\mathbf{v} \times (\nabla \times \mathbf{H}) - \mathbf{V} \times (\nabla \times \mathbf{h}))$$

(all differential operators are in the fast variables), $\mathscr{P}_{\text{sol}}$ is the projection of three-dimensional vector fields onto the space of solenoidal vector fields bounded together with the derivatives. The computational complexity of these problems is the same as that of the auxiliary problems of types II and III.

The mean quantities (7.71)–(7.80) in the eddy corrections can be expressed in the terms of solutions to the auxiliary problems of type I (6 problems), auxiliary problems for the adjoint operator (7.81)–(7.83) (6 problems) and (7.84)–(7.86) (3 problems), and the r.h.s. of Eqs. 7.39–7.52 in the statements of the auxiliary problems of type II and III, applying the formulae

$$\begin{aligned}
&\left\langle\!\!\left\langle -V_j\mathbf{q}^v - \mathbf{V}(q^v)_j + H_j\mathbf{q}^h + \mathbf{H}(q^h)_j \right\rangle\!\!\right\rangle_n \\
&= \langle\!\langle (V_n\mathbf{e}_j - V_j\mathbf{e}_n)\cdot\mathscr{P}_\nabla\mathbf{q}^v + (H_j\mathbf{e}_n - H_n\mathbf{e}_j)\cdot\mathscr{P}_\nabla\mathbf{q}^h \\
&\quad + (\mathscr{M}^*)^v(\mathbf{Z}^{vv}_{jn},\mathbf{Z}^{vh}_{jn})\cdot\mathscr{P}_{\rm sol}\mathbf{q}^v + (\mathscr{M}^*)^h(\mathbf{Z}^{vv}_{jn},\mathbf{Z}^{vh}_{jn})\cdot\mathscr{P}_{\rm sol}\mathbf{q}^h\rangle\!\rangle \\
&= \langle\!\langle (V_n\mathbf{e}_j - V_j\mathbf{e}_n)\cdot\mathscr{P}_\nabla\mathbf{q}^v + (H_j\mathbf{e}_n - H_n\mathbf{e}_j)\cdot\mathscr{P}_\nabla\mathbf{q}^h \\
&\quad + \mathbf{Z}^{vv}_{jn}\cdot\mathscr{M}^v(\mathscr{P}_{\rm sol}\mathbf{q}^v,\mathscr{P}_{\rm sol}\mathbf{q}^h) + \mathbf{Z}^{vh}_{jn}\cdot\mathscr{M}^h(\mathscr{P}_{\rm sol}\mathbf{q}^v,\mathscr{P}_{\rm sol}\mathbf{q}^h)\rangle\!\rangle
\end{aligned}$$

$$\begin{aligned}
&\langle\!\langle \mathbf{V}\times\mathbf{q}^h - \mathbf{H}\times\mathbf{q}^v\rangle\!\rangle_n \\
&= \langle\!\langle \mathbf{q}^v\cdot(\mathbf{H}\times\mathbf{e}_n) - \mathbf{q}^h\cdot(\mathbf{V}\times\mathbf{e}_n)\rangle\!\rangle \\
&= \langle\!\langle \mathscr{P}_\nabla\mathbf{q}^v\cdot(\mathbf{H}\times\mathbf{e}_n) - \mathscr{P}_\nabla\mathbf{q}^h\cdot(\mathbf{V}\times\mathbf{e}_n) \\
&\quad + \mathscr{P}_{\rm sol}\mathbf{q}^v\cdot(\mathscr{M}^*)^v(\mathbf{Z}^{hv}_n,\mathbf{Z}^{hh}_n) + \mathscr{P}_{\rm sol}\mathbf{q}^h\cdot(\mathscr{M}^*)^h(\mathbf{Z}^{hv}_n,\mathbf{Z}^{hh}_n)\rangle\!\rangle \\
&= \langle\!\langle \mathscr{P}_\nabla\mathbf{q}^v\times\mathbf{H} - \mathscr{P}_\nabla\mathbf{q}^h\times\mathbf{V}\rangle\!\rangle_n \\
&\quad + \langle\!\langle \mathscr{M}^v(\mathscr{P}_{\rm sol}\mathbf{q}^v,\mathscr{P}_{\rm sol}\mathbf{q}^h)\cdot\mathbf{Z}^{hv}_n + \mathscr{M}^h(\mathscr{P}_{\rm sol}\mathbf{q}^v,\mathscr{P}_{\rm sol}\mathbf{q}^h)\cdot\mathbf{Z}^{hh}_n\rangle\!\rangle. \qquad (7.88)
\end{aligned}$$

Here $\mathscr{P}_\nabla = \mathscr{I} - \mathscr{P}_{\rm sol}$ is the projection onto potential vector fields, $\mathscr{I}$ is the identity, $\mathbf{q}$ a solution to an auxiliary problem of type II (for computation of the entries of the tensor of combined eddy correction of magnetic diffusion and kinematic viscosity, $\mathbf{D}$) or III (for computation of the entries of the tensor of combined eddy correction of advection, $\mathbf{A}$). For the problems of type II, the quantities $\mathscr{P}_\nabla\mathbf{q}^v, \mathscr{P}_\nabla\mathbf{q}^h$ can be computed employing the expressions for the divergencies (7.40), (7.42), (7.44) and (7.46); for the problems of type III, $\mathscr{P}_\nabla\mathbf{q}^v = \mathscr{P}_\nabla\mathbf{q}^h = 0$. One can subsequently compute $\mathscr{M}(\mathscr{P}_{\rm sol}\mathbf{q}^v, \mathscr{P}_{\rm sol}\mathbf{q}^h)$ without solving the auxiliary problems of types II and III. (For space-periodic fields, it is convenient to compute the projection $\mathscr{P}_\nabla$ using the Fast Fourier Transform).

From the mathematical point of view, the auxiliary problems for the adjoint operator are correctly stated, e.g., if the regime $\mathbf{V}, \mathbf{H}, P$ is space-periodic, and steady or periodic in time, and the domain of $\mathscr{M}^*$ consists of vector fields with the same properties. In general, solving numerically the auxiliary problems (7.81)–(7.83) and (7.84)–(7.86) for the adjoint operator can be problematic, since the operator $\mathscr{M}^*$ is not parabolic. The following approach can be useful: Denote by $\mathbf{Z}(\tau; \mathbf{x}, t)$ solution, obtained setting for the unknown fields $\mathbf{Z}$ zero "initial" conditions at $t = \tau > 0$ and solving the problems (7.81)–(7.83) and (7.84)–(7.86) "backwards", for the time decreasing from $t = \tau 0$ to $t = 0$ (reversal of time transforms the operators $(\mathscr{M}^*)^v, (\mathscr{M}^*)^h$ into parabolic ones). Spatio-temporal averaging over the fast variables of the scalar products with the fields $\mathbf{Z}$ in the formulae (7.87) and (7.88) can be defined as

$$\langle\!\langle \mathbf{f} \cdot \mathbf{Z} \rangle\!\rangle = \lim_{\tau\to\infty} \lim_{\ell\to\infty} \frac{1}{\tau\ell^3} \int_0^{\tau} \int_{-\ell/2}^{\ell/2} \int_{-\ell/2}^{\ell/2} \int_{-\ell/2}^{\ell/2} \mathbf{f}(\mathbf{x}, t)\mathbf{Z}(\tau; \mathbf{x}, t)\mathrm{d}\mathbf{x}\,\mathrm{d}t,$$

provided this limit exists (for instance, if the regime $\mathbf{V}, \mathbf{H}, P$ is periodic or quasi-periodic in the fast time).

7.6 Conclusions

1. In this chapter we have considered the problem of weakly nonlinear stability of a short-scale MHD regime to perturbations involving large spatial and temporal scales. The amplitude of a perturbation is assumed to be the order of the ratio of spatial scales, ε. The perturbation has been expanded in a power series in ε, and the mean-field equations for the evolution of the leading terms of the expansion averaged over spatial and temporal short scales have been derived. The evolution of perturbations of a generic short-scale MHD regime is governed by linear partial differential equations (7.35)–(7.36), involving the operator of the combined α-effect (which arises in the linear stability problem). If the regime $\mathbf{V}, \mathbf{H}, P$ is parity-invariant, the α-effect does not emerge in the leading order, and the mean-field equations (7.69)–(7.70) generalise the Navier–Stokes and magnetic induction equations. They involve additional terms: the operator of the anisotropic combined MHD eddy diffusion (also emerging in the linear stability problem) and new quadratic terms, which are analogous to the advective ones and describe the so-called anisotropic combined MHD eddy advection.
2. We have proposed a method for computation of the entries of the combined MHD eddy diffusion (7.71)–(7.74) and the advection (7.75)–(7.80) tensors emerging in the mean-field equations (7.69)–(7.70). The method is based on the expressions for these entries in the terms of solutions to the auxiliary problems for the operator, adjoint to the operator of linearisation of the equations of magnetohydrodynamics in the vicinity of the short-scale MHD regime, whose weakly nonlinear stability is examined.

Chapter 8
Weakly Nonlinear Stability of Forced Thermal Hydromagnetic Convection

In this chapter we consider, following [330], weakly nonlinear stability to large-scale perturbations of convective hydromagnetic (CHM) regimes in a horizontal layer of electrically conducting fluid. (A study of the preceding linear stage of their evolution, restricted to the case of a non-rotating fluid, was carried out in [13].) Our subject here is *forced* thermal hydromagnetic convection, i.e., we assume that forces of an external origin are acting on the fluid (in addition to the buoyancy, Coriolis and Lorentz forces), and/or external magnetic fields are inducing electric currents in the volume occupied by the fluid, and/or distributed sources of heat are present in it. A similar assumption, made in Chaps. 6 and 7, was natural in the investigation of MHD regimes, since in the absence of external sources of energy any MHD regime would relax, due to the dissipation of energy by diffusion, to a trivial steady state with fluid at rest. The present physical system is different in that a sufficiently large difference of temperature at the boundaries of the fluid layer can sustain non-trivial short-scale MHD regimes. In the absence of such sources, the system is spatially and temporally invariant; this increases the algebraic complexity of the large-scale stability problem, because the dimension of the kernel of the operator of linearisation increases. Stability of free thermal hydromagnetic convection is considered in the next chapter.

The horizontal stress-free ideally electrically conducting boundaries are supposed to be kept at constant temperatures. The Boussinesq approximation is employed. The assumption that the boundaries are free may be perceived as not physically well-founded. However, it is inherently compatible with the solenoidality condition, and due to the ensuing computational convenience it is often employed in numerical studies of convection and convective dynamos (see, e.g., [55, 227, 229, 333] and references therein). Since rotation of fluid is an important feature of the astrophysical dynamos and geodynamo, in the present chapter, in contrast with Chaps. 6 and 7, we allow rotation about the vertical axis (and then the plane layer of fluid can be regarded as a segment of the spherical outer core of a

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planet in the equatorial region). Taking into account the Coriolis force results in a more complex analysis. In order to exclude the pressure from consideration, we apply the Navier–Stokes equation in the form for vorticity. Consequently, when the α-effect is insignificant, we derive the equation for the mean perturbation of the flow as the solvability condition for the equation in fast variables at order ε^3, and not ε^2 as in the problem considered in the previous chapter.

As in Chap. 7, we do not exclude the possibility of a time dependence of the short-scale regime, whose stability is investigated. Unless the regime is stable to short-scale perturbations, the permissible initial conditions for auxiliary problems constitute a restricted class. Like in Chap. 7, we do not demand the spatial periodicity or quasi-periodicity in horizontal directions of the short-scale CHM regimes that are perturbed. Whether there exist solutions to the auxiliary problems with the properties required for our constructions (smoothness and boundedness), as well as whether the spatial and spatio-temporal averaging over fast variables is well-defined, in such a general setup remain open questions. As in the previous chapters, they do not arise if the CHM regime under consideration is periodic in horizontal directions and it is steady or time-periodic.

As for the MHD regimes considered in the previous chapter, the mean-field equations for perturbations, that we construct here, constitute a closed system of equations. Generically they turn out to be linear and describe the anisotropic combined MHD α-effect. If the α-effect is absent or insignificant in the leading order (e.g., due to the parity invariance or the symmetry about a vertical axis of the CHM regime, whose stability is examined), the mean-field equations for perturbations generalise the fundamental equations of magnetohydrodynamics (the Navier–Stokes and magnetic induction equations). As in Chap. 7, they involve the operator of combined eddy diffusion, usually anisotropic and not necessarily negative-definite, and additional quadratic terms, analogous to the ones describing advection. Also, terms involving non-local pseudodifferential operators emerge in the mean-field equations, unless the α-effect is absent due to a symmetry without a time shift.

As in the MHD problem that we have investigated in Chap. 7, the mean-field equations for perturbations of individual CHM regimes, derived under the assumption of insignificance of the α-effect in the leading order, lack the operator of the α-effect. However, it coexists in these equations with the operators describing the effects of the combined eddy diffusion and advection, when one examines weakly nonlinear stability to large-scale perturbations of CHM regimes, depending on the small parameter ε (which denotes the spatial scale ratio assumed to be of the same order as the amplitude of the perturbations). This can happen, if the CHM regime whose stability is examined has a weakly (the order ε) antisymmetric part—for instance, when the external forces and/or sources acting in the layer are ε-dependent and weakly antisymmetric. If a bifurcation of a CHM regime, which is parity-invariant or symmetric about a vertical axis, results in the loss of this symmetry, then the mean-field equations are supplemented by amplitude equations involving a cubic nonlinearity.

8.1 Statement of the Problem

8.1.1 *The Governing Equations for the Thermal Hydromagnetic Convection*

The CHM regime $\mathbf{V}, \mathbf{H}, \mathscr{T}$, whose weakly nonlinear stability we investigate, in the Boussinesq approximation satisfies the equations

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \nu \nabla^2 \Omega + \nabla \times (\mathbf{V} \times \boldsymbol{\Omega} - \mathbf{H} \times (\nabla \times \mathbf{H}) + \mathbf{V} \times \tau \mathbf{e}_3 + \beta \mathscr{T} \mathbf{e}_3 + \mathbf{F}), \quad (8.1)$$

$$\frac{\partial \mathbf{H}}{\partial t} = \eta \nabla^2 \mathbf{H} + \nabla \times (\mathbf{V} \times \mathbf{H}) + \mathbf{J}, \quad (8.2)$$

$$\frac{\partial \mathscr{T}}{\partial t} = \kappa \nabla^2 \mathscr{T} - (\mathbf{V} \cdot \nabla) \mathscr{T} + S,$$

the solenoidality conditions

$$\nabla \cdot \mathbf{V} = \nabla \cdot \mathbf{H} = 0 \quad (8.3)$$

and relation for vorticity

$$\nabla \times \mathbf{V} = \boldsymbol{\Omega}. \quad (8.4)$$

Here $\mathbf{V}(\mathbf{x}, t)$ is the flow velocity of the conducting fluid, $\boldsymbol{\Omega}(\mathbf{x}, t)$ its vorticity, $\mathscr{T}(\mathbf{x}, t)$ temperature, $\mathbf{H}(\mathbf{x}, t)$ magnetic field, t time, ν molecular viscosity, η molecular magnetic diffusivity, κ molecular thermal diffusivity, $\tau/2$ the rate of rotation of the fluid, $\beta(\mathscr{T} - \mathscr{T}_2)\mathbf{e}_3$ is the buoyancy force, $\mathbf{F}(\mathbf{x}, t)$ an external body force, $\mathbf{J}(\mathbf{x}, t)$ reflects the presence of external currents in the fluid, $S(\mathbf{x}, t)$ is the distribution of the heat sources in the layer of fluid, $\mathbf{e}_3$ denotes the upward vertical unit vector. To simplify the notation, when suitable we use the ten-dimensional vector fields $(\boldsymbol{\omega}, \mathbf{v}, \mathbf{h}, \theta)$ and seven-dimensional fields $(\boldsymbol{\omega}, \mathbf{h}, \theta)$ or $(\mathbf{v}, \mathbf{h}, \theta)$.

Although, perhaps, the most interesting case for physical applications is $\mathbf{F} = 0$, $\mathbf{J} = 0, S = 0$ (considered in the next chapter), we assume in this chapter that at least one of the terms is not a function of the Cartesian vertical coordinate x_3 and time t only. When stability of an unsteady regime is examined, we assume in addition that at least one of the terms is unsteady. Then their presence breaks the temporal and spatial (in horizontal directions) invariance; this simplifies the analysis of stability.

8.1.2 *Boundary Conditions*

The following boundary conditions are imposed on the horizontal boundaries $x_3 = \pm L/2$:

- Stress-free boundaries:

$$\left.\frac{\partial V_1}{\partial x_3}\right|_{x_3=\pm L/2} = \left.\frac{\partial V_2}{\partial x_3}\right|_{x_3=\pm L/2} = 0, \quad V_3|_{x_3=\pm L/2} = 0 \tag{8.5}$$

(the subscript enumerates the vector components), which implies the boundary conditions for vorticity

$$\Omega_1|_{x_3=\pm L/2} = \Omega_2|_{x_3=\pm L/2} = 0, \quad \left.\frac{\partial \Omega_3}{\partial x_3}\right|_{x_3=\pm L/2} = 0; \tag{8.6}$$

- Perfectly electrically conducting boundaries:

$$\left.\frac{\partial H_1}{\partial x_3}\right|_{x_3=\pm L/2} = \left.\frac{\partial H_2}{\partial x_3}\right|_{x_3=\pm L/2} = 0, \quad H_3|_{x_3=\pm L/2} = 0; \tag{8.7}$$

- Isothermal boundaries:

$$\mathscr{T}|_{x_3=-L/2} = \mathscr{T}_1, \quad \mathscr{T}|_{x_3=L/2} = \mathscr{T}_2 \tag{8.8}$$

(thermal convection is possible only when $\mathscr{T}_1 > \mathscr{T}_2$). It is convenient to introduce a new variable

$$\Theta = \mathscr{T} - \mathscr{T}_1 + \delta(x_3 + L/2), \tag{8.9}$$

where $\delta = (\mathscr{T}_1 - \mathscr{T}_2)/L$, which satisfies the equation

$$\frac{\partial \Theta}{\partial t} = \kappa \nabla^2 \Theta - (\mathbf{V} \cdot \nabla)\Theta + \delta V_3 + S \tag{8.10}$$

and the homogeneous boundary conditions

$$\Theta|_{x_3=\pm L/2} = 0. \tag{8.11}$$

8.1.3 Equations for Perturbations of a CHM Regime

The operator of linearisation of the governing equations (8.1)–(8.4) and (8.10) around the CHM regime, whose stability is examined, $\mathscr{L} = (\mathscr{L}^\omega, \mathscr{L}^h, \mathscr{L}^\theta)$, has the components

$$\begin{aligned}\mathscr{L}^\omega(\boldsymbol{\omega}, \mathbf{v}, \mathbf{h}, \theta) \equiv & -\frac{\partial \boldsymbol{\omega}}{\partial t} + \nu \nabla^2 \boldsymbol{\omega} + \nabla \times (\mathbf{V} \times \boldsymbol{\omega} + \mathbf{v} \times \boldsymbol{\Omega} \\ & - \mathbf{H} \times (\nabla \times \mathbf{h}) - \mathbf{h} \times (\nabla \times \mathbf{H})) + \tau \frac{\partial \mathbf{v}}{\partial x_3} + \beta \nabla \theta \times \mathbf{e}_3,\end{aligned}$$

$$\mathscr{L}^h(\mathbf{v},\mathbf{h}) \equiv -\frac{\partial \mathbf{h}}{\partial t} + \eta\nabla^2\mathbf{h} + \nabla\times(\mathbf{v}\times\mathbf{H} + \mathbf{V}\times\mathbf{h}),$$
$$\mathscr{L}^\theta(\mathbf{v},\theta) \equiv -\frac{\partial \theta}{\partial t} + \kappa\nabla^2\theta - (\mathbf{V}\cdot\nabla)\theta - (\mathbf{v}\cdot\nabla)\Theta + \delta v_3.$$

Evidently, $\mathscr{L}^\omega = (\nabla\times)\mathscr{L}^v$ (provided $\nabla_\mathbf{x}\times\mathbf{v} = \boldsymbol{\omega}$), where

$$\mathscr{L}^v(\mathbf{v},\mathbf{h},\theta,p) \equiv -\frac{\partial \mathbf{v}}{\partial t} + \nu\nabla^2\mathbf{v} + \mathbf{V}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times\boldsymbol{\Omega}$$
$$- \mathbf{H}\times(\nabla\times\mathbf{h}) - \mathbf{h}\times(\nabla\times\mathbf{H}) + \tau\mathbf{v}\times\mathbf{e}_3 + \beta\theta\mathbf{e}_3 - \nabla p$$

is the linearisation of the Navier–Stokes equation in the form for the flow velocity

$$\frac{\partial \mathbf{V}}{\partial t} = \nu\nabla^2\mathbf{V} + \mathbf{V}\times\boldsymbol{\Omega} - \mathbf{H}\times(\nabla\times\mathbf{H}) + \tau\mathbf{V}\times\mathbf{e}_3 + \beta\Theta\mathbf{e}_3 - \nabla P. \qquad (8.12)$$

Here $P = P'/\rho + |\mathbf{V}|^2/2 + \tau^2(x_1^2 + x_2^2)/8$ is the modified pressure, P' pressure, ρ the fluid density. In this chapter we do not impose restrictions on the means of fields in the domain of the operator $\mathscr{L}$.

We will consider a weakly nonlinear stability problem for a perturbation, whose amplitude is of the order ε. The perturbed regime $\mathbf{V}+\varepsilon\mathbf{v}, \boldsymbol{\Omega}+\varepsilon\boldsymbol{\omega}, \mathbf{H}+\varepsilon\mathbf{h}, \Theta+\varepsilon\theta$ satisfies Eqs. 8.1–8.11, and hence the profiles of perturbations, $\boldsymbol{\omega}, \mathbf{v}, \mathbf{h}, \theta$ (which we will henceforth call just the perturbation) satisfy the equations

$$\mathscr{L}^\omega(\boldsymbol{\omega},\mathbf{v},\mathbf{h},\theta) = -\varepsilon\nabla\times(\mathbf{v}\times\boldsymbol{\omega} - \mathbf{h}\times(\nabla\times\mathbf{h})), \qquad (8.13)$$

$$\mathscr{L}^h(\mathbf{v},\mathbf{h}) = -\varepsilon\nabla\times(\mathbf{v}\times\mathbf{h}), \qquad (8.14)$$

$$\mathscr{L}^\theta(\mathbf{v},\theta) = \varepsilon(\mathbf{v}\cdot\nabla)\theta, \qquad (8.15)$$

$$\nabla\cdot\mathbf{h} = 0, \qquad (8.16)$$

$$\nabla\cdot\boldsymbol{\omega} = \nabla\cdot\mathbf{v} = 0, \qquad (8.17)$$

$$\nabla\times\mathbf{v} = \boldsymbol{\omega}. \qquad (8.18)$$

Let $\langle\!\langle\cdot\rangle\!\rangle_h$ denote the horizontal component of a vector field averaged over the volume of the layer. Subtracting the Navier–Stokes equation (8.12) from the same equation for the perturbed regime, we obtain the equation governing the evolution of the perturbation $\mathbf{v}$. Averaging the horizontal component of this equation over the layer and taking into account the solenoidality conditions (8.16)–(8.17) and the relation for vorticity (8.18), we find

$$\frac{\partial\langle\!\langle\mathbf{v}\rangle\!\rangle_h}{\partial t} = \langle\!\langle\mathbf{v}\rangle\!\rangle_h \times \tau\mathbf{e}_3 - \langle\!\langle\nabla p\rangle\!\rangle_h.$$

Thus, the evolution of the mean horizontal component of the flow velocity is controlled by the mean horizontal component of the gradient of the modified pressure p, which must be specified as a boundary condition in the horizontal directions. We assume that the growth of pressure in the horizontal directions is bounded so that $\langle\!\langle \nabla p \rangle\!\rangle_h = 0$, i.e., the jump of p at the infinity is not sufficiently high to cause pumping of the fluid through the layer. Then, if initially the mean horizontal component of the flow velocity is zero, this remains true at any time:

$$\langle\!\langle \mathbf{v} \rangle\!\rangle_h = 0. \tag{8.19}$$

8.1.4 Asymptotic Expansion of Weakly Nonlinear Large-Scale Perturbations

We introduce the fast spatial, $\mathbf{x}$, and temporal, t, variables, and the respective slow horizontal variables $\mathbf{X} = \varepsilon(x_1, x_2)$ and $T = \varepsilon^2 t$. The chosen scaling ε^2 is appropriate, if the combined α-effect is insignificant in the leading order (see a discussion at the end of Sect. 8.3.2). A solution to the problem (8.13)–(8.19) governing the evolution of the perturbation is sought in the form of power series in ε,

$$\boldsymbol{\omega} = \sum_{n=0}^{\infty} \boldsymbol{\omega}_n(\mathbf{X}, T, \mathbf{x}, t)\varepsilon^n, \tag{8.20}$$

$$\mathbf{v} = \sum_{n=0}^{\infty} \mathbf{v}_n(\mathbf{X}, T, \mathbf{x}, t)\varepsilon^n, \tag{8.21}$$

$$\mathbf{h} = \sum_{n=0}^{\infty} \mathbf{h}_n(\mathbf{X}, T, \mathbf{x}, t)\varepsilon^n, \tag{8.22}$$

$$\theta = \sum_{n=0}^{\infty} \theta_n(\mathbf{X}, T, \mathbf{x}, t)\varepsilon^n. \tag{8.23}$$

8.1.5 The Hierarchy of Equations for Perturbations

The spatial and spatio-temporal means of scalar and vector fields over the fast variables, $\langle\cdot\rangle$ and $\langle\!\langle\cdot\rangle\!\rangle$, and the respective fluctuating parts, $\{\cdot\}$ and $\{\!\{\cdot\}\!\}$, are defined in Sect. 7.1.1; $\langle\mathbf{f}\rangle_k$ and $\langle\!\langle\mathbf{f}\rangle\!\rangle_k$ denote the kth components of the means $\langle\mathbf{f}\rangle$ and $\langle\!\langle\mathbf{f}\rangle\!\rangle$, respectively. The subscripts v and h label the vertical and horizontal parts of three-dimensional mean vector fields:

$$\begin{aligned}
\langle \mathbf{f} \rangle_v &\equiv \langle \mathbf{f} \rangle_3 \mathbf{e}_3, & \{\mathbf{f}\}_v &\equiv \mathbf{f} - \langle \mathbf{f} \rangle_v; \\
\langle\!\langle \mathbf{f} \rangle\!\rangle_v &\equiv \langle\!\langle \mathbf{f} \rangle\!\rangle_3 \mathbf{e}_3, & \{\!\{\mathbf{f}\}\!\}_v &\equiv \mathbf{f} - \langle\!\langle \mathbf{f} \rangle\!\rangle_v; \\
\langle \mathbf{f} \rangle_h &\equiv \langle \mathbf{f} \rangle_1 \mathbf{e}_1 + \langle \mathbf{f} \rangle_2 \mathbf{e}_2, & \{\mathbf{f}\}_h &\equiv \mathbf{f} - \langle \mathbf{f} \rangle_h; \\
\langle\!\langle \mathbf{f} \rangle\!\rangle_h &\equiv \langle\!\langle \mathbf{f} \rangle\!\rangle_1 \mathbf{e}_1 + \langle\!\langle \mathbf{f} \rangle\!\rangle_2 \mathbf{e}_2, & \{\!\{\mathbf{f}\}\!\}_h &\equiv \mathbf{f} - \langle\!\langle \mathbf{f} \rangle\!\rangle_h.
\end{aligned}$$

(The superscripts v and h in the notation for various quantities refer to the flow velocity and magnetic field.)

Extracting the mean horizontal and the complimentary fluctuating parts from the solenoidality conditions (8.16) and (8.17) upon substitution of the series (8.21) and (8.22), we find at the order ε^n the following conditions for solenoidality of fields constituting a large-scale perturbation:

$$\nabla_{\mathbf{X}} \cdot \langle \mathbf{v}_n \rangle_h = \nabla_{\mathbf{X}} \cdot \langle \mathbf{h}_n \rangle_h = 0, \tag{8.24}$$

$$\nabla_{\mathbf{x}} \cdot \{\mathbf{v}_n\}_h + \nabla_{\mathbf{X}} \cdot \{\mathbf{v}_{n-1}\}_h = 0, \tag{8.25}$$

$$\nabla_{\mathbf{x}} \cdot \{\mathbf{h}_n\}_h + \nabla_{\mathbf{X}} \cdot \{\mathbf{h}_{n-1}\}_h = 0. \tag{8.26}$$

Evidently,

$$\nabla_{\mathbf{X}} \cdot \langle \boldsymbol{\omega}_n \rangle_v = 0, \tag{8.27}$$

and the identity $\nabla \cdot \boldsymbol{\omega} = 0$ implies

$$\nabla_{\mathbf{x}} \cdot \{\boldsymbol{\omega}_n\}_v + \nabla_{\mathbf{X}} \cdot \{\boldsymbol{\omega}_{n-1}\}_v = 0 \tag{8.28}$$

for all $n \geq 0$. (Any term in the expansions (8.20)–(8.23) for a negative index n is zero by definition.) As usual, differentiation in the fast and slow variables is assumed in the differential operators with the subscripts $\mathbf{x}$ and $\mathbf{X}$; $\nabla_{\mathbf{X}} = (\partial/\partial X_1, \partial/\partial X_2, 0)$. Differentiation in the fast variables only is assumed below in the operator of linearisation $\mathscr{L}$ defined in Sect. 8.1.3.

Substituting the power series (8.20)–(8.23) into the Eqs. 8.13–8.15 for the evolution of the perturbation, we transform them into equalities of power series in ε. Equating the coefficients at order ε^n, we obtain a hierarchy of systems of equations,

$$\begin{aligned}
&\mathscr{L}^{\omega}(\boldsymbol{\omega}_n, \mathbf{v}_n, \mathbf{h}_n, \theta_n) - \frac{\partial \boldsymbol{\omega}_{n-2}}{\partial T} + \nu(2(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\{\boldsymbol{\omega}_{n-1}\}_v + \nabla^2_{\mathbf{X}} \boldsymbol{\omega}_{n-2}) \\
&+ \nabla_{\mathbf{X}} \times (\mathbf{V} \times \boldsymbol{\omega}_{n-1} + \mathbf{v}_{n-1} \times \boldsymbol{\Omega} - \mathbf{H} \times (\nabla_{\mathbf{X}} \times \mathbf{h}_{n-2} + \nabla_{\mathbf{x}} \times \mathbf{h}_{n-1}) - \mathbf{h}_{n-1} \times (\nabla_{\mathbf{x}} \times \mathbf{H}) \\
&+ \sum_{k=0}^{n-2} (\mathbf{v}_k \times \boldsymbol{\omega}_{n-2-k} - \mathbf{h}_k \times (\nabla_{\mathbf{x}} \times \mathbf{h}_{n-2-k} + \nabla_{\mathbf{X}} \times \mathbf{h}_{n-3-k}))) \\
&+ \nabla_{\mathbf{x}} \times \left(\sum_{k=0}^{n-1} (\mathbf{v}_k \times \boldsymbol{\omega}_{n-1-k} - \mathbf{h}_k \times (\nabla_{\mathbf{x}} \times \mathbf{h}_{n-1-k} + \nabla_{\mathbf{X}} \times \mathbf{h}_{n-2-k})) \right. \\
&\left. - \mathbf{H} \times (\nabla_{\mathbf{X}} \times \mathbf{h}_{n-1}) \right) + \beta \nabla_{\mathbf{X}} \theta_{n-1} \times \mathbf{e}_3 = 0,
\end{aligned} \tag{8.29}$$

$$\mathscr{L}^h(\mathbf{v}_n,\mathbf{h}_n)-\frac{\partial \mathbf{h}_{n-2}}{\partial T}+\eta\big(2(\nabla_\mathbf{x}\cdot\nabla_\mathbf{X})\{\mathbf{h}_{n-1}\}_h+\nabla^2_\mathbf{X}\mathbf{h}_{n-2}\big)$$
$$+\nabla_\mathbf{X}\times\left(\mathbf{v}_{n-1}\times\mathbf{H}+\mathbf{V}\times\mathbf{h}_{n-1}+\sum_{k=0}^{n-2}\mathbf{v}_k\times\mathbf{h}_{n-2-k}\right)+\nabla_\mathbf{x}\times\sum_{k=0}^{n-1}\mathbf{v}_k\times\mathbf{h}_{n-1-k}=0, \tag{8.30}$$

$$\mathscr{L}^\theta(\mathbf{v}_n,\theta_n)-\frac{\partial \theta_{n-2}}{\partial T}+\kappa\big(2(\nabla_\mathbf{x}\cdot\nabla_\mathbf{X})\theta_{n-1}+\nabla^2_\mathbf{X}\theta_{n-2}\big)$$
$$-(\mathbf{V}\cdot\nabla_\mathbf{X})\theta_{n-1}-\sum_{k=0}^{n-1}(\mathbf{v}_k\cdot\nabla_\mathbf{x})\theta_{n-1-k}-\sum_{k=0}^{n-2}(\mathbf{v}_k\cdot\nabla_\mathbf{X})\theta_{n-2-k}=0. \tag{8.31}$$

8.2 Mathematical Tools

To derive the mean-field equations for perturbations, we consider successively the systems in the hierarchy and, as in the previous chapters, rely on the solvability conditions. The problem under consideration being more difficult than the ones considered so far, we need to extend our mathematical instrumentarium. Since we use the Navier–Stokes equation in the form for vorticity, we will have to calculate the terms in the expansion of the flow given the terms in the expansion of vorticity. To implement the condition (8.19) that the fluid is not pumped through the layer by the gradient of pressure at infinity for a flow depending on the fast and slow variables, we need a bound for vector fields, solenoidal in the fast variables. We consider these technical questions in the present section.

8.2.1 Reconstruction of the Flow from the Vorticity

Clearly, $\langle\boldsymbol{\omega}_n\rangle_v$, $\langle\mathbf{v}_n\rangle_h$ and $\langle\mathbf{h}_n\rangle_h$ satisfy the boundary conditions defined by (8.5)–(8.7). Due to solenoidality of $\langle\mathbf{v}_n\rangle_h$ in the slow variables (8.24), we can introduce a stream function $\psi_n(X_1,X_2,t,T)$:

$$\langle\mathbf{v}_n\rangle_h=\left(-\frac{\partial\psi_n}{\partial X_2},\frac{\partial\psi_n}{\partial X_1},0\right).$$

Substituting the series (8.20) and (8.21) into the identity (8.18) for the perturbation of vorticity, we find an equation relating vorticity and flow perturbations

$$\nabla_\mathbf{x}\times\{\mathbf{v}_n\}_h=\boldsymbol{\omega}_n-\nabla_\mathbf{X}\times\mathbf{v}_{n-1}. \tag{8.32}$$

We have to solve this equation in $\{\mathbf{v}_n\}_h$, the divergence of $\{\mathbf{v}_n\}_h$ in the fast variables being specified by (8.25).

Using the relation (8.32) for the modified index $n \to n-1$, and the expression (8.28) for the divergence of $\{\boldsymbol{\omega}_n\}_v$ in the fast variables, it is easy to show that the r.h.s. of (8.32) is solenoidal in the fast variables. Let us consider the operator curl which maps solenoidal vector fields, defined in the layer and satisfying the boundary conditions (8.5), to solenoidal fields. The domain of the adjoint operator, which is also the curl, consists of solenoidal vector fields satisfying the boundary conditions for vorticity, (8.6); its kernel is spanned by the constant vector field (0,0,1). Consequently, Eq. 8.32 is solvable in $\{\mathbf{v}_n\}_h$, if the spatial mean of the vertical component of the r.h.s. vanishes[1]:

$$\langle \boldsymbol{\omega}_n \rangle_3 = (\nabla_{\mathbf{X}} \times \langle \mathbf{v}_{n-1} \rangle) \cdot \mathbf{e}_3 \quad \Leftrightarrow \quad \langle \boldsymbol{\omega}_n \rangle_v = \nabla_{\mathbf{X}} \times \langle \mathbf{v}_{n-1} \rangle_h. \tag{8.33}$$

For $n = 0$, this relation between the mean vorticity and flow velocity reduces to $\langle \boldsymbol{\omega}_0 \rangle_v = 0$. In the terms of the stream function, Eq. 8.33 is equivalent to $\nabla^2_{\mathbf{X}} \psi_{n-1} = \langle \boldsymbol{\omega}_n \rangle_3$. One can determine $\langle \mathbf{v}_{n-1} \rangle_h$ from this equation and the condition of the global boundedness of ψ_{n-1} (guaranteeing that the mean of $\langle \mathbf{v}_{n-1} \rangle_h$ over the slow spatial variables vanishes), provided the mean of $\langle \boldsymbol{\omega}_n \rangle_3$ over the plane of the slow spatial variables is zero.

In view of relation (8.33), Eq. 8.32 can be transformed into

$$\nabla_{\mathbf{x}} \times \{\mathbf{v}_n\}_h = \{\boldsymbol{\omega}_n\}_v - \nabla_{\mathbf{X}} \times \{\mathbf{v}_{n-1}\}_h. \tag{8.34}$$

The substitution $\{\mathbf{v}_n\}_h = \mathbf{v} + \nabla_{\mathbf{x}} B$, where B is a globally bounded solution to the Neumann problem

$$\nabla^2_{\mathbf{x}} B = -\nabla_{\mathbf{X}} \cdot \{\mathbf{v}_{n-1}\}_h, \quad \left.\frac{\partial B}{\partial x_3}\right|_{x_3 = \pm L/2} = 0,$$

converts Eqs. 8.34 and 8.25 into the system (8.17)–(8.19). If the solvability condition is satisfied, this system has a solution $\mathbf{v} = \mathscr{P}_{\text{sol}} \{\mathbf{A}\}_h$, where $\mathbf{A}$ is a globally bounded solution to the Poisson equation

$$\nabla^2 \mathbf{A} = -\nabla \times \boldsymbol{\omega}, \quad \left.\frac{\partial A_1}{\partial x_3}\right|_{x_3 = \pm L/2} = \left.\frac{\partial A_2}{\partial x_3}\right|_{x_3 = \pm L/2} = 0, \quad A_3|_{x_3 = \pm L/2} = 0,$$

whose horizontal component is zero-mean, $\mathscr{P}_{\text{sol}}$ is the projection of three-dimensional vector fields onto the space of solenoidal fields,

$$\mathscr{P}_{\text{sol}} \mathbf{A} \equiv \mathbf{A} - \nabla a,$$

and a is a solution to the Neumann problem

[1] This condition is also sufficient for space-periodic regimes. If space periodicity is not imposed, the energy spectrum of the vector field to which the inverse curl is applied must fall off sufficiently fast near zero. Henceforth, the regime $\mathbf{V}, \mathbf{H}, \Theta$ is assumed to have the property that, in the course of solution of the auxiliary problems, the inverse curl can be applied to any vector field whose vertical component has a zero spatial mean.

$$\nabla^2 a = \nabla \cdot \mathbf{A}, \quad \left.\frac{\partial a}{\partial x_3}\right|_{x_3=\pm L/2} = 0.$$

This construction defines an operator $\mathscr{C} : \boldsymbol{\omega} \to \mathbf{v}$, inverse to the curl, which acts from the space of solenoidal vector fields that have a zero spatial mean of the vertical component and satisfy the boundary conditions (8.6) for vorticity, into the space of solenoidal fields that satisfy the boundary conditions (8.5) for the flow and the condition (8.19) of no pumping.

8.2.2 Solvability of Auxiliary Problems

We assume that the fields $\mathbf{V}, \mathbf{H}, \Theta$, constituting a CHM regime whose stability is examined, are smooth and globally bounded, and all spatial and spatio-temporal means over the fast variables, that we calculate in the course of derivation of the mean-field equations, are well-defined. These conditions are satisfied, if the smooth CHM regime is periodic in horizontal directions, and steady or periodic in time (since then the region, where the solutions are defined, is compact).

In construction of the asymptotic solutions we encounter problems of the following type:

$$\mathscr{L}^{\omega}(\boldsymbol{\omega}, \mathbf{v}, \mathbf{h}, \theta) = \mathbf{f}^{\omega}, \quad \mathscr{L}^{h}(\mathbf{v}, \mathbf{h}) = \mathbf{f}^{h}, \quad \mathscr{L}^{\theta}(\mathbf{v}, \theta) = f^{\theta}, \tag{8.35}$$

$$\nabla_{\mathbf{x}} \cdot \boldsymbol{\omega} = d^{\omega}, \quad \nabla_{\mathbf{x}} \cdot \mathbf{v} = d^{v}, \quad \nabla_{\mathbf{x}} \cdot \mathbf{h} = d^{h}, \tag{8.36}$$

$$\boldsymbol{\omega} - \nabla_{\mathbf{x}} \times \mathbf{v} = \mathbf{a}. \tag{8.37}$$

The r.h.s. of the equations are known. The consistency relations [obtained by combining the divergence of equation (8.37) and the first two equations in (8.35) with the conditions (8.36)],

$$\nabla_{\mathbf{x}} \cdot \mathbf{a} = d^{\omega}, \tag{8.38}$$

$$-\frac{\partial d^{\omega}}{\partial t} + \nu \nabla^2_{\mathbf{x}} d^{\omega} + \tau \frac{\partial d^{\omega}}{\partial x_3} = \nabla_{\mathbf{x}} \cdot \mathbf{f}^{\omega}, \tag{8.39}$$

$$-\frac{\partial d^{h}}{\partial t} + \eta \nabla^2_{\mathbf{x}} d^{h} = \nabla_{\mathbf{x}} \cdot \mathbf{f}^{h}, \tag{8.40}$$

are supposed to hold true, $\mathbf{a}$ satisfies the boundary conditions (8.6) for vorticity, and

$$\langle d^{v} \rangle = \langle d^{h} \rangle = 0. \tag{8.41}$$

The fields $\boldsymbol{\omega}, \mathbf{v}, \mathbf{h}$ and θ must satisfy the same boundary conditions as the CHM regime, (8.5)–(8.7) and (8.11), and have the prescribed means $\langle\!\langle \boldsymbol{\omega} \rangle\!\rangle_v$, $\langle\!\langle \mathbf{h} \rangle\!\rangle_h$ and $\langle \mathbf{v} \rangle_h$.

By virtue of the consistency relations (8.39) and (8.40), it suffices to demand that the conditions (8.36) for vorticity $\boldsymbol{\omega}$ and magnetic field $\mathbf{h}$ are satisfied at $t = 0$.

Suppose now that $\boldsymbol{\omega}, \mathbf{v}, \mathbf{h}$ and θ are arbitrary fields bounded together with the derivatives and satisfying the boundary conditions defined by (8.5)–(8.7) and (8.11). Since $\mathbf{V}$ and $\mathbf{H}$ belong to this class of fields as well, we find

$$\langle \mathscr{L}^{\omega}(\boldsymbol{\omega}, \mathbf{v}, \mathbf{h}, \theta)\rangle_v = -\frac{\partial \langle \boldsymbol{\omega}\rangle_v}{\partial t}, \tag{8.42}$$

$$\langle \mathscr{L}^{h}(\mathbf{v}, \mathbf{h})\rangle_h = -\frac{\partial \langle \mathbf{h}\rangle_h}{\partial t}, \tag{8.43}$$

$$\Rightarrow \quad \langle\!\langle \mathscr{L}^{\omega}(\boldsymbol{\omega}, \mathbf{v}, \mathbf{h}, \theta)\rangle\!\rangle_v = 0, \quad \langle\!\langle \mathscr{L}^{h}(\mathbf{v}, \mathbf{h})\rangle\!\rangle_h = 0. \tag{8.44}$$

Relations (8.44) imply that the conditions

$$\langle\!\langle \mathbf{f}^{\omega}(\mathbf{x}, t)\rangle\!\rangle_v = 0, \tag{8.45}$$

$$\langle\!\langle \mathbf{f}^{h}(\mathbf{x}, t)\rangle\!\rangle_h = 0 \tag{8.46}$$

are necessary for existence of solutions to the problem (8.35)–(8.41) from the considered class. Integrating equations (8.42) and (8.43) in the fast time, we obtain

$$\langle \boldsymbol{\omega}\rangle_v\big|_{t=0} - \langle \boldsymbol{\omega}\rangle_v = \int_0^t \langle \mathbf{f}^{\omega}\rangle_v \mathrm{d}t, \quad \langle \mathbf{h}\rangle_h\big|_{t=0} - \langle \mathbf{h}\rangle_h = \int_0^t \langle \mathbf{f}^{h}\rangle_h \mathrm{d}t, \tag{8.47}$$

whereby

$$\langle \boldsymbol{\omega}\rangle_v = \langle\!\langle \boldsymbol{\omega}\rangle\!\rangle_v - \left\{\!\!\left\{ \int_0^t \langle \mathbf{f}^{\omega}\rangle_v \mathrm{d}t \right\}\!\!\right\}, \quad \langle \mathbf{h}\rangle_h = \langle\!\langle \mathbf{h}\rangle\!\rangle_h - \left\{\!\!\left\{ \int_0^t \langle \mathbf{f}^{h}\rangle_h \mathrm{d}t \right\}\!\!\right\} \tag{8.48}$$

(although here the integrals in the r.h.s. fluctuate only in time, we continue to employ the notation $\{\!\{\cdot\}\!\}$). Thus, the means $\langle \boldsymbol{\omega}\rangle_v$ and $\langle \mathbf{h}\rangle_h$ are well-defined, if the means $\langle\!\langle \int_0^t \langle \mathbf{f}^{\omega}\rangle_v \mathrm{d}t\rangle\!\rangle$ and $\langle\!\langle \int_0^t \langle \mathbf{f}^{h}\rangle_h \mathrm{d}t\rangle\!\rangle$ are. It easy to check, that relations (8.45) and (8.46) follow from existence of these integrals, provided the means $\langle\!\langle \mathbf{f}^{\omega}\rangle\!\rangle_v$ and $\langle\!\langle \mathbf{f}^{h}\rangle\!\rangle_h$ are well-defined. Averaging the vertical component of equation (8.37) over the fast spatial variables, we obtain a necessary condition for its solvability (see the previous section):

$$\langle \mathbf{a}\rangle_v = \langle\!\langle \boldsymbol{\omega}\rangle\!\rangle_v - \left\{\!\!\left\{ \int_0^t \langle \mathbf{f}^{\omega}\rangle_v \mathrm{d}t \right\}\!\!\right\}. \tag{8.49}$$

Till the end of Sect. 8.5 the following solvability criterion is assumed: if the consistency conditions (8.49) are satisfied, then for arbitrary $\mathbf{f}^{\omega}(\mathbf{x}, t)$, $\mathbf{f}^{h}(\mathbf{x}, t)$ and $f^{\theta}(\mathbf{x}, t)$, which are zero-mean [as specified by (8.45) and (8.46)] and globally bounded together with the derivatives, the system of Eqs. 8.35–8.41 has a solution $\boldsymbol{\omega}, \mathbf{v}, \mathbf{h}, \theta$ in this class at least for some initial conditions satisfying (8.48) at $t = 0$. A solution to the problem (8.35)–(8.41) can be constructed for any initial conditions as a solution to a parabolic equation, but it is not guaranteed to be globally bounded. Expressions (8.42) and (8.43) for the spatial mean of the image of the operator of linearisation imply that for a solution (from the class of fields under consideration) to the problem

$$\mathscr{L}^{\omega}(\boldsymbol{\omega}, \mathbf{v}, \mathbf{h}, \theta) = 0, \quad \mathscr{L}^{h}(\mathbf{v}, \mathbf{h}) = 0, \quad \mathscr{L}^{\theta}(\mathbf{v}, \theta) = 0 \tag{8.50}$$

supplemented by conditions (8.16)–(8.19), the means $\langle\boldsymbol{\omega}\rangle_v$ and $\langle\mathbf{h}\rangle_h$ remain constant in time. A CHM regime is *stable to short-scale perturbations*, if for any globally bounded initial conditions a solution to the problem (8.50), (8.16)–(8.19) exponentially decays in time, provided $\langle\boldsymbol{\omega}\rangle_v = \langle\mathbf{h}\rangle_h = 0$. As we have shown in Sect. 7.4.2, if the CHM regime is space-periodic and stable to short-scale perturbations, then for any smooth initial conditions satisfying relations (8.36), (8.37) and (8.48) at $t = 0$ there exists a globally bounded solution to the system (8.35)–(8.41) [the arguments of Sect. 7.4.2 do not rely on the specific form of the considered equations and they are applicable for a wider class of linear evolutionary problems—in particular, for the system considered here]. As in Chap. 7, the CHM regime $\mathbf{V}, \mathbf{H}, \boldsymbol{\Theta}$ is not required to be stable to short-scale perturbations (despite the small-scale instability develops in the fast time, i.e., faster than the large-scale one), so that the mean-field equations constructed in this chapter were applicable for examination of the large-scale stability of chaotic CHM regimes.

In the remainder of this section we justify, using the arguments similar to the ones presented in Chaps. 2 and 3, the solvability criterion for the problem (8.35)–(8.41) for CHM regimes that are steady or time-periodic and periodic in horizontal directions: we show that generically equalities (8.45) and (8.46) are sufficient for existence of a solution, which is steady or time-periodic and possesses all the periodicities of the CHM regime. Consider the operator of linearisation, $\mathscr{M}' = (\mathscr{M}'^{\omega}, \mathscr{M}'^{h}, \mathscr{M}'^{\theta})$, in the form not involving explicitly the flow velocity:

$$\begin{aligned}
\mathscr{M}'^{\omega}(\boldsymbol{\omega}', \mathbf{h}', \theta) &\equiv \mathscr{L}^{\omega}(\boldsymbol{\omega}', \mathscr{C}\boldsymbol{\omega}', \mathbf{h}', \theta),\\
\mathscr{M}'^{h}(\boldsymbol{\omega}', \mathbf{h}') &\equiv \mathscr{L}^{h}(\mathscr{C}\boldsymbol{\omega}', \mathbf{h}'),\\
\mathscr{M}'^{\theta}(\boldsymbol{\omega}', \theta) &\equiv \mathscr{L}^{\theta}(\mathscr{C}\boldsymbol{\omega}', \theta).
\end{aligned}$$

$\mathscr{M}'$ is defined in the space of seven-dimensional fields $(\boldsymbol{\omega}', \mathbf{h}', \theta)$ which satisfy the boundary conditions stated by (8.6), (8.7) and (8.11), the vorticity, $\boldsymbol{\omega}'$, and magnetic, $\mathbf{h}'$, components are solenoidal, and $\langle\boldsymbol{\omega}'\rangle_v = 0$. Let $\mathscr{M}$ denote the restriction of $\mathscr{M}'$ to the subspace of the domain of $\mathscr{M}$, in which $\langle\mathbf{h}'\rangle_h = 0$. The substitutions

$$\boldsymbol{\omega} = \boldsymbol{\omega}' + \mathbf{a}, \quad \mathbf{v} = \mathbf{v}' + \nabla_{\mathbf{x}}A^{v} + \langle\mathbf{v}\rangle_h,$$

$$\mathbf{h} = \mathbf{h}' + \nabla_{\mathbf{x}}A^{h} + \langle\!\langle\mathbf{h}\rangle\!\rangle_h - \left\{\!\!\left\{\int_0^t \langle\mathbf{f}^h\rangle_h \mathrm{d}t\right\}\!\!\right\},$$

where A^v and A^h are globally bounded solutions to the Neumann problems

$$\nabla_{\mathbf{x}}^2 A^v = d^v, \quad \left.\frac{\partial A^v}{\partial x_3}\right|_{x_3=\pm L/2} = 0;$$
$$\nabla_{\mathbf{x}}^2 A^h = d^h, \quad \left.\frac{\partial A^h}{\partial x_3}\right|_{x_3=\pm L/2} = 0,$$

transform the problem (8.35)–(8.41) into an equivalent form

$$\mathscr{M}^{\omega}(\boldsymbol{\omega}', \mathbf{h}', \theta) = \mathbf{f}'^{\omega}, \tag{8.51}$$

$$\mathscr{M}^{h}(\boldsymbol{\omega}', \mathbf{h}') = \mathbf{f}'^{h}, \tag{8.52}$$

$$\mathscr{M}^{\theta}(\boldsymbol{\omega}', \theta) = f'^{\theta}, \tag{8.53}$$

$$\nabla_{\mathbf{x}} \cdot \boldsymbol{\omega}' = \nabla_{\mathbf{x}} \cdot \mathbf{h}' = 0, \tag{8.54}$$

$$\langle \boldsymbol{\omega}' \rangle_v = 0, \quad \langle \mathbf{h}' \rangle_h = 0, \tag{8.55}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{f}'^{\omega} = \nabla_{\mathbf{x}} \cdot \mathbf{f}'^{h} = 0, \tag{8.56}$$

$$\langle \mathbf{f}'^{\omega} \rangle_v = \langle \mathbf{f}'^{h} \rangle_h = 0. \tag{8.57}$$

The operator $\widetilde{\mathscr{L}^*} = ((\tilde{\mathscr{L}}^*)^v, (\tilde{\mathscr{L}}^*)^h, (\tilde{\mathscr{L}}^*)^\theta)$, adjoint to $\widetilde{\mathscr{L}} = (\mathscr{L}^v, \mathscr{L}^h, \mathscr{L}^\theta)$, can be derived performing integration by parts in the defining identity

$$\langle \widetilde{\mathscr{L}^*}(\mathbf{v}, \mathbf{h}, \theta) \cdot (\mathbf{v}', \mathbf{h}', \theta') \rangle \equiv \langle (\mathbf{v}, \mathbf{h}, \theta) \cdot \tilde{\mathscr{L}}(\mathbf{v}', \mathbf{h}', \theta') \rangle :$$

$$(\widetilde{\mathscr{L}^*})^v(\mathbf{v}, \mathbf{h}, \theta) = \frac{\partial \mathbf{v}}{\partial t} + \nu \nabla^2 \mathbf{v} - \nabla \times (\mathbf{V} \times \mathbf{v}) + \mathscr{P}_{\mathrm{sol}}\{\mathbf{H} \times (\nabla \times \mathbf{h}) - \mathbf{v} \times \boldsymbol{\Omega} - \tau \mathbf{v} \times \mathbf{e}_3 + \delta\theta \mathbf{e}_3 - \theta \nabla \Theta\}_h, \tag{8.58}$$

$$(\widetilde{\mathscr{L}^*})^h(\mathbf{v}, \mathbf{h}) = \frac{\partial \mathbf{h}}{\partial t} + \eta \nabla^2 \mathbf{h} + \nabla \times (\mathbf{H} \times \mathbf{v}) + \mathscr{P}_{\mathrm{sol}}(\mathbf{v} \times (\nabla \times \mathbf{H}) - \mathbf{V} \times (\nabla \times \mathbf{h})), \tag{8.59}$$

$$(\widetilde{\mathscr{L}^*})^\theta(\mathbf{v}, \theta) = \frac{\partial \theta}{\partial t} + \kappa \nabla^2 \theta + (\mathbf{V} \cdot \nabla)\theta + \beta v_3. \tag{8.60}$$

The conditions on the horizontal boundaries for the fields in the domain of $\widetilde{\mathscr{L}^*}$ are such that the surface integrals emerging in the integration by parts vanish. It is easy to check that if the boundary conditions for the domain of $\widetilde{\mathscr{L}}$ are stated by (8.5), (8.7) and (8.11), the same conditions hold for the domain of $\widetilde{\mathscr{L}^*}$. $\widetilde{\mathscr{L}^*}$ is thus defined in the class of vector fields, globally bounded with the derivatives,[2] that satisfy the boundary conditions (8.5), (8.7) and (8.11) and have solenoidal flow velocity and magnetic components.

The operator $\mathscr{M}'^{*} = ((\mathscr{M}'^{*})^{\omega}, (\mathscr{M}'^{*})^{h}, (\mathscr{M}'^{*})^{\theta})$, adjoint to $\mathscr{M}'$, can be determined as an adjoint to a composition of $\widetilde{\mathscr{L}}$ with other operators:

$$(\mathscr{M}'^{*})^{\omega}(\boldsymbol{\omega},\mathbf{h},\theta) = \frac{\partial\boldsymbol{\omega}}{\partial t} + \nu\nabla^2\boldsymbol{\omega} + \mathscr{C}'\{\nabla\times(-\mathbf{V}\times(\nabla\times\boldsymbol{\omega})) + \mathbf{H}\times(\nabla\times\mathbf{h}) + \boldsymbol{\Omega}\times(\nabla\times\boldsymbol{\omega}) - \tau\frac{\partial\boldsymbol{\omega}}{\partial x_3} + \delta\theta\mathbf{e}_3 - \theta\nabla\Theta\}_h, \tag{8.61}$$

$$(\mathscr{M}'^{*})^{h}(\boldsymbol{\omega},\mathbf{h}) = \frac{\partial\mathbf{h}}{\partial t} + \eta\nabla^2\mathbf{h} + \nabla\times(\mathbf{H}\times(\nabla\times\boldsymbol{\omega})) + \mathscr{P}_{\mathrm{sol}}((\nabla\times\boldsymbol{\omega})\times(\nabla\times\mathbf{H}) - \mathbf{V}\times(\nabla\times\mathbf{h})), \tag{8.62}$$

$$(\mathscr{M}'^{*})^{\theta}(\boldsymbol{\omega},\theta) = \frac{\partial\theta}{\partial t} + \kappa\nabla^2\theta + (\mathbf{V}\cdot\nabla)\theta + \beta\mathbf{e}_3\cdot(\nabla\times\boldsymbol{\omega}). \tag{8.63}$$

Here $\mathscr{C}'$ is also an inverse curl, as $\mathscr{C}$, but it is defined for different boundary conditions: if $\boldsymbol{\omega}$ is a solution to the problem

$$\nabla\times\boldsymbol{\omega} = \mathbf{v} - \nabla p \qquad \Leftrightarrow \qquad \nabla^2\boldsymbol{\omega} = -\nabla\times\mathbf{v},$$
$$\nabla\cdot\boldsymbol{\omega} = 0, \qquad \langle\boldsymbol{\omega}\rangle_v = 0,$$

satisfying the boundary conditions for vorticity, then $\mathscr{C}'\mathbf{v} \equiv \boldsymbol{\omega}$. Since $\mathscr{M}$ is a composition of $\mathscr{M}'$ and the projection onto the subspace of vector fields with the zero spatial mean of the horizontal part of the magnetic component, we find

$$\mathscr{M}^{*} = ((\mathscr{M}'^{*})^{\omega}, \{(\mathscr{M}'^{*})^{h}\}_h, (\mathscr{M}'^{*})^{\theta}). \tag{8.64}$$

The operators $\mathscr{M}'^{*}$ and $\mathscr{M}^{*}$ have the same domains as $\mathscr{M}'$ and $\mathscr{M}$, respectively.

Clearly, $(0,(C_1,C_2,0),0) \in \ker\mathscr{M}'^{*}$ for any constant C_1 and C_2. We consider the generic case, in which the kernel of $\mathscr{M}'^{*}$ is comprised of such constant vectors. (This is not so, when free hydromagnetic convection is considered, i.e., no additional external forces or sources of electric currents or heat act in the layer of fluid.) Then the kernel of $\mathscr{M}^{*}$ is trivial: by virtue of the expression (8.64) for the adjoint operator,

$$\mathscr{M}^{*}(\boldsymbol{\omega},\mathbf{h},\theta) = (0,(c_1(t),c_2(t),0),0)$$

[2] In this class the bilinear form $\langle\mathbf{a}\cdot\mathbf{b}\rangle$ is not a genuine scalar product, because, for instance, $\langle a^2(\mathbf{x},t)\rangle = 0$ for any smooth field a with a compact support.

is equivalent to the relation

$$\mathscr{M}^*\left(\boldsymbol{\omega}, \mathbf{h} - \sum_{k=1}^{2}\int_0^t c_k(t')\mathrm{d}t'\mathbf{e}_k, \theta\right) = 0$$

for any $(\boldsymbol{\omega}, \mathbf{h}, \theta) \in \ker\mathscr{M}^*$, and thus

$$\boldsymbol{\omega} = 0, \quad \mathbf{h} - \sum_{k=1}^{2}\int_0^t c_k(t')\mathrm{d}t'\mathbf{e}_k = (C_1, C_2, 0), \quad \theta = 0$$

in view of the structure of $\ker\mathscr{M}'^*$. Consequently, $\mathbf{h} = 0$, since $\langle\mathbf{h}\rangle_h = 0$ for any field from the domains of $\mathscr{M}$ and $\mathscr{M}^*$.

Application of the operators $(-\partial/\partial t + \nu\nabla^2)^{-1}, (-\partial/\partial t + \eta\nabla^2)^{-1}$ and $(-\partial/\partial t \ + \kappa\nabla^2)^{-1}$ to Eqs. 8.51–8.53, respectively [the r.h.s. of Eqs. 8.51 and 8.52 are in the domain of the operators by virtue of condition (8.57)] transforms the problem (8.51)–(8.55) into an equivalent one:

$$\mathscr{M}^\diamond(\boldsymbol{\omega}', \mathbf{h}', \theta) = \mathbf{f}''. \tag{8.65}$$

Here the operator $\mathscr{M}^\diamond$ is a sum of the identity and a compact operator (for a time-periodic problem a proof of this statement is similar to the one for the dynamo problem in Sect. 4.1.3). It has the same domain as $\mathscr{M}$, and the r.h.s. of (8.65) belongs to the domain [solenoidality is guaranteed by (8.65), and the mean vertical component of the r.h.s. of the vorticity equation and the mean horizontal component of the r.h.s. of the equation for the magnetic field remain zero after the respective operators were applied in the course of derivation of the problem (8.65)]. Thus, the Fredholm alternative theorem can be applied to equation (8.65). Since $\ker\mathscr{M}$ is trivial, the kernel of $(\mathscr{M}^\diamond)^*$ is also trivial, and by the theorem the problem (8.65) has a unique solution. Consequently, the problem (8.35)–(8.41), equivalent to (8.65), generically has a unique solution with the required periodicities.

As in Chap. 7, in Sects. 8.3–8.5 we consider the generic case, where the problem (8.35)–(8.41) has a solution for an arbitrary r.h.s. $\mathbf{f}^\omega, \mathbf{f}^h, f^\theta$ satisfying the conditions (8.45) and (8.46), without imposing the steadiness and/or periodicity conditions.

8.2.3 *A Bound for a Smooth Vector Field, Solenoidal in Fast Variables*

We obtain in this section a bound, which will be suitable hereafter: if the vertical component of a smooth vector field $\mathbf{v}(\mathbf{x}, \mathbf{X})$, solenoidal in the fast variables, vanishes at the horizontal boundaries, then the field $\langle\{\mathbf{v}\}_h|_{\mathbf{X}=\varepsilon(x_1,x_2)}\rangle$ is asymptotically smaller than any power of ε.

Let $[f]$ denote in this section the mean of f over a plane of the fast horizontal variables x_1, x_2. By the definition of $\{\cdot\}_h$, the horizontal component of the integral $\int_{-L/2}^{L/2}[\{\mathbf{v}\}_h]\mathrm{d}x_3$ vanishes for all $\mathbf{X}$. Averaging the solenoidality condition over the fast horizontal variables and integrating it in the vertical direction, we also find that $[v_3] = 0$ for all $\mathbf{X}$. Thus, we need to show that $\langle(\{\mathbf{v}\}_h - [\{\mathbf{v}\}_h])|_{\mathbf{X}=\varepsilon(x_1,x_2)}\rangle$ is asymptotically smaller than any power of ε.

The mean $[\cdot]$ of the r.h.s. of the equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\boldsymbol{\varphi} = \{\mathbf{v}\}_h - [\{\mathbf{v}\}_h]$$

is zero, and hence it has a solution globally bounded with the derivatives, such that $[\boldsymbol{\varphi}] = 0$. Denote $\boldsymbol{\varphi}_i = \partial\boldsymbol{\varphi}/\partial x_i$. By the chain rule,

$$\begin{aligned}
\left\langle(\{\mathbf{v}\}_h - [\{\mathbf{v}\}_h])|_{\mathbf{X}=\varepsilon(x_1,x_2)}\right\rangle &= \left\langle\left(\frac{\partial\boldsymbol{\varphi}_1}{\partial x_1} + \frac{\partial\boldsymbol{\varphi}_2}{\partial x_2}\right)\Bigg|_{\mathbf{X}=\varepsilon(x_1,x_2)}\right\rangle \\
&= \left\langle\frac{d\boldsymbol{\varphi}_1}{dx_1} + \frac{d\boldsymbol{\varphi}_2}{dx_2} - \varepsilon\left(\frac{\partial\boldsymbol{\varphi}_1}{\partial X_1} + \frac{\partial\boldsymbol{\varphi}_2}{\partial X_2}\right)\Bigg|_{\mathbf{X}=\varepsilon(x_1,x_2)}\right\rangle \\
&= -\varepsilon\left\langle\left(\frac{\partial\boldsymbol{\varphi}_1}{\partial X_1} + \frac{\partial\boldsymbol{\varphi}_2}{\partial X_2}\right)\Bigg|_{\mathbf{X}=\varepsilon(x_1,x_2)}\right\rangle.
\end{aligned}$$

Since $[\boldsymbol{\varphi}_i] = 0$, such a transformation can be repeated an arbitrary number of times. This proves our statement.

An analogous statement can be proved similarly: if a zero-mean ($\langle f\rangle = 0$) smooth field $f(\mathbf{X}, T, \mathbf{x}, t)$ is globally bounded together with the derivatives, then $\langle f|_{\mathbf{X}=\varepsilon(x_1,x_2)}\rangle = o(\varepsilon^n)$ for any $n > 0$.

8.3 The Combined Magnetic and Kinematic α-Effects

We start to solve successively the systems of Eqs. 8.29–8.31 for $n \geq 0$ together with the respective solenoidality and vorticity conditions. The flow of calculations is now more intricate than in the problems that we have studied so far. Because we use the Navier–Stokes equation in the form for vorticity, when solving the Eqs. 8.29–8.31 obtained at order ε^n we also have to consider the vorticity equation (8.29) obtained at order ε^{n+1} in order to determine $\langle\boldsymbol{\omega}_{n+1}\rangle_v$ and $\langle\mathbf{v}_n\rangle_h$. We advance by the following steps:

1°. Considering the mean over the fast spatial variables of the horizontal component of (8.30) at order ε^n, calculate $\langle\mathbf{h}_n\rangle_h$; derive a PDE in the slow variables in $\langle\!\langle\mathbf{h}_{n-2}\rangle\!\rangle_h$.

2°. Considering the mean over the fast spatial variables of the vertical component of (8.29) at order ε^{n+1}, calculate $\langle \mathbf{v}_n \rangle_h$; derive a PDE in the slow variables in $\langle\!\langle \mathbf{v}_{n-2} \rangle\!\rangle_h$.

3°. Derive a system of PDEs in the fast variables in $\{\boldsymbol{\omega}_n\}_v$, $\mathbf{h}_n$ and θ_n from the Eqs. 8.29–8.31 obtained at order ε^n.

4°. Following the procedure for solution of the problem (8.35)–(8.41) discussed in Sect. 8.2.2, express a solution $\boldsymbol{\omega}_n, \mathbf{v}_n, \mathbf{h}_n$ and θ_n to the problem obtained in step 3° in the terms of solutions to auxiliary problems and the spatio-temporal means of the terms of expansions (8.21)–(8.22).

This concludes solution of the Eqs. 8.29–8.31 obtained at order n, and we proceed to solution of the next system in the hierarchy.

8.3.1 Solution of Order ε^0 Equations

For $n = 0$, Eqs. 8.29–8.31 reduce to

$$\mathscr{L}(\boldsymbol{\omega}_0, \mathbf{v}_0, \mathbf{h}_0, \theta_0) = 0. \tag{8.66}$$

Steps 1° and 2° for $n = 0$. Averaging over the fast spatial variables the vertical component of the equation for vorticity and the horizontal component of the equation for magnetic field from the system (8.66), and the vertical component of the equation for vorticity (8.29) for $n = 1$, we find, respectively

$$\frac{\partial \langle \boldsymbol{\omega}_0 \rangle_v}{\partial t} = 0 \qquad \Rightarrow \qquad \langle \boldsymbol{\omega}_0 \rangle_v = 0$$

(provided $\langle \boldsymbol{\omega}_0 \rangle_v |_{t=0} = 0$—this is just the relation (8.33) for $n = 0$, which must be satisfied for solvability of the Eq. 8.32 for $n = 0$) and

$$\frac{\partial \langle \mathbf{h}_0 \rangle_h}{\partial t} = 0 \qquad \Rightarrow \qquad \langle \mathbf{h}_0 \rangle_h = \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h, \tag{8.67}$$

$$\frac{\partial \langle \boldsymbol{\omega}_1 \rangle_v}{\partial t} = 0 \qquad \Rightarrow \qquad \langle \boldsymbol{\omega}_1 \rangle_v = \langle\!\langle \boldsymbol{\omega}_1 \rangle\!\rangle_v. \tag{8.68}$$

The latter relation, the condition of solenoidality in the slow variables (8.24) for $n = 0$, and the vorticity relation (8.33) for $n = 1$ imply

$$\langle \mathbf{v}_0 \rangle_h = \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h + \mathbf{v}'_0(t, T).$$

We have shown in Sect. 8.2.3 that $\langle \{\mathbf{v}_0\}_h |_{\mathbf{X}=\varepsilon(x_1,x_2)} \rangle$ is asymptotically smaller than any power of ε. Thus, the condition of no-pumping (8.19) holds asymptotically to any power of ε, if $\mathbf{v}'_0 = 0$, i.e.

$$\langle \mathbf{v}_0 \rangle_h = \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h, \tag{8.69}$$

and the mean of $\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h$ over the plane of the slow spatial variables vanishes.

Step 3° for $n = 0$. Equations (8.66) can be expressed as

$$\mathscr{L}^{\omega}(\{\boldsymbol{\omega}_0\}_v, \{\mathbf{v}_0\}_h, \{\mathbf{h}_0\}_h, \theta_0) = \nabla_{\mathbf{x}} \times (-\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h \times \boldsymbol{\Omega} + \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h \times (\nabla_{\mathbf{x}} \times \mathbf{H})), \quad (8.70)$$

$$\mathscr{L}^{h}(\{\mathbf{v}_0\}_h, \{\mathbf{h}_0\}_h) = -(\langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h \cdot \nabla_{\mathbf{x}})\mathbf{V} + (\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h \cdot \nabla_{\mathbf{x}})\mathbf{H}, \quad (8.71)$$

$$\mathscr{L}^{\theta}(\{\mathbf{v}_0\}_h, \theta_0) = (\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h \cdot \nabla_{\mathbf{x}})\Theta. \quad (8.72)$$

This is a closed system together with the relations for $n = 0$ for the divergencies, (8.25)–(8.26) and (8.28), and for vorticity (8.34). Evidently, the solvability conditions (8.45)–(8.46) for a generic problem (8.35)–(8.41) considered in Sect. 8.2.2 are satisfied for this system.

Step 4° for $n = 0$. The means $\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h$ and $\langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h$. The means $\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h$ and $\langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h$ are independent of the fast variables, and differentiation in the operator $\mathscr{L}$ is performed in the fast variables only. By linearity, a solution has the following structure:

$$(\{\boldsymbol{\omega}_0\}_v, \{\mathbf{v}_0\}_h, \{\mathbf{h}_0\}_h, \theta_0) = \boldsymbol{\xi}_0 + \sum_{k=1}^{2} (\mathbf{S}_k^{v\cdot} \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k + \mathbf{S}_k^{h\cdot} \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k). \quad (8.73)$$

The ten-dimensional fields $\mathbf{S}(\mathbf{x}, t)$ satisfy the boundary conditions defined by (8.5)–(8.7) and (8.11), and solve the auxiliary problems of type I:

Auxiliary problem I.1 in $\mathbf{S}_k^{v\cdot} = (\mathbf{S}_k^{v\omega}, \mathbf{S}_k^{vv}, \mathbf{S}_k^{vh}, S_k^{v\theta})$.

$$\mathscr{L}(\mathbf{S}_k^{v\cdot}) = \left(\frac{\partial \boldsymbol{\Omega}}{\partial x_k}, \frac{\partial \mathbf{H}}{\partial x_k}, \frac{\partial \Theta}{\partial x_k}\right), \quad (8.74)$$

$$\nabla_{\mathbf{x}} \times \mathbf{S}_k^{vv} = \mathbf{S}_k^{v\omega}, \quad \nabla_{\mathbf{x}} \cdot \mathbf{S}_k^{vv} = 0, \quad \langle \mathbf{S}_k^{vv} \rangle_h = 0, \quad (8.75)$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{S}_k^{v\omega} = \nabla_{\mathbf{x}} \cdot \mathbf{S}_k^{vh} = 0, \quad (8.76)$$

$$\langle \mathbf{S}_k^{v\omega} \rangle_v = 0, \quad \langle \mathbf{S}_k^{vh} \rangle_h = 0. \quad (8.77)$$

Auxiliary problem I.2 in $\mathbf{S}_k^{h\cdot} = (\mathbf{S}_k^{h\omega}, \mathbf{S}_k^{hv}, \mathbf{S}_k^{hh}, S_k^{h\theta})$.

$$\mathscr{L}(\mathbf{S}_k^{h\cdot}) = \left(-\nabla_{\mathbf{x}} \times \frac{\partial \mathbf{H}}{\partial x_k}, -\frac{\partial \mathbf{V}}{\partial x_k}, 0\right), \quad (8.78)$$

$$\nabla_{\mathbf{x}} \times \mathbf{S}_k^{hv} = \mathbf{S}_k^{h\omega}, \quad \nabla_{\mathbf{x}} \cdot \mathbf{S}_k^{hv} = 0, \quad \langle \mathbf{S}_k^{hv} \rangle_h = 0, \quad (8.79)$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{S}_k^{h\omega} = \nabla_{\mathbf{x}} \cdot \mathbf{S}_k^{hh} = 0, \quad (8.80)$$

$$\langle \mathbf{S}_k^{h\omega} \rangle_v = 0, \qquad \langle \mathbf{S}_k^{hh} \rangle_h = 0. \quad (8.81)$$

Clearly, the auxiliary problems (8.74)–(8.77) and (8.78)–(8.81) are instances of the problem (8.35)–(8.41), for which the consistency relations (8.38)–(8.41) and the solvability conditions (8.45)–(8.46) and (8.49) are trivially satisfied. The solenoidality conditions in (8.75)–(8.76) and (8.79)–(8.80) stem from the equations for the divergencies (8.25)–(8.26) and (8.28) for $n=0$, and the vorticity relations in (8.75) and (8.79) from the relation (8.32) for $n=0$. The divergencies (8.76) and (8.80), and the means (8.77) and (8.81) vanish at all times as long as they do at $t=0$: by the standard arguments, these quantities are time-independent. Thus, any smooth vector fields, globally bounded with their derivatives, satisfying (8.75)–(8.77) and (8.79)–(8.81), and such that the solutions are globally bounded in space and time, can serve as initial conditions for the auxiliary problems of type I. (Recall that the solutions are globally bounded for any initial conditions, if the CHM regime $\mathbf{V},\mathbf{H},\Theta$ is stable to short-scale perturbations.)

Equations (8.74)–(8.77) and (8.78)–(8.81) are equivalent to the equalities

$$\mathscr{L}(\mathbf{S}_k^{v\omega},\mathbf{S}_k^{vv}+\mathbf{e}_k,\mathbf{S}_k^{vh},S_k^{v\theta})=0;\quad \mathscr{L}(\mathbf{S}_k^{h\omega},\mathbf{S}_k^{hv},\mathbf{S}_k^{hh}+\mathbf{e}_k,S_k^{h\theta})=0,$$

respectively. Hence in the auxiliary problem I.2 (but not I.1) one is supposed to find eigenfunctions from the kernel of $\mathscr{M}'$. Since for all constant C_1 and C_2, $(0,(C_1,C_2,0),0)\in\ker\mathscr{M}'^*$, the solutions exist for $k=1,2$ independently of whether $\ker\mathscr{M}$ is trivial (which we assume in Sects. 8.3)–(8.5). In computations it is natural to employ the vector potentials of the vorticity components of (8.74) and (8.78),

$$\mathscr{L}^v(\mathbf{S}_k^{vv},\mathbf{S}_k^{vh},S_k^{v\theta},S_k^{vp})=\frac{\partial\mathbf{V}}{\partial x_k},\quad \mathscr{L}^v(\mathbf{S}_k^{hv},\mathbf{S}_k^{hh},S_k^{h\theta},S_k^{hp})=-\frac{\partial\mathbf{H}}{\partial x_k},$$

respectively, and solve the auxiliary problems in $\mathbf{S}_k^{\cdot v},\mathbf{S}_k^{\cdot h},S_k^{\cdot\theta}$ and $S_k^{\cdot p}$ (in the two problems $\langle\nabla_{\mathbf{x}}S_k^{\cdot p}\rangle=0$ and $\langle S_k^{\cdot p}\rangle=0$).

The field $\xi_0(\mathbf{X},T,\mathbf{x},t)=(\xi_0^\omega,\xi_0^v,\xi_0^h,\xi_0^\theta)$, exponentially decaying in the fast time, satisfies the equations

$$\mathscr{L}(\xi_0)=0, \tag{8.82}$$

$$\nabla_{\mathbf{x}}\times\xi_0^v=\xi_0^\omega,\quad \nabla_{\mathbf{x}}\cdot\xi_0^v=0,\quad \langle\xi_0^v\rangle_h=0, \tag{8.83}$$

$$\nabla_{\mathbf{x}}\cdot\xi_0^\omega=\nabla_{\mathbf{x}}\cdot\xi_0^h=0, \tag{8.84}$$

$$\langle\xi_0^\omega\rangle_v=0,\quad \langle\xi_0^h\rangle_h=0. \tag{8.85}$$

Together, conditions (8.76), (8.80) and (8.84) guarantee that (8.26) and (8.28) hold true for $n=0$, and conditions (8.75), (8.79) and (8.83) that (8.32) for $n=0$ is satisfied. As for the auxiliary problems of type I, it suffices that (8.84) and (8.85) are satisfied at $t=0$. Initial conditions for the problem (8.82)–(8.85) can be determined from the expressions (8.73) for the leading terms in the expansions (8.20)–(8.23) at $t=0$. The initial conditions for ξ_0 must belong to the stable manifold of the CHM regime $\mathbf{V},\mathbf{H},\Theta$, i.e. ξ_0 must decay exponentially in the fast time; if the CHM regime is stable to short-scale perturbations, this is guaranteed for any smooth globally bounded initial conditions. A change in initial conditions for $\mathbf{S}$ in the class of permissible initial conditions is compensated by the respective change in initial conditions for ξ_0; the induced changes in $\mathbf{S}$ exponentially decay in time. The second relation in (8.67) implies $\langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h|_{T=0} = \langle \mathbf{h}_0 \rangle_h|_{t=0}$.

If the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ is steady, periodic or quasi-periodic in time and/or space, it is natural to demand that solutions to the auxiliary problems of type I, $\mathbf{S}$, have the same properties—the steadiness, periodicity or quasi-periodicity. The auxiliary problems being instances of the problem (8.51)–(8.57), their solvability is guaranteed for generic space-periodic CHM regimes, as shown in Sect. 8.2.2. ξ_0 is a transient dying out at the onset of the saturated (in the fast time) regime.

8.3.2 Solvability of Order ε^1 Equations

Step 1° for $n = 1$. Upon averaging over the fast spatial variables, the horizontal component of the Eq. 8.30 for $n = 1$, obtained from the magnetic induction equation at order ε^1, reduces to

$$\frac{\partial \langle \mathbf{h}_1 \rangle_h}{\partial t} = \nabla_{\mathbf{X}} \times \left(\sum_{k=1}^{2} \left(\boldsymbol{\alpha}_k^v \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k + \boldsymbol{\alpha}_k^h \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k \right) + \widetilde{\xi}^h \right). \tag{8.86}$$

To derive this equation, we have used the expression (8.43) for the mean of the image of the operator of linearisation, and the expressions (8.73) for the leading terms in the asymptotic expansions (8.21)–(8.22) of the flow and magnetic field perturbations. We have denoted

$$\boldsymbol{\alpha}_k^v \equiv \langle \mathbf{V} \times \mathbf{S}_k^{vh} + (\mathbf{S}_k^{vv} + \mathbf{e}_k) \times \mathbf{H} \rangle_v, \quad \boldsymbol{\alpha}_k^h \equiv \langle \mathbf{V} \times (\mathbf{S}_k^{hh} + \mathbf{e}_k) + \mathbf{S}_k^{hv} \times \mathbf{H} \rangle_v,$$
$$\widetilde{\xi}^h \equiv \langle \xi_0^v \times \mathbf{H} + \mathbf{V} \times \xi_0^h \rangle_v. \tag{8.87}$$

The operator $(\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k, \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k) \to \nabla_{\mathbf{X}} \times \sum_{k=1}^{2} \left(\langle\!\langle \boldsymbol{\alpha}_k^v \rangle\!\rangle \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k + \langle\!\langle \boldsymbol{\alpha}_k^h \rangle\!\rangle \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k \right)$, emerging in the r.h.s. of (8.86) upon averaging over the fast time, is called the *operator of the combined magnetic* α*-effect*.

As we have shown in Sect. 8.2.2 (see Eq. 8.48), (8.86) implies

$$\langle \mathbf{h}_1 \rangle_h = \langle\!\langle \mathbf{h}_1 \rangle\!\rangle_h + \nabla_{\mathbf{X}} \times \left(\left\{\!\!\left\{ \int_0^t \widetilde{\xi}^h \mathrm{d}t \right\}\!\!\right\} \right.$$
$$\left. + \sum_{k=1}^{2} \left(\left\{\!\!\left\{ \int_0^t \boldsymbol{\alpha}_k^v \mathrm{d}t \right\}\!\!\right\} \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k + \left\{\!\!\left\{ \int_0^t \boldsymbol{\alpha}_k^h \mathrm{d}t \right\}\!\!\right\} \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k \right) \right). \tag{8.88}$$

The magnetic α-effect is *insignificant in the leading order*, if the means $\langle\!\langle \int_0^t \boldsymbol{\alpha}_k \mathrm{d}t \rangle\!\rangle$ exist and thus the r.h.s. of this equation is correctly defined (the mean $\langle\!\langle \int_0^t \widetilde{\xi}^h \mathrm{d}t \rangle\!\rangle$ is well-defined, since $\widetilde{\xi}^h$ decays exponentially together with ξ_0). This implies the relations

$$\langle\!\langle \boldsymbol{\alpha}_k^v \rangle\!\rangle = \langle\!\langle \boldsymbol{\alpha}_k^h \rangle\!\rangle = 0, \tag{8.89}$$

analogous to the solvability condition (8.46) for the Eq. 8.30 for $n = 1$.

Step 2° for $n = 1$. The vertical component of the Eq. 8.29 for $n = 2$, obtained from the Navier–Stokes equation for vorticity at order ε^2, after averaging over the fast spatial variables becomes

$$\frac{\partial \langle \boldsymbol{\omega}_2 \rangle_v}{\partial t} = \nabla_{\mathbf{X}} \times \big\langle \mathbf{V} \times (\nabla_{\mathbf{X}} \times \mathbf{v}_0) - \mathbf{V} \nabla_{\mathbf{X}} \cdot \{\mathbf{v}_0\}_h - \mathbf{H} \times (\nabla_{\mathbf{X}} \times \mathbf{h}_0) + \mathbf{H} \nabla_{\mathbf{X}} \cdot \{\mathbf{h}_0\}_h \big\rangle_h. \tag{8.90}$$

Derivation of this equation exploits the boundary conditions for the flow and magnetic field, the expression (8.42) for the mean of the image of the operator of linearisation, expressions for the divergencies (8.24)–(8.26) and the vorticity (8.32) for $n = 0$ and 1, and the condition (8.33) for $n = 0$ under which recovery of the flow is possible. Substituting now the expressions (8.73) for the leading terms in the asymptotic expansions (8.21)–(8.22) of the flow and magnetic field perturbations, we obtain from (8.90)

$$\frac{\partial \langle \boldsymbol{\omega}_2 \rangle_v}{\partial t} = \nabla_{\mathbf{X}} \times \left(\sum_{k=1}^{2} \sum_{m=1}^{2} \left(\mathbf{a}_{mk}^v \frac{\partial \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k}{\partial X_m} + \mathbf{a}_{mk}^h \frac{\partial \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k}{\partial X_m} \right) + \widetilde{\xi}^\omega \right),$$

which is equivalent, by virtue of the solenoidality condition (8.24) for $n = 0$, to the equation

$$\frac{\partial \langle \boldsymbol{\omega}_2 \rangle_v}{\partial t} = \nabla_{\mathbf{X}} \times \left(\sum_{k=1}^{2} \sum_{m=1}^{2} \left(\alpha_{mk}^v \frac{\partial \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k}{\partial X_m} + \alpha_{mk}^h \frac{\partial \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k}{\partial X_m} \right) \mathbf{e}_k + \widetilde{\xi}^\omega \right). \tag{8.91}$$

We have denoted here

$$\mathbf{a}_{mk}^v \equiv \big\langle \mathbf{V} \times (\mathbf{e}_m \times (\mathbf{S}_k^{vv} + \mathbf{e}_k)) - \mathbf{H} \times (\mathbf{e}_m \times \mathbf{S}_k^{vh}) - \mathbf{V} (S_k^{vv})_m + \mathbf{H} (S_k^{vh})_m \big\rangle_h, \tag{8.92}$$

$$\mathbf{a}_{mk}^h \equiv \big\langle \mathbf{V} \times (\mathbf{e}_m \times \mathbf{S}_k^{hv}) - \mathbf{H} \times (\mathbf{e}_m \times (\mathbf{S}_k^{hh} + \mathbf{e}_k)) - \mathbf{V} (S_k^{hv})_m + \mathbf{H} (S_k^{hh})_m \big\rangle_h; \tag{8.93}$$

$$\alpha_{11}^v \equiv (a_{11}^v)_1 - (a_{21}^v)_2 - (a_{22}^v)_1, \quad \alpha_{21}^v \equiv (a_{21}^v)_1, \tag{8.94}$$

$$\alpha_{12}^v \equiv (a_{12}^v)_2, \quad \alpha_{22}^v \equiv (a_{22}^v)_2 - (a_{12}^v)_1 - (a_{11}^v)_2, \tag{8.95}$$

$$\alpha_{11}^h \equiv (a_{11}^h)_1 - (a_{21}^h)_2 - (a_{22}^h)_1, \quad \alpha_{21}^h \equiv (a_{21}^h)_1, \tag{8.96}$$

$$\alpha^h_{12} \equiv (a^h_{12})_2, \quad \alpha^h_{22} \equiv (a^h_{22})_2 - (a^h_{12})_1 - (a^h_{11})_2; \tag{8.97}$$

$$\widetilde{\xi}^\omega \equiv \left\langle \mathbf{V} \times (\nabla_{\mathbf{X}} \times \xi^v_0) - \mathbf{V}\nabla_{\mathbf{X}} \cdot \xi^v_0 - \mathbf{H} \times (\nabla_{\mathbf{X}} \times \xi^h_0) + \mathbf{H}\nabla_{\mathbf{X}} \cdot \xi^h_0 \right\rangle_h. \tag{8.98}$$

The operator

$$(\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k, \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k) \to \sum_{k=1}^{2}\sum_{m=1}^{2}\left(\langle\!\langle \mathbf{a}^v_{mk} \rangle\!\rangle \frac{\partial \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k}{\partial X_m} + \langle\!\langle \mathbf{a}^h_{mk} \rangle\!\rangle \frac{\partial \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k}{\partial X_m} \right),$$

appearing in the r.h.s. of (8.91) upon averaging over the fast time, is called the *operator of the combined AKA-effect* (anisotropic kinematic α-effect).

As shown in Sect. 8.2.2 (see (8.48)), (8.49) implies

$$\langle \boldsymbol{\omega}_2 \rangle_v = \langle\!\langle \boldsymbol{\omega}_2 \rangle\!\rangle_v + \nabla_{\mathbf{X}} \times \left(\left\{\!\!\left\{ \int_0^{t} \widetilde{\xi}^\omega \mathrm{d}t \right\}\!\!\right\}_h \right.$$
$$\left. + \sum_{k=1}^{2}\sum_{m=1}^{2}\left(\left\{\!\!\left\{ \int_0^t \alpha^v_{mk}\mathrm{d}t \right\}\!\!\right\} \frac{\partial \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k}{\partial X_m} + \left\{\!\!\left\{ \int_0^t \alpha^h_{mk}\mathrm{d}t \right\}\!\!\right\} \frac{\partial \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k}{\partial X_m} \right) \mathbf{e}_k \right). \tag{8.99}$$

The AKA-effect is *insignificant in the leading order*, if the r.h.s. of (8.99) is correctly defined, i.e. the means $\langle\!\langle \int_0^t \alpha_{mk}\mathrm{d}t \rangle\!\rangle$ exist ($\langle\!\langle \int_0^t \widetilde{\xi}^\omega \mathrm{d}t \rangle\!\rangle$ is well-defined, since $\widetilde{\xi}^\omega$ decays exponentially together with ξ_0). In particular, the insignificance implies that the analogues of the solvability condition (8.46) hold:

$$\langle\!\langle \alpha^v_{mk} \rangle\!\rangle = \langle\!\langle \alpha^h_{mk} \rangle\!\rangle = 0. \tag{8.100}$$

If the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ is steady or periodic in time, then insignificance in the leading order of the magnetic α-effect and the AKA-effect follows from the four scalar conditions (8.89) and eight scalar conditions (8.100), respectively.

Neither the horizontal components of the magnetic α-tensors, nor the vertical components of the kinematic α-tensors are required to vanish (and therefore we speak of the insignificance and not the absence of the combined α-effects).

Instead of the boundary condition (8.8) for temperature that we consider in this section, we might consider the condition of a constant heat flux through the horizontal boundaries of the layer:

$$\left.\frac{\partial \mathscr{T}}{\partial x_3}\right|_{x_3=\pm L/2} = \delta. \tag{8.101}$$

Then the system of mean-field equations is augmented by an equation for the mean perturbation of temperature, which we derive in this paragraph. We set now $\Theta = \mathscr{T} + \delta x_3$; the new variable satisfies (8.10). The seven-dimensional vector $(0, 0, C_\theta)$, where C_θ is a constant, now also belongs to the kernel of $\mathscr{M}^*$, and hence we have a new solvability condition for the problem (8.51)–(8.57):

$$\langle\!\langle f^{\theta}(\mathbf{x},t)\rangle\!\rangle = 0.$$

It is evidently satisfied for the auxiliary problems of type I. Averaging over the fast spatial variables the Eq. 8.31 for $n = 0$ obtained from the heat equation at the leading order, we find $\langle\theta_0\rangle = \langle\!\langle\theta_0\rangle\!\rangle$. The Eq. 8.31 for $n = 1$, averaged over the fast spatial variables, reduces to

$$-\frac{\partial\langle\theta_1\rangle}{\partial t} + \langle\delta(v_1)_3 - (\mathbf{v}_1\cdot\nabla_{\mathbf{x}})\Theta - (\mathbf{V}\cdot\nabla_{\mathbf{X}})\theta_0\rangle = 0. \tag{8.102}$$

By virtue of the relation (8.25) for $n = 1$ for the divergencies,

$$(v_1)_3 = -\int_{-L/2}^{x_3}\left(\frac{\partial(v_1)_1}{\partial x_1} + \frac{\partial(v_1)_2}{\partial x_2} + \frac{\partial(v_0)_1}{\partial X_1} + \frac{\partial(v_0)_2}{\partial X_2}\right)\mathrm{d}x_3.$$

Substituting this together with the expressions (8.73) for the leading terms in the expansions of the flow and magnetic field perturbations (which remain unaltered, except for the problems (8.74)–(8.77) and (8.78)–(8.81) must be solved for the new boundary conditions), we find from (8.102):

$$\frac{\partial\langle\theta_1\rangle}{\partial t} = -\sum_{k=1}^{2}\sum_{m=1}^{2}\left(\frac{\partial\langle\!\langle\mathbf{v}_0\rangle\!\rangle_k}{\partial X_m}\left\langle\delta\int_{-L/2}^{x_3}(S_k^{vv})_m\mathrm{d}x_3 + (S_k^{vv})_m\Theta + V_m S_k^{v\theta}\right\rangle\right.$$
$$\left. + \frac{\partial\langle\!\langle\mathbf{h}_0\rangle\!\rangle_k}{\partial X_m}\left\langle\delta\int_{-L/2}^{x_3}(S_k^{hv})_m\mathrm{d}x_3 + (S_k^{hv})_m\Theta + V_m S_k^{h\theta}\right\rangle\right) + \tilde{\xi}^{\theta}. \tag{8.103}$$

The differential operator in the r.h.s. of this equation is the so-called *operator of the combined thermal α-effect*. The mean $\langle\theta_1\rangle$ is well-defined, if the thermal α-effect is insignificant in the leading order, i.e. the means

$$\left\langle\!\!\left\langle\int_0^t\left(\delta\int_{-L/2}^{x_3}(S_k^{vv})_m\mathrm{d}x_3 + V_m S_k^{v\theta} + (S_k^{vv})_m\Theta\right)\mathrm{d}t\right\rangle\!\!\right\rangle$$

and

$$\left\langle\!\!\left\langle\int_0^t\left(\delta\int_{-L/2}^{x_3}(S_k^{hv})_m\mathrm{d}x_3 + V_m S_k^{h\theta} + (S_k^{hv})_m\Theta\right)\mathrm{d}t\right\rangle\!\!\right\rangle$$

are correctly defined (this can be easily established by the arguments similar to the ones presented above). In particular, this implies

$$\left\langle\!\!\left\langle \delta \int_{-L/2}^{x_3} (S_k^{vv})_m \mathrm{d}x_3 + V_m S_k^{v\theta} + (S_k^{vv})_m \Theta \right\rangle\!\!\right\rangle$$
$$= \left\langle\!\!\left\langle \delta \int_{-L/2}^{x_3} (S_k^{hv})_m \mathrm{d}x_3 + V_m S_k^{h\theta} + (S_k^{hv})_m \Theta \right\rangle\!\!\right\rangle = 0. \qquad (8.104)$$

If some of the solvability conditions (8.89), (8.100) and (8.104) are not satisfied, the asymptotic techniques for multiscale systems remain applicable, provided the slow time is of the order $1/\varepsilon$: $T = \varepsilon t$. Then the new terms $\partial\langle \mathbf{h}_0\rangle_h/\partial T$, $\partial\langle \boldsymbol{\omega}_1\rangle_v/\partial T$ and, in case of the boundary conditions (8.101), $\partial\langle\theta_0\rangle/\partial T$ emerge in the r.h.s. of (8.86), (8.91) and (8.103), respectively. Upon averaging over the fast time, Eqs. 8.86, 8.91 (and 8.103, if appropriate) constitute a closed system of the mean-field equations (which is supplemented by the solenoidality conditions (8.24) for $n = 0$ and the vorticity relation (8.33) for $n = 1$). As discussed in Chap. 2, generically a system of linear partial differential equations of the first order of such a structure has superexponentially growing solutions, and the unbounded growth of the perturbation results in the onset of a new CHM regime. Consequently, we end here the study of stability of CHM regimes exhibiting the α-effects (although, as in Chap. 6, we might continue solving the systems from the hierarchy and construct complete formal asymptotic expansions of the weakly nonlinear perturbations). The remainder of the chapter is devoted to a potentially more interesting case, where the α-effects are insignificant in the leading order, and thus the possible indefinite growth of perturbations in principle can be offset by other eddy effects and saturate.

8.4 The Symmetries of the CHM Regime Guaranteeing Insignificance of the MHD α-Effect

In this section we define the symmetries of a CHM regime, that guarantee the absence or insignificance of the α-effect. They are compatible with the Eqs. 8.1–8.4 and (8.10) governing forced convective dynamos (when the source terms $\nabla \times \mathbf{F}, \mathbf{J}$ and S possess the respective symmetry), and the boundary conditions (8.5)–(8.7) and (8.11).

A CHM regime is *parity-invariant with a time shift* $\widetilde{T}$, if the fields $\mathbf{V}$ and $\mathbf{H}$ are parity-invariant with the time shift, and $\boldsymbol{\Omega}$ and Θ are parity-antiinvariant. A three-dimensional vector field $\mathbf{f}$ is parity-invariant with a time shift $\widetilde{T}$, if

$$\mathbf{f}(-\mathbf{x}, t) = -\mathbf{f}(\mathbf{x}, t + \widetilde{T}),$$

and parity-antiinvariant, if

$$\mathbf{f}(-\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t + \widetilde{T}).$$

A scalar field f is parity-invariant with a time shift $\widetilde{T}$, if

$$f(-\mathbf{x},t)=f(\mathbf{x},t+\widetilde{T}),$$

and parity-antiinvariant, if

$$f(-\mathbf{x},t)=-f(\mathbf{x},t+\widetilde{T}).$$

(We assume that the origin of the coordinate system is at the centre of the symmetry, which is located on the mid-plane of the fluid layer and does not move.) For this symmetry, we call a set of fields $(\boldsymbol{\omega},\mathbf{v},\mathbf{h},\theta)$ *symmetric*, if $\mathbf{v}$ and $\mathbf{h}$ are parity-invariant, and $\boldsymbol{\omega}$ and θ parity-antiinvariant, and we call the set *antisymmetric*, if $\mathbf{v}$ and $\mathbf{h}$ are parity-antiinvariant, and $\boldsymbol{\omega}$ and θ parity-invariant.

A CHM regime is *symmetric about a vertical axis with a time shift* $\widetilde{T}$, if all fields $\mathbf{V},\mathbf{H},\boldsymbol{\Omega}$ and Θ are symmetric about the axis. A three-dimensional vector field $\mathbf{f}$ is symmetric about a vertical axis with a time shift $\widetilde{T}$, if

$$\begin{aligned}
f_1(-x_1,-x_2,x_3,t)&=-f_1(x_1,x_2,x_3,t+\widetilde{T}),\\
f_2(-x_1,-x_2,x_3,t)&=-f_2(x_1,x_2,x_3,t+\widetilde{T}),\\
f_3(-x_1,-x_2,x_3,t)&=f_3(x_1,x_2,x_3,t+\widetilde{T}),
\end{aligned}$$

and it is antisymmetric, if

$$\begin{aligned}
f_1(-x_1,-x_2,x_3,t)&=f_1(x_1,x_2,x_3,t+\widetilde{T}),\\
f_2(-x_1,-x_2,x_3,t)&=f_2(x_1,x_2,x_3,t+\widetilde{T}),\\
f_3(-x_1,-x_2,x_3,t)&=-f_3(x_1,x_2,x_3,t+\widetilde{T}).
\end{aligned}$$

A scalar field f is symmetric about the vertical axis with a time shift $\widetilde{T}$, if

$$f(-x_1,-x_2,x_3,t)=f(x_1,x_2,x_3,t+\widetilde{T}),$$

and antisymmetric, if

$$f(-x_1,-x_2,x_3,t)=-f(x_1,x_2,x_3,t+\widetilde{T}).$$

(Without any loss of generality we assume that the origin of the coordinate system is located on the axis under consideration, which does not move.) For this symmetry, a set of fields $(\boldsymbol{\omega},\mathbf{v},\mathbf{h},\theta)$ is called *symmetric*, if the four fields have the symmetry about the axis, and *antisymmetric*, if all four fields are antisymmetric.

The symmetries without a time shift (i.e., for $\widetilde{T}=0$) are spatial and do not affect the pattern of dependence of the field on time—they only have to be satisfied at any time. By contrast, only a time-periodic field with the period $2\widetilde{T}$ can possess a symmetry with a time shift $\widetilde{T}\neq 0$. For instance, a travelling wave may have such a symmetry.

Suppose a symmetry is defined by means of a mapping $\mathscr{S}$, i.e. symmetric fields satisfy the condition $\mathscr{S}\mathbf{f} = \mathbf{f}$, and antisymmetric ones the condition $\mathscr{S}\mathbf{f} = -\mathbf{f}$. Since all the symmetries that we consider in this chapter are of the second order (i.e., $\mathscr{S}^2$ is the identity), any scalar or vector field $\mathbf{f}$ can be decomposed into a sum of a symmetric, $(\mathbf{f} + \mathscr{S}\mathbf{f})/2$, and antisymmetric, $(\mathbf{f} - \mathscr{S}\mathbf{f})/2$, field.

It easy to establish that if the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ has a symmetry of the type discussed above, then the subspaces of symmetric (for the given symmetry) and antisymmetric sets of fields are invariant for the operator $\mathscr{L} = (\mathscr{L}^{\omega}, \mathscr{L}^{h}, \mathscr{L}^{\theta})$. In this case, the r.h.s. of the auxiliary problems of type I are antisymmetric sets, and hence their solutions $\mathbf{S}$ can be also regarded as antisymmetric sets (for a symmetry without a time shift, this requires antisymmetric initial conditions; however, the symmetric component of a solution is inessential, since by construction it must decay in the fast time). Consequently, the expressions for the entries of the tensors of the combined magnetic and kinematic α-effects, derived in the previous section, imply that both α-effects are insignificant in the leading order. By contrast, the presence of these symmetries does not guarantee the disappearance of the thermal α-effect. For this reason, we do not consider further the conditions of a constant thermal flux through the horizontal boundaries (8.101), but limit ourselves to the numerically more convenient conditions for the boundaries held at constant temperatures (8.8).

The definitions (8.92)–(8.97) of the quantities $\alpha^{\cdot}_{mk}$ imply

$$\alpha^{\cdot}_{mk}(t + \widetilde{T}) = -\alpha^{\cdot}_{mk}(t),$$

provided the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ has a symmetry with a time shift $\widetilde{T}$ of the type considered in this section. It is then easy to verify that $\vartheta^{\cdot}(t) \equiv \left\{\int_0^t \alpha^{\cdot}_{mk} \mathrm{d}t\right\}$ has a similar property $\vartheta^{\cdot}(t + \widetilde{T}) = -\vartheta^{\cdot}(t)$:

$$\vartheta^{\cdot}(t) = \int_0^t \alpha^{\cdot}_{mk}\mathrm{d}t - \frac{1}{2\widetilde{T}}\int_0^{2\widetilde{T}}\int_0^{\tilde{t}} \alpha^{\cdot}_{mk}(t')\mathrm{d}t'\mathrm{d}\tilde{t} = \frac{1}{2\widetilde{T}}\int_0^{2\widetilde{T}}\int_{\tilde{t}}^{t} \alpha^{\cdot}_{mk}(t')\mathrm{d}t'\mathrm{d}\tilde{t},$$

implying

$$\vartheta^{\cdot}(t) + \vartheta^{\cdot}(t + \widetilde{T}) = \frac{1}{2\widetilde{T}}\int_0^{2\widetilde{T}}\left(\int_{\tilde{t}}^{t} \alpha^{\cdot}_{mk}(t')\mathrm{d}t' + \int_{\tilde{t}}^{t+\widetilde{T}} \alpha^{\cdot}_{mk}(t')\mathrm{d}t'\right)\mathrm{d}\tilde{t}$$

$$= \frac{1}{2\widetilde{T}}\int_0^{2\widetilde{T}}\left(\int_{\tilde{t}}^{t} \alpha^{\cdot}_{mk}(t')\mathrm{d}t' - \int_{\tilde{t}-\widetilde{T}}^{t} \alpha^{\cdot}_{mk}(t')\mathrm{d}t'\right)\mathrm{d}\tilde{t} = -\frac{1}{2\widetilde{T}}\int_0^{2\widetilde{T}}\int_{\tilde{t}-\widetilde{T}}^{\tilde{t}} \alpha^{\cdot}_{mk}(t')\mathrm{d}t'\mathrm{d}\tilde{t}$$

$$= -\frac{1}{2\widetilde{T}}\int_0^{\widetilde{T}}\left(\int_{\tilde{t}-\widetilde{T}}^{\tilde{t}} \alpha^{\cdot}_{mk}(t')\mathrm{d}t' + \int_{\tilde{t}}^{\tilde{t}+\widetilde{T}} \alpha^{\cdot}_{mk}(t')\mathrm{d}t'\right)\mathrm{d}\tilde{t} = 0.$$

8.5 Mean-Field Equations for Large-Scale Perturbations in the Case of Insignificant α-Effect

In this section we construct the mean-field equations, assuming that the α-effects are insignificant in the leading order—for instance, if the CHM regime has a symmetry with a time shift discussed in the previous section.

8.5.1 Solution of Order ε^1 Equations

As discussed in Sect. 8.3.2, insignificance of the α-effect in the leading order implies solvability (8.29)–(8.31) for $n = 1$, obtained from the equations governing the evolution of a perturbation at order ε^1.

Step 2° for $n = 1$ *continued*. We have calculated in Sect. 8.3.2 the expression (8.99) for $\langle \boldsymbol{\omega}_2 \rangle_v$. Using the condition (8.24) for $n = 1$ of solenoidality of the mean flow in the slow variables, we find from the relation between the mean vorticity and flow velocity (8.33) for $n = 2$ the mean flow $\langle \mathbf{v}_1 \rangle_h$:

$$\langle \mathbf{v}_1 \rangle_h = \langle\!\langle \mathbf{v}_1 \rangle\!\rangle_h + \sum_{k=1}^{2} \sum_{m=1}^{2} \left(\left\{\!\!\left\{ \int_0^t \alpha^v_{mk} \mathrm{d}t \right\}\!\!\right\} \frac{\partial \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k}{\partial X_m} + \left\{\!\!\left\{ \int_0^t \alpha^h_{mk} \mathrm{d}t \right\}\!\!\right\} \frac{\partial \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k}{\partial X_m} \right) \mathbf{e}_k$$
$$- \nabla_{\mathbf{X}} \sum_{m=1}^{2} \left(\left\{\!\!\left\{ \int_0^t (\alpha^v_{m1} - \alpha^v_{m2}) \mathrm{d}t \right\}\!\!\right\} \frac{\partial^2 \nabla_{\mathbf{X}}^{-2} \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_1}{\partial X_m \partial X_1} \right.$$
$$\left. + \left\{\!\!\left\{ \int_0^t (\alpha^h_{m1} - \alpha^h_{m2}) \mathrm{d}t \right\}\!\!\right\} \frac{\partial^2 \nabla_{\mathbf{X}}^{-2} \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_1}{\partial X_m \partial X_1} \right) + \widetilde{\boldsymbol{\xi}}^v. \tag{8.105}$$

Here $\nabla_{\mathbf{X}}^{-2}$ denotes the inverse Laplacian in the slow variables: for $\mathbf{f}(\mathbf{X})$ with a vanishing mean over the plane of the slow variables, $\mathbf{g}(\mathbf{X}) = \nabla_{\mathbf{X}}^{-2} \mathbf{f}$ is a solution to the equation $\nabla_{\mathbf{X}}^2 \mathbf{g} = \mathbf{f}$, globally bounded together with the derivatives and having a zero mean over the plane of the slow variables.

In (8.105),

$$\widetilde{\boldsymbol{\xi}}^v = \left\{\!\!\left\{ \int_0^t \boldsymbol{\xi}' \mathrm{d}t \right\}\!\!\right\}, \quad \boldsymbol{\xi}'(\mathbf{X}, T, t) = \widetilde{\boldsymbol{\xi}}^\omega - \nabla_{\mathbf{X}} \nabla_{\mathbf{X}}^{-2} (\nabla_{\mathbf{X}} \cdot \widetilde{\boldsymbol{\xi}}^\omega). \tag{8.106}$$

We show now, following the proof in the first part of Sect. 7.4.2, that $\widetilde{\boldsymbol{\xi}}^v$ exponentially decays in the fast time. $\boldsymbol{\xi}'$ exponentially decays (inheriting this property from $\boldsymbol{\xi}_0$):

$$|\boldsymbol{\xi}'(\mathbf{X}, T, t)| \le \xi'_0 e^{-\alpha t},$$

and therefore

$$\begin{aligned}
\widetilde{\boldsymbol{\xi}}^{v} &= \int_{0}^{t} \boldsymbol{\xi}'(\mathbf{X},T,t')\mathrm{d}t' - \lim_{\hat{t}\to\infty}\frac{1}{\hat{t}}\int_{0}^{\hat{t}}\int_{0}^{t''} \boldsymbol{\xi}'(\mathbf{X},T,t')\mathrm{d}t'\mathrm{d}t'' \\
&= \lim_{\hat{t}\to\infty}\frac{1}{\hat{t}}\int_{0}^{\hat{t}}\int_{t''}^{t} \boldsymbol{\xi}'(\mathbf{X},T,t')\mathrm{d}t'\mathrm{d}t'' = -\lim_{\hat{t}\to\infty}\frac{1}{\hat{t}}\int_{t}^{\hat{t}}\int_{t}^{t''} \boldsymbol{\xi}'(\mathbf{X},T,t')\mathrm{d}t'\mathrm{d}t'' \\
&\Rightarrow |\widetilde{\boldsymbol{\xi}}^{v}| \leq \lim_{\hat{t}\to\infty}\frac{1}{\hat{t}}\int_{t}^{\hat{t}}\int_{t}^{t''} \xi_0' e^{-\alpha t'}\mathrm{d}t'\mathrm{d}t'' \leq \int_{t}^{\infty} \xi_0' e^{-\alpha t'}\mathrm{d}t' = (\xi_0'/\alpha)e^{-\alpha t}.
\end{aligned}$$

(Bounds for derivatives of $\widetilde{\boldsymbol{\xi}}^{v}$ in the slow variables can be derived similarly.)

Since the solvability condition (8.33) for the Eq. (8.32) for $n = 1$ is now satisfied, we can use this equation to find $\{\mathbf{v}_1\}_h$ (see Sect. 8.2.1). As we have shown in Sect. 8.2.3, the condition of no-pumping through the layer (8.19) is asymptotically satisfied for $\mathbf{v}_1|_{\mathbf{X}=\varepsilon(x_1,x_2)}$ up to an arbitrary power of ε, because by construction the mean of $\langle\mathbf{v}_1\rangle_h$ over the plane of the slow variables is zero.

Step 3° for $n = 1$. We substitute $\boldsymbol{\omega}_1 = \{\boldsymbol{\omega}_1\}_v + \nabla_{\mathbf{X}} \times \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h$ into (8.29). Upon splitting the unknown fields into the mean and fluctuating parts, the Eqs. 8.29–8.31 for $n = 1$ become:

$$\begin{aligned}
&\mathscr{L}^{\omega}(\{\boldsymbol{\omega}_1\}_v, \{\mathbf{v}_1\}_h, \{\!\{\mathbf{h}_1\}\!\}_h, \theta_1) + (\nabla_{\mathbf{X}} \times \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h)_3 \frac{\partial \mathbf{V}}{\partial x_3} - (\langle \mathbf{v}_1 \rangle_h \cdot \nabla_{\mathbf{x}})\boldsymbol{\Omega} \\
&\quad + (\langle\!\langle \mathbf{h}_1 \rangle\!\rangle_h \cdot \nabla_{\mathbf{x}})(\nabla_{\mathbf{x}} \times \mathbf{H}) + 2\nu(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\{\boldsymbol{\omega}_0\}_v - \nabla_{\mathbf{x}} \times (\mathbf{H} \times (\nabla_{\mathbf{X}} \times \mathbf{h}_0)) \\
&\quad + \nabla_{\mathbf{X}} \times (\mathbf{V} \times \boldsymbol{\omega}_0 + \mathbf{v}_0 \times \boldsymbol{\Omega} - \mathbf{H} \times (\nabla_{\mathbf{x}} \times \mathbf{h}_0) - \mathbf{h}_0 \times (\nabla_{\mathbf{x}} \times \mathbf{H})) \\
&\quad + \nabla_{\mathbf{x}} \times (\mathbf{v}_0 \times \boldsymbol{\omega}_0 - \mathbf{h}_0 \times (\nabla_{\mathbf{x}} \times \mathbf{h}_0)) + \beta \nabla_{\mathbf{X}} \theta_0 \times \mathbf{e}_3 = 0;
\end{aligned} \tag{8.107}$$

$$\begin{aligned}
&\mathscr{L}^{h}(\{\mathbf{v}_1\}_h, \{\!\{\mathbf{h}_1\}\!\}_h) - (\langle \mathbf{v}_1 \rangle_h \cdot \nabla_{\mathbf{x}})\mathbf{H} + (\langle\!\langle \mathbf{h}_1 \rangle\!\rangle_h \cdot \nabla_{\mathbf{x}})\mathbf{V} + 2\eta(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\{\mathbf{h}_0\}_h \\
&\quad + \nabla_{\mathbf{X}} \times (\mathbf{v}_0 \times \mathbf{H} + \mathbf{V} \times \mathbf{h}_0) + \nabla_{\mathbf{x}} \times (\mathbf{v}_0 \times \mathbf{h}_0) = 0;
\end{aligned} \tag{8.108}$$

$$\begin{aligned}
&\mathscr{L}^{\theta}(\{\mathbf{v}_1\}_h, \theta_1) - (\langle \mathbf{v}_1 \rangle_h \cdot \nabla_{\mathbf{x}})\Theta + 2\kappa(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\theta_0 - (\mathbf{V} \cdot \nabla_{\mathbf{X}})\theta_0 \\
&\quad - (\mathbf{v}_0 \cdot \nabla_{\mathbf{x}})\theta_0 = 0.
\end{aligned} \tag{8.109}$$

Step 4° for $n = 1$. We substitute into Eqs. 8.107–8.109 the expressions (8.73) for the leading terms of the expansion of the perturbation (8.20)–(8.23), and the expression (8.105) for the spatial mean of the leading term of the expansion of the flow velocity, and use the supplementary relations for $n = 1$ for the divergencies, (8.25), (8.26) and (8.28), and for vorticity (8.34). By linearity, solutions to Eqs. 8.107–8.109 have the following structure:

$$(\{\boldsymbol{\omega}_1\}_v, \{\mathbf{v}_1\}_h, \{\!\{\mathbf{h}_1\}\!\}_h, \theta_1) = \xi_1 + \sum_{k=1}^{2}\Bigg(\mathbf{S}_k^{v\cdot}\langle\!\langle \mathbf{v}_1\rangle\!\rangle_k + \mathbf{S}_k^{h\cdot}\langle\!\langle \mathbf{h}_1\rangle\!\rangle_k$$
$$+\sum_{m=1}^{2}\Bigg(\mathbf{G}_{mk}^{v\cdot}\frac{\partial\langle\!\langle \mathbf{v}_0\rangle\!\rangle_k}{\partial X_m} + \mathbf{G}_{mk}^{h\cdot}\frac{\partial\langle\!\langle \mathbf{h}_0\rangle\!\rangle_k}{\partial X_m} + \mathbf{Y}_{mk}^{v\cdot}\frac{\partial^3\nabla_{\mathbf{X}}^{-2}\langle\!\langle \mathbf{v}_0\rangle\!\rangle_1}{\partial X_k\partial X_m\partial X_1} + \mathbf{Y}_{mk}^{h\cdot}\frac{\partial^3\nabla_{\mathbf{X}}^{-2}\langle\!\langle \mathbf{h}_0\rangle\!\rangle_1}{\partial X_k\partial X_m\partial X_1}$$
$$+\mathbf{Q}_{mk}^{vv\cdot}\langle\!\langle \mathbf{v}_0\rangle\!\rangle_k\langle\!\langle \mathbf{v}_0\rangle\!\rangle_m + \mathbf{Q}_{mk}^{vh\cdot}\langle\!\langle \mathbf{v}_0\rangle\!\rangle_k\langle\!\langle \mathbf{h}_0\rangle\!\rangle_m + \mathbf{Q}_{mk}^{hh\cdot}\langle\!\langle \mathbf{h}_0\rangle\!\rangle_k\langle\!\langle \mathbf{h}_0\rangle\!\rangle_m\Bigg)\Bigg) \qquad (8.110)$$

Here the fields $\mathbf{G}(\mathbf{x},t)$ are solutions to the auxiliary problems of type II:

Auxiliary problem II.1 in $\mathbf{G}_{mk}^{v\cdot} = (\mathbf{G}_{mk}^{v\omega}, \mathbf{G}_{mk}^{vv}, \mathbf{G}_{mk}^{vh}, G_{mk}^{v\theta})$

$$\mathscr{L}^{\omega}(\mathbf{G}_{mk}^{v\omega}, \mathbf{G}_{mk}^{vv}, \mathbf{G}_{mk}^{vh}, G_{mk}^{v\theta}) = -\varepsilon_{mk3}\frac{\partial\mathbf{V}}{\partial x_3} - 2\nu\frac{\partial\mathbf{S}_k^{v\omega}}{\partial x_m} - \mathbf{e}_m\times\big(\mathbf{V}\times\mathbf{S}_k^{v\omega}$$
$$+(\mathbf{S}_k^{vv}+\mathbf{e}_k)\times\boldsymbol{\Omega} - \mathbf{H}\times(\nabla_{\mathbf{x}}\times\mathbf{S}_k^{vh}) - \mathbf{S}_k^{vh}\times(\nabla_{\mathbf{x}}\times\mathbf{H}) + \beta S_k^{v\theta}\mathbf{e}_3\big)$$
$$+\nabla_{\mathbf{x}}\times(\mathbf{H}\times(\mathbf{e}_m\times\mathbf{S}_k^{vh})) + \left\{\!\!\left\{\int_0^t \alpha_{mk}^{v}\,\mathrm{d}t\right\}\!\!\right\}\frac{\partial\boldsymbol{\Omega}}{\partial x_k}, \qquad (8.111)$$

$$\mathscr{L}^{h}(\mathbf{G}_{mk}^{vv}, \mathbf{G}_{mk}^{vh}) = -2\eta\frac{\partial\mathbf{S}_k^{vh}}{\partial x_m} - \mathbf{e}_m\times\big(\mathbf{V}\times\mathbf{S}_k^{vh} + (\mathbf{S}_k^{vv}+\mathbf{e}_k)\times\mathbf{H}\big) + \left\{\!\!\left\{\int_0^t \alpha_{mk}^{v}\,\mathrm{d}t\right\}\!\!\right\}\frac{\partial\mathbf{H}}{\partial x_k}, \qquad (8.112)$$

$$\mathscr{L}^{\theta}(\mathbf{G}_{mk}^{vv}, G_{mk}^{v\theta}) = -2\kappa\frac{\partial S_k^{v\theta}}{\partial x_m} + V_m S_k^{v\theta} + \left\{\!\!\left\{\int_0^t \alpha_{mk}^{v}\,\mathrm{d}t\right\}\!\!\right\}\frac{\partial\Theta}{\partial x_k}, \qquad (8.113)$$

$$\nabla_{\mathbf{x}}\cdot\mathbf{G}_{mk}^{v\omega} = -(S_k^{v\omega})_m, \quad \nabla_{\mathbf{x}}\cdot\mathbf{G}_{mk}^{vv} = -(S_k^{vv})_m, \quad \nabla_{\mathbf{x}}\cdot\mathbf{G}_{mk}^{vh} = -(S_k^{vh})_m, \qquad (8.114)$$

$$\nabla_{\mathbf{x}}\times\mathbf{G}_{mk}^{vv} = \mathbf{G}_{mk}^{v\omega} - \mathbf{e}_m\times\mathbf{S}_k^{vv}; \qquad (8.115)$$

here ε_{mkj} is the antisymmetric unit tensor.

Auxiliary problem II.2 in $\mathbf{G}_{mk}^{h\cdot} = (\mathbf{G}_{mk}^{h\omega}, \mathbf{G}_{mk}^{hv}, \mathbf{G}_{mk}^{hh}, G_{mk}^{h\theta})$.

$$\mathscr{L}^{\omega}(\mathbf{G}_{mk}^{h\omega}, \mathbf{G}_{mk}^{hv}, \mathbf{G}_{mk}^{hh}, G_{mk}^{h\theta}) = -2\nu\frac{\partial\mathbf{S}_k^{h\omega}}{\partial x_m} - \mathbf{e}_m\times\big(\mathbf{V}\times\mathbf{S}_k^{h\omega}$$
$$+\mathbf{S}_k^{hv}\times\boldsymbol{\Omega} - \mathbf{H}\times(\nabla_{\mathbf{x}}\times\mathbf{S}_k^{hh}) - (\mathbf{S}_k^{hh}+\mathbf{e}_k)\times(\nabla_{\mathbf{x}}\times\mathbf{H}) + \beta S_k^{h\theta}\mathbf{e}_3\big)$$
$$+\nabla_{\mathbf{x}}\times(\mathbf{H}\times(\mathbf{e}_m\times(\mathbf{S}_k^{hh}+\mathbf{e}_k))) + \left\{\!\!\left\{\int_0^t \alpha_{mk}^{h}\,\mathrm{d}t\right\}\!\!\right\}\frac{\partial\boldsymbol{\Omega}}{\partial x_k}, \qquad (8.116)$$

$$\mathscr{L}^h(\mathbf{G}_{mk}^{hv}, \mathbf{G}_{mk}^{hh}) = -2\eta \frac{\partial \mathbf{S}_k^{hh}}{\partial x_m} - \mathbf{e}_m \times \big(\mathbf{V} \times (\mathbf{S}_k^{hh} + \mathbf{e}_k) + \mathbf{S}_k^{hv} \times \mathbf{H}\big) + \left\{\!\!\left\{ \int_0^t \alpha_{mk}^h \mathrm{d}t \right\}\!\!\right\} \frac{\partial \mathbf{H}}{\partial x_k}, \tag{8.117}$$

$$\mathscr{L}^\theta(\mathbf{G}_{mk}^{hv}, G_{mk}^{h\theta}) = -2\kappa \frac{\partial S_k^{h\theta}}{\partial x_m} + V_m S_k^{h\theta} + \left\{\!\!\left\{ \int_0^t \alpha_{mk}^h \mathrm{d}t \right\}\!\!\right\} \frac{\partial \Theta}{\partial x_k}, \tag{8.118}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{G}_{mk}^{h\omega} = -(S_k^{h\omega})_m, \quad \nabla_{\mathbf{x}} \cdot \mathbf{G}_{mk}^{hv} = -(S_k^{hv})_m, \quad \nabla_{\mathbf{x}} \cdot \mathbf{G}_{mk}^{hh} = -(S_k^{hh})_m, \tag{8.119}$$

$$\nabla_{\mathbf{x}} \times \mathbf{G}_{mk}^{hv} = \mathbf{G}_{mk}^{h\omega} - \mathbf{e}_m \times \mathbf{S}_k^{hv}. \tag{8.120}$$

The fields $\mathbf{Q}(\mathbf{x}, t)$ are solutions to the auxiliary problems of type III:
Auxiliary problem III.1 in $\mathbf{Q}_{mk}^{vv\cdot} = (\mathbf{Q}_{mk}^{vv\omega}, \mathbf{Q}_{mk}^{vvv}, \mathbf{Q}_{mk}^{vvh}, Q_{mk}^{vv\theta})$.

$$\mathscr{L}^\omega(\mathbf{Q}_{mk}^{vv\omega}, \mathbf{Q}_{mk}^{vvv}, \mathbf{Q}_{mk}^{vvh}, Q_{mk}^{vv\theta}) = \rho_{mk} \nabla_{\mathbf{x}} \times \Big(-(\mathbf{S}_k^{vv} + \mathbf{e}_k) \times \mathbf{S}_m^{v\omega}$$
$$+ \mathbf{S}_k^{vh} \times (\nabla_{\mathbf{x}} \times \mathbf{S}_m^{vh}) - (\mathbf{S}_m^{vv} + \mathbf{e}_m) \times \mathbf{S}_k^{v\omega} + \mathbf{S}_m^{vh} \times (\nabla_{\mathbf{x}} \times \mathbf{S}_k^{vh}) \Big), \tag{8.121}$$

$$\nabla_{\mathbf{x}} \times \mathbf{Q}_{mk}^{vvv} = \mathbf{Q}_{mk}^{vv\omega}, \tag{8.122}$$

$$\mathscr{L}^h(\mathbf{Q}_{mk}^{vvv}, \mathbf{Q}_{mk}^{vvh}) = -\rho_{mk} \nabla_{\mathbf{x}} \times \Big((\mathbf{S}_k^{vv} + \mathbf{e}_k) \times \mathbf{S}_m^{vh} + (\mathbf{S}_m^{vv} + \mathbf{e}_m) \times \mathbf{S}_k^{vh} \Big), \tag{8.123}$$

$$\mathscr{L}^\theta(\mathbf{Q}_{mk}^{vvv}, Q_{mk}^{vv\theta}) = \rho_{mk} \Big(((\mathbf{S}_k^{vv} + \mathbf{e}_k) \cdot \nabla_{\mathbf{x}}) S_m^{v\theta} + ((\mathbf{S}_m^{vv} + \mathbf{e}_m) \cdot \nabla_{\mathbf{x}}) S_k^{v\theta} \Big). \tag{8.124}$$

Auxiliary problem III.2 in $\mathbf{Q}_{mk}^{vh\cdot} = (\mathbf{Q}_{mk}^{vh\omega}, \mathbf{Q}_{mk}^{vhv}, \mathbf{Q}_{mk}^{vhh}, Q_{mk}^{vh\theta})$.

$$\mathscr{L}^\omega(\mathbf{Q}_{mk}^{vh\omega}, \mathbf{Q}_{mk}^{vhv}, \mathbf{Q}_{mk}^{vhh}, Q_{mk}^{vh\theta}) = -\nabla_{\mathbf{x}} \times \Big((\mathbf{S}_k^{vv} + \mathbf{e}_k) \times \mathbf{S}_m^{h\omega} + \mathbf{S}_m^{hv} \times \mathbf{S}_k^{v\omega}$$
$$- \mathbf{S}_k^{vh} \times (\nabla_{\mathbf{x}} \times \mathbf{S}_m^{hh}) - (\mathbf{S}_m^{hh} + \mathbf{e}_m) \times (\nabla_{\mathbf{x}} \times \mathbf{S}_k^{vh}) \Big), \tag{8.125}$$

$$\nabla_{\mathbf{x}} \times \mathbf{Q}_{mk}^{vhv} = \mathbf{Q}_{mk}^{vh\omega}, \tag{8.126}$$

$$\mathscr{L}^h(\mathbf{Q}_{mk}^{vhv}, \mathbf{Q}_{mk}^{vhh}) = -\nabla_{\mathbf{x}} \times \Big((\mathbf{S}_k^{vv} + \mathbf{e}_k) \times (\mathbf{S}_m^{hh} + \mathbf{e}_m) + \mathbf{S}_m^{hv} \times \mathbf{S}_k^{vh} \Big), \tag{8.127}$$

$$\mathscr{L}^\theta(\mathbf{Q}_{mk}^{vhv}, Q_{mk}^{vh\theta}) = ((\mathbf{S}_k^{vv} + \mathbf{e}_k) \cdot \nabla_{\mathbf{x}}) S_m^{h\theta} + (\mathbf{S}_m^{hv} \cdot \nabla_{\mathbf{x}}) S_k^{v\theta}. \tag{8.128}$$

Auxiliary problem III.3 in $\mathbf{Q}_{mk}^{hh\cdot} = (\mathbf{Q}_{mk}^{hh\omega}, \mathbf{Q}_{mk}^{hhv}, \mathbf{Q}_{mk}^{hhh}, Q_{mk}^{hh\theta})$.

$$\mathscr{L}^{\omega}(\mathbf{Q}^{hh\omega}_{mk}, \mathbf{Q}^{hhv}_{mk}, \mathbf{Q}^{hhh}_{mk}, Q^{hh\theta}_{mk}) = \rho_{mk}\nabla_{\mathbf{x}} \times \Big(-\mathbf{S}^{hv}_{k} \times \mathbf{S}^{h\omega}_{m}$$
$$+ (\mathbf{S}^{hh}_{k} + \mathbf{e}_{k}) \times (\nabla_{\mathbf{x}} \times \mathbf{S}^{hh}_{m}) - \mathbf{S}^{hv}_{m} \times \mathbf{S}^{h\omega}_{k} + (\mathbf{S}^{hh}_{m} + \mathbf{e}_{m}) \times (\nabla_{\mathbf{x}} \times \mathbf{S}^{hh}_{k})\Big), \qquad (8.129)$$

$$\nabla_{\mathbf{x}} \times \mathbf{Q}^{hhv}_{mk} = \mathbf{Q}^{hh\omega}_{mk}, \qquad (8.130)$$

$$\mathscr{L}^{h}(\mathbf{Q}^{hhv}_{mk}, \mathbf{Q}^{hhh}_{mk}) = -\rho_{mk}\nabla_{\mathbf{x}} \times \Big(\mathbf{S}^{hv}_{k} \times (\mathbf{S}^{hh}_{m} + \mathbf{e}_{m}) + \mathbf{S}^{hv}_{m} \times (\mathbf{S}^{hh}_{k} + \mathbf{e}_{k})\Big), \qquad (8.131)$$

$$\mathscr{L}^{\theta}(\mathbf{Q}^{hhv}_{mk}, Q^{hh\theta}_{mk}) = \rho_{mk}\Big((\mathbf{S}^{hv}_{k} \cdot \nabla_{\mathbf{x}})S^{h\theta}_{m} + (\mathbf{S}^{hv}_{m} \cdot \nabla_{\mathbf{x}})S^{h\theta}_{k}\Big). \qquad (8.132)$$

The auxiliary problems of type III are supplemented by the solenoidality conditions:

$$\nabla_{\mathbf{x}} \cdot \mathbf{Q}^{\cdots}_{mk} = 0. \qquad (8.133)$$

The problems III.1 (8.121)–(8.124) and III.3 (8.129)–(8.132) are stated for $m \le k$. We have denoted $\rho_{mk} = 1$ for $m < k$, and $\rho_{mk} = 1/2$ for $m = k$; for $m > k$ by definition $\rho_{mk} = 0$ and $\mathbf{Q}^{vv\cdot}_{mk} = \mathbf{Q}^{hh\cdot}_{mk} = 0$.

The fields $\mathbf{Y}(\mathbf{x}, t)$ are solutions to the auxiliary problems of type IV:

Auxiliary problem IV.1 in $\mathbf{Y}^{v\cdot}_{mk} = (\mathbf{Y}^{v\omega}_{mk}, \mathbf{Y}^{vv}_{mk}, \mathbf{Y}^{vh}_{mk}, Y^{v\theta}_{mk})$.

$$\mathscr{M}(\mathbf{Y}^{v\cdot}_{mk}) = -\rho_{mk}\left(\left\{\!\!\left\{\int_0^t (\alpha^{v}_{m1} - \alpha^{v}_{m2})\,\mathrm{d}t\right\}\!\!\right\}\frac{\partial}{\partial x_k} + \left\{\!\!\left\{\int_0^t (\alpha^{v}_{k1} - \alpha^{v}_{k2})\,\mathrm{d}t\right\}\!\!\right\}\frac{\partial}{\partial x_m}\right)(\boldsymbol{\Omega}, \mathbf{H}, \Theta), \qquad (8.134)$$

$$\nabla_{\mathbf{x}} \times \mathbf{Y}^{vv}_{mk} = \mathbf{Y}^{v\omega}_{mk}. \qquad (8.135)$$

Auxiliary problem IV.2 in $\mathbf{Y}^{h\cdot}_{mk} = (\mathbf{Y}^{h\omega}_{mk}, \mathbf{Y}^{hv}_{mk}, \mathbf{Y}^{hh}_{mk}, Y^{h\theta}_{mk})$.

$$\mathscr{M}(\mathbf{Y}^{h\cdot}_{mk}) = -\rho_{mk}\left(\left\{\!\!\left\{\int_0^t (\alpha^{h}_{m1} - \alpha^{h}_{m2})\,\mathrm{d}t\right\}\!\!\right\}\frac{\partial}{\partial x_k} + \left\{\!\!\left\{\int_0^t (\alpha^{h}_{k1} - \alpha^{h}_{k2})\,\mathrm{d}t\right\}\!\!\right\}\frac{\partial}{\partial x_m}\right)(\boldsymbol{\Omega}, \mathbf{H}, \Theta), \qquad (8.136)$$

$$\nabla_{\mathbf{x}} \times \mathbf{Y}^{hv}_{mk} = \mathbf{Y}^{h\omega}_{mk}. \qquad (8.137)$$

The auxiliary problems of type IV are supplemented by the solenoidality conditions:

$$\nabla_{\mathbf{x}} \cdot \mathbf{Y}^{\cdot\cdot}_{mk} = 0. \qquad (8.138)$$

$\mathbf{Y}^{\cdot\cdot}_{mk} = 0$ for $m > k$.

The transient $\boldsymbol{\xi}_1(\mathbf{x}, t, \mathbf{X}, T)$, exponentially decaying in time, is governed by the equations

$$\mathscr{L}^{\omega}(\boldsymbol{\xi}_1) = -2\nu(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\boldsymbol{\xi}_0^{\omega} - \nabla_{\mathbf{X}}\times\big(\mathbf{V}\times\boldsymbol{\xi}_0^{\omega} + \boldsymbol{\xi}_0^{v}\times\boldsymbol{\Omega} - \mathbf{H}\times(\nabla_{\mathbf{x}}\times\boldsymbol{\xi}_0^{h})$$
$$- \boldsymbol{\xi}_0^{h}\times(\nabla_{\mathbf{x}}\times\mathbf{H})\big) - \nabla_{\mathbf{x}}\times\big(\mathbf{v}_0\times\boldsymbol{\xi}_0^{\omega} + \boldsymbol{\xi}_0^{v}\times(\boldsymbol{\omega}_0 - \boldsymbol{\xi}_0^{\omega}) - \mathbf{H}\times(\nabla_{\mathbf{X}}\times\boldsymbol{\xi}_0^{h})$$
$$-\mathbf{h}_0\times(\nabla_{\mathbf{X}}\times\boldsymbol{\xi}_0^{h}) - \boldsymbol{\xi}_0^{h}\times(\nabla_{\mathbf{x}}\times(\mathbf{h}_0 - \boldsymbol{\xi}_0^{h}))\big) - \beta\nabla_{\mathbf{X}}\xi_0^{\theta}\times\mathbf{e}_3, \tag{8.139}$$

$$\mathscr{L}^{h}(\boldsymbol{\xi}_1) = -2\eta(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\boldsymbol{\xi}_1^{h} - \nabla_{\mathbf{X}}\times(\boldsymbol{\xi}_0^{v}\times\mathbf{H} + \mathbf{V}\times\boldsymbol{\xi}_0^{h})$$
$$- \nabla_{\mathbf{x}}\times(\mathbf{v}_0\times\boldsymbol{\xi}_0^{h} + \boldsymbol{\xi}_0^{v}\times(\mathbf{h}_0 - \boldsymbol{\xi}_0^{h})), \tag{8.140}$$

$$\mathscr{L}^{\theta}(\boldsymbol{\xi}_1) = -2\kappa(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\xi_0^{\theta} + (\mathbf{V}\cdot\nabla_{\mathbf{X}})\xi_0^{\theta} + (\mathbf{v}_0\cdot\nabla_{\mathbf{x}})\xi_0^{\theta} + (\boldsymbol{\xi}_0^{v}\cdot\nabla_{\mathbf{x}})(\theta_0 - \xi_0^{\theta}), \tag{8.141}$$

$$\nabla_{\mathbf{x}}\cdot\boldsymbol{\xi}_1^{\omega} = -\nabla_{\mathbf{X}}\cdot\boldsymbol{\xi}_0^{\omega},\quad \nabla_{\mathbf{x}}\cdot\boldsymbol{\xi}_1^{v} = -\nabla_{\mathbf{X}}\cdot\boldsymbol{\xi}_0^{v},\quad \nabla_{\mathbf{x}}\cdot\boldsymbol{\xi}_1^{h} = -\nabla_{\mathbf{X}}\cdot\boldsymbol{\xi}_0^{h}, \tag{8.142}$$

$$\nabla_{\mathbf{x}}\times\boldsymbol{\xi}_1^{v} - \boldsymbol{\xi}_1^{\omega} = -\nabla_{\mathbf{X}}\times\boldsymbol{\xi}_0^{v}. \tag{8.143}$$

The fields $\mathbf{G}, \mathbf{Q}, \mathbf{Y}$ and $\boldsymbol{\xi}_1$ satisfy the boundary conditions stated by (8.5)–(8.7) and (8.11). The spatial means of the horizontal component of the flow velocity and the vertical component of the vorticity, as well as the spatio-temporal means of the horizontal component of the magnetic fields must vanish for any of these fields.

The vorticity equations (8.121), (8.125) and (8.129) in the auxiliary problems of type III are equivalent to the respective equations for their vector potentials,

$$\mathscr{L}^{v}\big(\mathbf{Q}_{mk}^{vvv}, \mathbf{Q}_{mk}^{vvh}, Q_{mk}^{vv\theta}, Q_{mk}^{vvp}\big) = \rho_{mk}\big(-(\mathbf{S}_k^{vv} + \mathbf{e}_k)\times\mathbf{S}_m^{v\omega}$$
$$+\mathbf{S}_k^{vh}\times(\nabla_{\mathbf{x}}\times\mathbf{S}_m^{vh}) - (\mathbf{S}_m^{vv} + \mathbf{e}_m)\times\mathbf{S}_k^{v\omega} + \mathbf{S}_m^{vh}\times(\nabla_{\mathbf{x}}\times\mathbf{S}_k^{vh})\big), \tag{8.144}$$

$$\mathscr{L}^{v}\Big(\mathbf{Q}_{mk}^{vhv}, \mathbf{Q}_{mk}^{vhh}, Q_{mk}^{vh\theta}, Q_{mk}^{vhp}\Big) = -(\mathbf{S}_k^{vv} + \mathbf{e}_k)\times\mathbf{S}_m^{h\omega} - \mathbf{S}_m^{hv}\times\mathbf{S}_k^{v\omega}$$
$$+\mathbf{S}_k^{vh}\times(\nabla_{\mathbf{x}}\times\mathbf{S}_m^{hh}) + (\mathbf{S}_m^{hh} + \mathbf{e}_m)\times(\nabla_{\mathbf{x}}\times\mathbf{S}_k^{vh}), \tag{8.145}$$

$$\mathscr{L}^{v}\Big(\mathbf{Q}_{mk}^{hhv}, \mathbf{Q}_{mk}^{hhh}, Q_{mk}^{hh\theta}, Q_{mk}^{hhp}\Big) = \rho_{mk}\big(-\mathbf{S}_k^{hv}\times\mathbf{S}_m^{h\omega} + (\mathbf{S}_k^{hh} + \mathbf{e}_k)\times(\nabla_{\mathbf{x}}\times\mathbf{S}_m^{hh})$$
$$-\mathbf{S}_m^{hv}\times\mathbf{S}_k^{h\omega} + (\mathbf{S}_m^{hh} + \mathbf{e}_m)\times(\nabla_{\mathbf{x}}\times\mathbf{S}_k^{hh})\big). \tag{8.146}$$

Here $\langle\nabla_{\mathbf{x}}Q_{mk}^{\cdot\cdot p}\rangle_h = 0$. Note that the means over the fast spatial variables of the horizontal components of the r.h.s. of the above equations are zero.

Together, the relations for vorticity (8.115), (8.120), (8.122), (8.126), (8.130), (8.135), (8.137) and (8.143) are equivalent to the equation for determination of the flow, (8.34) for $n = 1$, where the flow velocity and vorticity (8.110) is substituted. Similarly, for the flow velocity, vorticity and magnetic field (8.110), the relations for divergence (8.114), (8.119), (8.113), (8.138) and (8.142) are together equivalent to the relations (8.25), (8.26) and (8.28) for $n = 1$. It suffices that the relations for the divergence of vorticity and magnetic components hold at $t = 0$, because the consistency relations (8.39) and (8.40) are satisfied for the data in the auxiliary problems [in particular, this can be easily demonstrated for the auxiliary problems of type II using Eqs. 8.74, 8.76, 8.78 and 8.80].

Initial conditions for the spatial means of solutions to the auxiliary problems and the transient $\boldsymbol{\xi}_1$ can be found by averaging the equalities (8.47) in the fast time:

$$\langle \mathbf{G}^{vh}_{mk} \rangle_h \Big|_{t=0} = -\left\langle\!\!\left\langle \mathbf{e}_m \times \int_0^t \boldsymbol{\alpha}^v_k \mathrm{d}t \right\rangle\!\!\right\rangle_h, \quad \langle \mathbf{G}^{hh}_{mk} \rangle_h \Big|_{t=0} = -\left\langle\!\!\left\langle \mathbf{e}_m \times \int_0^t \boldsymbol{\alpha}^h_k \mathrm{d}t \right\rangle\!\!\right\rangle_h,$$
$$\langle \boldsymbol{\xi}^h_1 \rangle_h \Big|_{t=0} = -\left\langle\!\!\left\langle \int_0^t \nabla_{\mathbf{x}} \times \widetilde{\boldsymbol{\xi}}^h \mathrm{d}t \right\rangle\!\!\right\rangle_h. \tag{8.147}$$

For all $t \geq 0$,

$$\langle \mathbf{G}^{\cdot\omega}_{mk} \rangle_v = \langle \mathbf{Q}^{\cdot\cdot\omega}_{mk} \rangle_v = \langle \mathbf{Y}^{\cdot\omega}_{mk} \rangle_v = \langle \boldsymbol{\xi}^{\omega}_1 \rangle_v = \langle \mathbf{Q}^{\cdot\cdot h}_{mk} \rangle_h = \langle \mathbf{Y}^{\cdot h}_{mk} \rangle_h = 0. \tag{8.148}$$

Initial conditions in the slow time for the mean vorticity can be found, using (8.68):

$$\langle\!\langle \boldsymbol{\omega}_1 \rangle\!\rangle_v \Big|_{T=0} = \langle \boldsymbol{\omega}_1 \rangle_v \Big|_{t=0}.$$

Averaging over the fast spatial variables the horizontal magnetic component of the equation (8.110) expressing the second term in the expansion of the perturbation, where $\{\!\{\mathbf{h}_1\}\!\}_h = \mathbf{h}_1 - \langle\!\langle \mathbf{h}_1 \rangle\!\rangle_h$, we obtain at $t = 0$

$$\langle\!\langle \mathbf{h}_1 \rangle\!\rangle_h \Big|_{T=0} = \langle \mathbf{h}_1 \rangle_h \Big|_{t=0} - \langle \boldsymbol{\xi}^h_1 \rangle_h \Big|_{t=0}$$
$$- \sum_{k=1}^{2} \sum_{m=1}^{2} \left(\langle \mathbf{G}^{vh}_{mk} \rangle_h \Big|_{t=0} \frac{\partial \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k}{\partial X_m} \Big|_{T=0} + \langle \mathbf{G}^{hh}_{mk} \rangle_h \Big|_{t=0} \frac{\partial \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k}{\partial X_m} \Big|_{T=0} \right).$$

Initial conditions for the problem (8.139)–(8.143) can now be determined from the vorticity, magnetic field and temperature components of equation (8.110) at $t = 0$.

The initial conditions must guarantee the global boundedness of solutions to each auxiliary problem. (As shown in Sect. 7.4.2, the solutions are bounded with the derivatives for any initial conditions, if the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ is space-periodic and stable to short-scale perturbations.) Any modification of the initial conditions for $\mathbf{G}$, $\mathbf{Q}$ and $\mathbf{Y}$ in the class of the permissible initial conditions is offset by a respective modification of the initial conditions for $\boldsymbol{\xi}_1$. (The permissible changes in the initial conditions belong to the stable manifold of the CHM regime, i.e. the changes in the solutions to the auxiliary problems caused by these changes must exponentially decay in the fast time).

Since the fields $\boldsymbol{\xi}_0$ and their derivatives exponentially decay in the fast time (see Sect. 8.3.1), the same holds true for the r.h.s. of (8.139)–(8.143). It was shown in Sect. 7.4.2, that this implies the exponential decay of $\boldsymbol{\xi}_1$ and the changes in $\mathbf{G}$ and $\mathbf{Q}$ due to permissible changes in the initial conditions for $\mathbf{S}$, if the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ is space-periodic and stable to short-scale perturbations. For the CHM regimes which do not have these properties the exponential decay of $\boldsymbol{\xi}_1$ must be guaranteed by an appropriate choice of the initial conditions.

Suppose now the CHM regime is a symmetric set of fields for a symmetry of the type considered in the previous section. Then $\mathbf{S}$ is essentially an antisymmetric set and thus the r.h.s. of the equations in the formulation of the auxiliary problems of types II–IV are symmetric sets (i.e., have the same symmetry as the CHM regime). Since the subspaces of symmetric and antisymmetric sets of fields are invariant for the operator of linearisation $\mathscr{L} = (\mathscr{L}^{\omega}, \mathscr{L}^{h}, \mathscr{L}^{\theta})$, the fields $\mathbf{G}$, $\mathbf{Q}$ and $\mathbf{Y}$ are symmetric sets (by construction, their antisymmetric components exponentially decay and thus can be ignored). Therefore, if the symmetry of the CHM regime does not involve a time shift ($\widetilde{T} = 0$), then the relations (8.147) and (8.148) for the averaged initial conditions are trivially satisfied (except for vanishing of the mean vertical components of the vorticity). Also, for $\widetilde{T} = 0$ we find $\mathbf{a}^{v}_{mk} = \mathbf{a}^{h}_{mk} = 0$; consequently, the r.h.s. of Eqs. 8.134 and 8.136 in the statement of the auxiliary problems of type IV are zero, and hence essentially $\mathbf{Y} = 0$.

If the CHM regime is a steady state, or it is periodic or quasi-periodic in the fast time, it is natural to demand that the fields $\mathbf{G}, \mathbf{Q}$ and $\mathbf{Y}$ are, respectively, steady, periodic or quasi-periodic solutions to the auxiliary problems of types II–IV; then $\boldsymbol{\xi}_1$ is a decaying transient, bringing the solution to the saturated behaviour. We have shown in Sect. 8.2.2 that such solutions do exist for generic space-periodic CHM regimes, that are steady or periodic in time.

8.5.2 A Solvability Condition for Order ε^2 Equations: The Equation for the Mean Magnetic Perturbation

The Eqs. 8.29–8.31, obtained at order ε^2 from the Eqs. 8.13–8.15 governing the evolution of perturbations, are

$$\begin{aligned}&\mathscr{L}^{\omega}(\boldsymbol{\omega}_2,\mathbf{v}_2,\mathbf{h}_2,\theta_2)-\frac{\partial\boldsymbol{\omega}_0}{\partial T}+\nu(2(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\{\boldsymbol{\omega}_1\}_{\nu}+\nabla^2_{\mathbf{X}}\boldsymbol{\omega}_0)+\nabla_{\mathbf{X}}\times\Big(\mathbf{V}\times\boldsymbol{\omega}_1\\&\quad+\mathbf{v}_1\times\boldsymbol{\Omega}-\mathbf{H}\times(\nabla_{\mathbf{X}}\times\mathbf{h}_0+\nabla_{\mathbf{x}}\times\mathbf{h}_1)-\mathbf{h}_1\times(\nabla_{\mathbf{x}}\times\mathbf{H})+\mathbf{v}_0\times\boldsymbol{\omega}_0-\mathbf{h}_0\times(\nabla_{\mathbf{x}}\times\mathbf{h}_0)\Big)\\&\quad+\nabla_{\mathbf{x}}\times\Big(-\mathbf{H}\times(\nabla_{\mathbf{X}}\times\mathbf{h}_1)+\mathbf{v}_1\times\boldsymbol{\omega}_0+\mathbf{v}_0\times\boldsymbol{\omega}_1-\mathbf{h}_1\times(\nabla_{\mathbf{x}}\times\mathbf{h}_0)\\&\quad-\mathbf{h}_0\times(\nabla_{\mathbf{x}}\times\mathbf{h}_1+\nabla_{\mathbf{X}}\times\mathbf{h}_0)\Big)+\beta\nabla_{\mathbf{X}}\theta_1\times\mathbf{e}_3=0;\end{aligned}\tag{8.149}$$

$$\begin{aligned}&\mathscr{L}^{h}(\mathbf{v}_2,\mathbf{h}_2)-\frac{\partial\mathbf{h}_0}{\partial T}+\eta(2(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\{\mathbf{h}_1\}_h+\nabla^2_{\mathbf{X}}\mathbf{h}_0)\\&\quad+\nabla_{\mathbf{X}}\times(\mathbf{v}_1\times\mathbf{H}+\mathbf{V}\times\mathbf{h}_1+\mathbf{v}_0\times\mathbf{h}_0)+\nabla_{\mathbf{x}}\times(\mathbf{v}_1\times\mathbf{h}_0+\mathbf{v}_0\times\mathbf{h}_1)=0;\end{aligned}\tag{8.150}$$

$$\begin{aligned}&\mathscr{L}^{\theta}(\mathbf{v}_2,\theta_2)-\frac{\partial\theta_0}{\partial T}+\kappa(2(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\theta_1+\nabla^2_{\mathbf{X}}\theta_0)\\&\quad-(\mathbf{V}\cdot\nabla_{\mathbf{X}})\theta_1-(\mathbf{v}_1\cdot\nabla_{\mathbf{x}})\theta_0-(\mathbf{v}_0\cdot\nabla_{\mathbf{x}})\theta_1-(\mathbf{v}_0\cdot\nabla_{\mathbf{X}})\theta_0=0.\end{aligned}\tag{8.151}$$

Step 1° for $n = 2$. We average the horizontal component of (8.150) in the fast spatial variables, and take into account the expression (8.43) for the mean of the image of the operator of linearisation, the flow and magnetic components of the expressions (8.73) for the leading terms of the expansion of the perturbation, the conditions of insignificance of the α-effect (8.89), the boundary conditions for the CHM regime and its perturbation, and the exponential decay of ξ_0 and ξ_1. The result is the *equation for the mean perturbation of magnetic field*:

$$-\frac{\partial}{\partial T}\langle\!\langle \mathbf{h}_0\rangle\!\rangle_h + \eta\nabla^2_{\mathbf{X}}\langle\!\langle \mathbf{h}_0\rangle\!\rangle_h + \nabla_{\mathbf{X}}\times\Bigg(\langle\!\langle \mathbf{v}_0\rangle\!\rangle_h \times \langle\!\langle \mathbf{h}_0\rangle\!\rangle_h$$
$$+\sum_{m=1}^{2}\sum_{k=1}^{2}\Bigg(\frac{\partial}{\partial X_m}\Bigg(\mathbf{D}^{vh}_{mk}\langle\!\langle \mathbf{v}_0\rangle\!\rangle_k + \mathbf{D}^{hh}_{mk}\langle\!\langle \mathbf{h}_0\rangle\!\rangle_k + \frac{\partial^2}{\partial X_k \partial X_1}\nabla^{-2}_{\mathbf{X}}\left(\mathbf{d}^{vh}_{mk}\langle\!\langle \mathbf{v}_0\rangle\!\rangle_1 + \mathbf{d}^{hh}_{mk}\langle\!\langle \mathbf{h}_0\rangle\!\rangle_1\right)\Bigg)$$
$$+\mathbf{A}^{vvh}_{mk}\langle\!\langle \mathbf{v}_0\rangle\!\rangle_k\langle\!\langle \mathbf{v}_0\rangle\!\rangle_m + \mathbf{A}^{vhh}_{mk}\langle\!\langle \mathbf{v}_0\rangle\!\rangle_k\langle\!\langle \mathbf{h}_0\rangle\!\rangle_m + \mathbf{A}^{hhh}_{mk}\langle\!\langle \mathbf{h}_0\rangle\!\rangle_k\langle\!\langle \mathbf{h}_0\rangle\!\rangle_m\Bigg)\Bigg) = 0. \qquad (8.152)$$

The constant vectors $\mathbf{D}$ and $\mathbf{A}$ enter into the operator of *combined eddy correction of magnetic diffusion* and the quadratic operator of *combined eddy correction of magnetic field advection*, respectively:

$$\mathbf{D}^{vh}_{mk} \equiv \langle\!\langle \mathbf{V}\times\mathbf{G}^{vh}_{mk} - \mathbf{H}\times\widetilde{\mathbf{G}}^{vv}_{mk}\rangle\!\rangle_v, \quad \mathbf{D}^{hh}_{mk} \equiv \langle\!\langle \mathbf{V}\times\mathbf{G}^{hh}_{mk} - \mathbf{H}\times\widetilde{\mathbf{G}}^{hv}_{mk}\rangle\!\rangle_v, \qquad (8.153)$$

$$\mathbf{d}^{vh}_{mk} \equiv \langle\!\langle \mathbf{V}\times\mathbf{Y}^{vh}_{mk} - \mathbf{H}\times\widetilde{\mathbf{Y}}^{vv}_{mk}\rangle\!\rangle_v, \quad \mathbf{d}^{hh}_{mk} \equiv \langle\!\langle \mathbf{V}\times\mathbf{Y}^{hh}_{mk} - \mathbf{H}\times\widetilde{\mathbf{Y}}^{hv}_{mk}\rangle\!\rangle_v, \qquad (8.154)$$

where

$$\widetilde{\mathbf{G}}^{vv}_{mk} \equiv \mathbf{G}^{vv}_{mk} + \left\{\!\!\left\{\int_0^t \alpha^v_{mk}\,\mathrm{d}t\right\}\!\!\right\}\mathbf{e}_k, \quad \widetilde{\mathbf{G}}^{hv}_{mk} \equiv \mathbf{G}^{hv}_{mk} + \left\{\!\!\left\{\int_0^t \alpha^h_{mk}\,\mathrm{d}t\right\}\!\!\right\}\mathbf{e}_k,$$

$$\widetilde{\mathbf{Y}}^{vv}_{mk} \equiv \mathbf{Y}^{vv}_{mk} - \left\{\!\!\left\{\int_0^t (\alpha^v_{m1} - \alpha^v_{m2})\,\mathrm{d}t\right\}\!\!\right\}\mathbf{e}_k, \quad \widetilde{\mathbf{Y}}^{hv}_{mk} \equiv \mathbf{Y}^{hv}_{mk} - \left\{\!\!\left\{\int_0^t (\alpha^h_{m1} - \alpha^h_{m2})\,\mathrm{d}t\right\}\!\!\right\}\mathbf{e}_k;$$

$$\mathbf{A}^{vvh}_{mk} \equiv \langle\!\langle \mathbf{V}\times\mathbf{Q}^{vvh}_{mk} - \mathbf{H}\times\mathbf{Q}^{vvv}_{mk} + \mathbf{S}^{vv}_k \times \mathbf{S}^{vh}_m\rangle\!\rangle_v, \qquad (8.155)$$

$$\mathbf{A}^{vhh}_{mk} \equiv \langle\!\langle \mathbf{V}\times\mathbf{Q}^{vhh}_{mk} - \mathbf{H}\times\mathbf{Q}^{vhv}_{mk} + \mathbf{S}^{vv}_k \times \mathbf{S}^{hh}_m + \mathbf{S}^{hv}_m \times \mathbf{S}^{vh}_k\rangle\!\rangle_v, \qquad (8.156)$$

$$\mathbf{A}^{hhh}_{mk} \equiv \langle\!\langle \mathbf{V}\times\mathbf{Q}^{hhh}_{mk} - \mathbf{H}\times\mathbf{Q}^{hhv}_{mk} + \mathbf{S}^{hv}_k \times \mathbf{S}^{hh}_m\rangle\!\rangle_v. \qquad (8.157)$$

One can simplify the operator of eddy correction of magnetic diffusion, exploiting solenoidality of $\langle\!\langle \mathbf{v}_0\rangle\!\rangle_h$ and $\langle\!\langle \mathbf{h}_0\rangle\!\rangle_h$:

$$\nabla_{\mathbf{X}}\times\sum_{m=1}^{2}\sum_{k=1}^{2}\frac{\partial}{\partial X_m}\left(\mathbf{D}^{vh}_{mk}\langle\!\langle \mathbf{v}_0\rangle\!\rangle_k + \mathbf{D}^{hh}_{mk}\langle\!\langle \mathbf{h}_0\rangle\!\rangle_k\right)$$
$$= \nabla_{\mathbf{X}}\times\sum_{m=1}^{2}\sum_{k=1}^{2}\frac{\partial}{\partial X_m}\left(\widetilde{D}^{vh}_{mk}\langle\!\langle \mathbf{v}_0\rangle\!\rangle_k + \widetilde{D}^{hh}_{mk}\langle\!\langle \mathbf{h}_0\rangle\!\rangle_k\right)\mathbf{e}_k, \qquad (8.158)$$

$$\nabla_{\mathbf{X}}\times\sum_{m=1}^{2}\sum_{k=1}^{2}\frac{\partial}{\partial X_m}\left(\mathbf{D}^{vh}_{mk}\langle\!\langle\mathbf{v}_0\rangle\!\rangle_k+\mathbf{D}^{hh}_{mk}\langle\!\langle\mathbf{h}_0\rangle\!\rangle_k\right)$$
$$=\nabla_{\mathbf{X}}\times\sum_{m=1}^{2}\sum_{k=1}^{2}\frac{\partial}{\partial X_m}\left(\widetilde{D}^{vh}_{mk}\langle\!\langle\mathbf{v}_0\rangle\!\rangle_k+\widetilde{D}^{hh}_{mk}\langle\!\langle\mathbf{h}_0\rangle\!\rangle_k\right)\mathbf{e}_k, \tag{8.159}$$

where, for $n=3-k$,

$$\widetilde{D}^{vh}_{nk}\equiv(D^{vh}_{nk})_k,\quad \widetilde{D}^{vh}_{kk}\equiv(D^{vh}_{kk})_k-(D^{vh}_{nn})_k-(D^{vh}_{nk})_n, \tag{8.160}$$

$$\widetilde{D}^{hh}_{nk}\equiv(D^{hh}_{nk})_k,\quad \widetilde{D}^{hh}_{kk}\equiv(D^{hh}_{kk})_k-(D^{hh}_{nn})_k-(D^{hh}_{nk})_n. \tag{8.161}$$

Thus, the solvability conditions for Eqs. 8.149–8.151 give rise to the expression (8.99) for the mean vorticity $\langle\boldsymbol{\omega}_2\rangle_v$ and the equation (8.152) for the mean perturbation of magnetic field. Since they are now satisfied, Eqs. 8.149–8.151 (supplemented by the appropriate relations) can be solved in $\{\boldsymbol{\omega}_2\}_v$, $\{\!\{\mathbf{h}_2\}\!\}_h$ and θ_2. However, these quantities do not contribute to the mean-field equation that remains to be derived, and we do not consider (8.149)–(8.151) further.

8.5.3 A Solvability Condition for Order ε^3 Equations: The Equation for the Mean Flow Perturbation

The Eqs. 8.29–8.31, obtained at order ε^3 from the equations for the evolution of perturbations, are

$$\begin{aligned}&\mathscr{L}^{\omega}(\boldsymbol{\omega}_3,\mathbf{v}_3,\mathbf{h}_3,\theta_3)-\frac{\partial\boldsymbol{\omega}_1}{\partial T}+\nu(2(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\{\boldsymbol{\omega}_2\}_v+\nabla^2_{\mathbf{X}}\boldsymbol{\omega}_1)\\&+\nabla_{\mathbf{X}}\times\Big(\mathbf{V}\times\boldsymbol{\omega}_2+\mathbf{v}_2\times\boldsymbol{\Omega}-\mathbf{H}\times(\nabla_{\mathbf{X}}\times\mathbf{h}_1+\nabla_{\mathbf{x}}\times\mathbf{h}_2)-\mathbf{h}_2\times(\nabla_{\mathbf{x}}\times\mathbf{H})\\&+\mathbf{v}_1\times\boldsymbol{\omega}_0+\mathbf{v}_0\times\boldsymbol{\omega}_1-\mathbf{h}_1\times(\nabla_{\mathbf{x}}\times\mathbf{h}_0)-\mathbf{h}_0\times(\nabla_{\mathbf{x}}\times\mathbf{h}_1+\nabla_{\mathbf{X}}\times\mathbf{h}_0)\Big)\\&+\nabla_{\mathbf{x}}\times\Big(-\mathbf{H}\times(\nabla_{\mathbf{X}}\times\mathbf{h}_2)+\mathbf{v}_2\times\boldsymbol{\omega}_0+\mathbf{v}_1\times\boldsymbol{\omega}_1+\mathbf{v}_0\times\boldsymbol{\omega}_2-\mathbf{h}_2\times(\nabla_{\mathbf{x}}\times\mathbf{h}_0)\\&-\mathbf{h}_1\times(\nabla_{\mathbf{x}}\times\mathbf{h}_1+\nabla_{\mathbf{X}}\times\mathbf{h}_0)-\mathbf{h}_0\times(\nabla_{\mathbf{x}}\times\mathbf{h}_2+\nabla_{\mathbf{X}}\times\mathbf{h}_1)\Big)+\beta\nabla_{\mathbf{X}}\theta_2\times\mathbf{e}_3=0;\end{aligned} \tag{8.162}$$

$$\begin{aligned}&\mathscr{L}^{h}(\mathbf{v}_3,\mathbf{h}_3)-\frac{\partial\mathbf{h}_1}{\partial T}+\eta(2(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\{\!\{\mathbf{h}_2\}\!\}_h+\nabla^2_{\mathbf{X}}\mathbf{h}_1)+\nabla_{\mathbf{X}}\times(\mathbf{v}_2\times\mathbf{H}+\mathbf{V}\times\mathbf{h}_2+\mathbf{v}_1\times\mathbf{h}_0\\&+\mathbf{v}_0\times\mathbf{h}_1)+\nabla_{\mathbf{x}}\times(\mathbf{v}_2\times\mathbf{h}_0+\mathbf{v}_1\times\mathbf{h}_1+\mathbf{v}_0\times\mathbf{h}_2)=0;\end{aligned} \tag{8.163}$$

$$\begin{aligned}&\mathscr{L}^{\theta}(\mathbf{v}_3,\theta_3)-\frac{\partial\theta_1}{\partial T}+\kappa(2(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\theta_2+\nabla^2_{\mathbf{X}}\theta_1)-(\mathbf{v}_1\cdot\nabla_{\mathbf{X}})\theta_0-(\mathbf{v}_0\cdot\nabla_{\mathbf{X}})\theta_1\\&-(\mathbf{V}\cdot\nabla_{\mathbf{X}})\theta_2-(\mathbf{v}_2\cdot\nabla_{\mathbf{x}})\theta_0-(\mathbf{v}_1\cdot\nabla_{\mathbf{x}})\theta_1-(\mathbf{v}_0\cdot\nabla_{\mathbf{x}})\theta_2=0.\end{aligned} \tag{8.164}$$

Step 1° for $n = 2$. We average the vertical component of the vorticity equation (8.162) in the fast variables, and employ the relations (8.24)–(8.26) for the divergencies and (8.32) for the vorticity for $n = 0, 1, 2$, the expression (8.42) for the mean of the image of the operator of linearisation, and the boundary conditions for the flow and magnetic field. This yields

$$
\begin{aligned}
&-\frac{\partial \langle\langle \boldsymbol{\omega}_1 \rangle\rangle_v}{\partial T} + \nu \nabla^2_{\mathbf{X}} \langle\langle \boldsymbol{\omega}_1 \rangle\rangle_v \\
&+ \nabla_{\mathbf{X}} \times \langle\langle \mathbf{V} \times (\nabla_{\mathbf{X}} \times \mathbf{v}_1) - \mathbf{V}\nabla_{\mathbf{X}} \cdot \{\mathbf{v}_1\}_h - \mathbf{H} \times (\nabla_{\mathbf{X}} \times \mathbf{h}_1) + \mathbf{H}\nabla_{\mathbf{X}} \cdot \{\mathbf{h}_1\}_h \\
&+ \mathbf{v}_0 \times (\nabla_{\mathbf{X}} \times \mathbf{v}_0) - \mathbf{v}_0 \nabla_{\mathbf{X}} \cdot \{\mathbf{v}_0\}_h - \mathbf{h}_0 \times (\nabla_{\mathbf{X}} \times \mathbf{h}_0) + \mathbf{h}_0 \nabla_{\mathbf{X}} \cdot \{\mathbf{h}_0\}_h \rangle\rangle_h = 0.
\end{aligned}
$$

We substitute here the expressions (8.73) and (8.110) for the fluctuating parts of the first two terms in the expansions of the perturbations of the flow and magnetic field, expression (8.105) for the spatial mean $\langle \mathbf{v}_1 \rangle_h$, use the vorticity relation in the slow variables (8.33) for $n = 1$, and, finally, note that ξ_0 and ξ_1 decay exponentially. The result is the *equation for the mean perturbation of the flow velocity*:

$$
\begin{aligned}
\nabla_{\mathbf{X}} \times \Bigg(&-\frac{\partial}{\partial T} \langle\langle \mathbf{v}_0 \rangle\rangle_h + \nu \nabla^2_{\mathbf{X}} \langle\langle \mathbf{v}_0 \rangle\rangle_h - (\langle\langle \mathbf{v}_0 \rangle\rangle_h \cdot \nabla_{\mathbf{X}}) \langle\langle \mathbf{v}_0 \rangle\rangle_h + (\langle\langle \mathbf{h}_0 \rangle\rangle_h \cdot \nabla_{\mathbf{X}}) \langle\langle \mathbf{h}_0 \rangle\rangle_h \\
&+ \sum_{j=1}^{2} \sum_{m=1}^{2} \sum_{k=1}^{2} \Bigg(\frac{\partial^2}{\partial X_j \partial X_m} \Big(\mathbf{D}^{vv}_{mkj} \langle\langle \mathbf{v}_0 \rangle\rangle_k + \mathbf{D}^{hv}_{mkj} \langle\langle \mathbf{h}_0 \rangle\rangle_k \\
&+ \frac{\partial^2}{\partial X_k \partial X_1} \nabla^{-2}_{\mathbf{X}} \big(\mathbf{d}^{vv}_{mkj} \langle\langle \mathbf{v}_0 \rangle\rangle_1 + \mathbf{d}^{hv}_{mkj} \langle\langle \mathbf{h}_0 \rangle\rangle_1 \big) \Big) \\
&+ \frac{\partial}{\partial X_j} \Big(\mathbf{A}^{vvv}_{mkj} \langle\langle \mathbf{v}_0 \rangle\rangle_k \langle\langle \mathbf{v}_0 \rangle\rangle_m + \mathbf{A}^{vhv}_{mkj} \langle\langle \mathbf{v}_0 \rangle\rangle_k \langle\langle \mathbf{h}_0 \rangle\rangle_m + \mathbf{A}^{hhv}_{mkj} \langle\langle \mathbf{h}_0 \rangle\rangle_k \langle\langle \mathbf{h}_0 \rangle\rangle_m \Big) \Bigg) \Bigg) = 0.
\end{aligned}
\tag{8.165}
$$

The constant vectors **D** and **A** enter into the operator of *combined eddy correction of kinematic viscosity* and the quadratic operator of *combined eddy correction of fluid advection*, respectively:

$$
\mathbf{D}^{vv}_{mkj} \equiv \langle\langle -V_j \widetilde{\mathbf{G}}^{vv}_{mk} - \mathbf{V}(G^{vv}_{mk})_j + H_j \mathbf{G}^{vh}_{mk} + \mathbf{H}(G^{vh}_{mk})_j \rangle\rangle_h, \tag{8.166}
$$

$$
\mathbf{D}^{hv}_{mkj} \equiv \langle\langle -V_j \widetilde{\mathbf{G}}^{hv}_{mk} - \mathbf{V}(G^{hv}_{mk})_j + H_j \mathbf{G}^{hh}_{mk} + \mathbf{H}(G^{hh}_{mk})_j \rangle\rangle_h, \tag{8.167}
$$

$$
\mathbf{d}^{vv}_{mkj} \equiv \langle\langle -V_j \widetilde{\mathbf{Y}}^{vv}_{mk} - \mathbf{V}(Y^{vv}_{mk})_j + H_j \mathbf{Y}^{vh}_{mk} + \mathbf{H}(Y^{vh}_{mk})_j \rangle\rangle_h, \tag{8.168}
$$

$$
\mathbf{d}^{hv}_{mkj} \equiv \langle\langle -V_j \widetilde{\mathbf{Y}}^{hv}_{mk} - \mathbf{V}(Y^{hv}_{mk})_j + H_j \mathbf{Y}^{hh}_{mk} + \mathbf{H}(Y^{hh}_{mk})_j \rangle\rangle_h; \tag{8.169}
$$

$$
\begin{aligned}
\mathbf{A}^{vvv}_{mkj} \equiv \langle\langle &-V_j \mathbf{Q}^{vvv}_{mk} - \mathbf{V}(Q^{vvv}_{mk})_j + H_j \mathbf{Q}^{vvh}_{mk} + \mathbf{H}(Q^{vvh}_{mk})_j \\
&- (S^{vv}_k)_j \mathbf{S}^{vv}_m + (S^{vh}_k)_j \mathbf{S}^{vh}_m \rangle\rangle_h,
\end{aligned}
\tag{8.170}
$$

$$\mathbf{A}^{vhv}_{mkj} \equiv \langle\!\langle -V_j\mathbf{Q}^{vhv}_{mk} - \mathbf{V}(Q^{vhv}_{mk})_j + H_j\mathbf{Q}^{vhh}_{mk} + \mathbf{H}(Q^{vhh}_{mk})_j \\ - (S^{vv}_k)_j\mathbf{S}^{hv}_m - (S^{hv}_m)_j\mathbf{S}^{vv}_k + (S^{vh}_k)_j\mathbf{S}^{vh}_m + (S^{hh}_m)_j\mathbf{S}^{vh}_k \rangle\!\rangle_h, \tag{8.171}$$

$$\mathbf{A}^{hhv}_{mkj} \equiv \langle\!\langle -V_j\mathbf{Q}^{hhv}_{mk} - \mathbf{V}(Q^{hhv}_{mk})_j + H_j\mathbf{Q}^{hhh}_{mk} + \mathbf{H}(Q^{hhh}_{mk})_j \\ - (S^{hv}_k)_j\mathbf{S}^{hv}_m + (S^{hh}_k)_j\mathbf{S}^{hh}_m \rangle\!\rangle_h. \tag{8.172}$$

By virtue of solenoidality of $\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h$ and $\langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h$, the operator of eddy correction of kinematic viscosity can be simplified:

$$\nabla_{\mathbf{X}} \times \sum_{j=1}^{2}\sum_{m=1}^{2}\sum_{k=1}^{2} \frac{\partial^2}{\partial X_j \partial X_m}\left(\mathbf{D}^{vv}_{mkj}\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k + \mathbf{D}^{hv}_{mkj}\langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k\right) \\ = \nabla_{\mathbf{X}} \times \sum_{k=1}^{2}\left(\left(\widetilde{D}^{vv}_{1k}\frac{\partial^2}{\partial X_k^2} + \widetilde{D}^{vv}_{2k}\frac{\partial^2}{\partial X_{3-k}^2} + \widetilde{D}^{vv}_{3k}\frac{\partial^2}{\partial X_1 \partial X_2}\right)\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_k \right. \\ \left. + \left(\widetilde{D}^{hv}_{1k}\frac{\partial^2}{\partial X_k^2} + \widetilde{D}^{hv}_{2k}\frac{\partial^2}{\partial X_{3-k}^2} + \widetilde{D}^{hv}_{3k}\frac{\partial^2}{\partial X_1 \partial X_2}\right)\langle\!\langle \mathbf{h}_0 \rangle\!\rangle_k\right)\mathbf{e}_k, \tag{8.173}$$

where, for $k = 1, 2$ and $n = 3 - k$,

$$\widetilde{D}^{vv}_{1k} \equiv \frac{1}{2}\left((D^{vv}_{111})_1 - (D^{vv}_{122})_1 - (D^{vv}_{221})_1 + (D^{vv}_{222})_2 - (D^{vv}_{112})_2 - (D^{vv}_{211})_2\right), \tag{8.174}$$

$$\widetilde{D}^{vv}_{2k} \equiv (D^{vv}_{nkn})_k, \quad \widetilde{D}^{vv}_{3k} \equiv (D^{vv}_{1k2})_k + (D^{vv}_{2k1})_k - (D^{vv}_{nnn})_k - (D^{vv}_{nkn})_n, \tag{8.175}$$

$$\widetilde{D}^{hv}_{1k} \equiv \frac{1}{2}\left((D^{hv}_{111})_1 - (D^{hv}_{122})_1 - (D^{hv}_{221})_1 + (D^{hv}_{222})_2 - (D^{hv}_{112})_2 - (D^{hv}_{211})_2\right), \tag{8.176}$$

$$\widetilde{D}^{hv}_{2k} \equiv (D^{hv}_{nkn})_k, \quad \widetilde{D}^{hv}_{3k} \equiv (D^{hv}_{1k2})_k + (D^{hv}_{2k1})_k - (D^{hv}_{nnn})_k - (D^{hv}_{nkn})_n. \tag{8.177}$$

Evidently, the values (8.153)–(8.157) and (8.166)–(8.172) of the coefficients in the newly emerging eddy terms in the mean-field equations for perturbations are not altered by exponentially decaying changes in the solutions to the auxiliary problems. In other words, the permissible modifications of the initial conditions for the auxiliary problems do not affect the mean-field equations.

The constant vectors **d** enter into a pseudodifferential, in general, operator formally of the second order, which thus can be regarded as describing a non-standard non-local anisotropic combined eddy diffusion. This operator emerges only if the CHM regime is unsteady and does not possess a symmetry without a time shift (see Sect. 8.4). To the best of our knowledge, this physical effect was discussed for the first time in [330].

Equations (8.152) and (8.165) constitute, together with the solenoidality conditions (8.24) for $n = 0$,

$$\nabla_{\mathbf{X}} \cdot \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h = \nabla_{\mathbf{X}} \cdot \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h = 0,$$

a closed system of mean-field equations for the leading terms in the expansions (8.21)–(8.22) of perturbations of the flow and magnetic field. The vertical component of (8.152) and the horizontal component of (8.165) vanish identically, since the slow spatial variables are horizontal. The outer curl in (8.165) can be removed, with the large-scale pressure introduced. However, in view of solenoidality of $\langle\langle \mathbf{v}_0 \rangle\rangle_h$ and $\langle\langle \mathbf{h}_0 \rangle\rangle_h$, it is natural to solve the mean-field equations in the terms of the stream function for the flow $\langle\langle \mathbf{v}_0 \rangle\rangle_h$ and an analogous vector potential for $\langle\langle \mathbf{h}_0 \rangle\rangle_h$, considering the only non-trivial vertical components of (8.165) and of the curl of (8.152).

As we have done for some linear stability problems, in principle we can successively find, by a further application of the procedure outlined in the beginning of Sect. 8.3, solutions to the systems (8.29)–(8.31) for $n \geq 2$, and thus determine any number of terms in the asymptotic series (8.20)–(8.23) for the large-scale perturbations. Equations for the averaged subsequent terms in the series are derived as the solvability conditions for the equations obtained at the respective order; they have an increasing order of nonlinearity and complexity. Since these equations are quite bulky, they are not considered here.

Note that the Coriolis force enters only into the statement of the auxiliary problems and is missing in the mean-field equations. This is not surprising, because for a solenoidal vector field $\langle\langle \mathbf{v}_0 \rangle\rangle_h$ the Coriolis force $\mathbf{e}_3 \times \langle \mathbf{h}_0 \rangle_h$ is a potential field, and hence rotation about the vertical axis affects only the large-scale pressure, which we do not track. The derivation of the mean-field equations is not affected, if additionally the fluid is rotating in the slow time and the total vector of the angular velocity is equal to $\tau \mathbf{e}_3 + \varepsilon^2 \boldsymbol{\tau}$. Then a new term, describing the Coriolis force due to rotation with the angular velocity $\boldsymbol{\tau}$, emerges in the mean-field equation (8.165) (by the remark above, only the horizontal component of $\boldsymbol{\tau}$ is essential).

As in the previous chapters, we do not specify the region in the plane of the slow variables, where the fluid is located, nor the boundary conditions for the mean perturbation governed by our mean-field equations. When a solution fails to be globally bounded in space, the expansion loses the asymptotic nature, and hence the equations cease to describe a large-scale perturbation. Hence the only essential requirement for a solution to the mean-field equations is its global boundedness in space on the entire interval of the slow time, where the perturbation is monitored.

8.6 Large-Scale Perturbations of Convective Hydromagnetic Regimes Near a Point of Bifurcation

We have shown in Sect. 8.3.2 that the system of mean-field equations for a large-scale perturbation is nonlinear, only if the α-effect is absent or insignificant in the leading order. As a consequence, the operator of the α-effect does not enter into the mean-field Eqs. 8.152 and 8.165. In the present section we consider large-scale perturbations of CHM regimes depending on a small parameter, and show that the operator of the α-effect appears in the mean-field equations even if the α-effect is insignificant in the leading order. We assume that the CHM regimes are parameterised by ε, which is simultaneously the spatial scale ratio and the amplitude of the perturbation, as in the case of an isolated CHM regime considered above.

For instance, suppose a pair of complex-conjugate eigenvalues of the operator of linearisation $\mathscr{L} = (\mathscr{L}^v, \mathscr{L}^h, \mathscr{L}^\theta)$ crosses the real axis for the critical value β_0 of parameter[3] β, and a Hopf bifurcation occurs. The family of CHM regimes emerging in this bifurcation for

$$\beta = \beta_0 + \beta_2 \varepsilon^2 \tag{8.178}$$

can be expanded in the asymptotic power series (see, e.g., [134])

$$\mathbf{V} = \sum_{n=0}^{\infty} \mathbf{V}_n(\mathbf{x}, t)\varepsilon^n, \tag{8.179}$$

$$\mathbf{H} = \sum_{n=0}^{\infty} \mathbf{H}_n(\mathbf{x}, t)\varepsilon^n, \tag{8.180}$$

$$\Theta = \sum_{n=0}^{\infty} \Theta_n(\mathbf{x}, t)\varepsilon^n. \tag{8.181}$$

The family branches in the direction $\mathbf{V}_1, \mathbf{H}_1, \Theta_1$ belonging to the invariant subspace of the operator of linearisation $\mathscr{L} = (\mathscr{L}^v, \mathscr{L}^h, \mathscr{L}^\theta)$ spanned by the oscillatory modes associated with the pair of the imaginary complex-conjugate eigenvalues (generically this subspace is two-dimensional).

Suppose now a steady short-scale mode $\mathbf{S} = (\mathbf{S}^v, \mathbf{S}^h, S^\theta)$ belongs to the kernel of the operator of linearisation $\mathscr{L}$ around the CHM regime for $\beta = \beta_0$,

$$\mathscr{L}(\mathbf{S}^v, \mathbf{S}^h, S^\theta, S^p) = 0, \quad \nabla_{\mathbf{x}} \cdot \mathbf{S}^v = \nabla_{\mathbf{x}} \cdot \mathbf{S}^h = 0. \tag{8.182}$$

Generically, if exists, such an eigenspace is one-dimensional. Its existence can result in emergence of a family of regimes in a saddle-node or pitchfork bifurcations. The family is branching in the direction of the eigenfunction $\mathbf{S}$ and can be expanded in the asymptotic power series (8.179)–(8.181) (see, e.g., [134]).

We understand the bifurcations in a broad sense as bifurcations (not necessarily of a CHM steady state or periodic regime) occurring when the dominant eigenvalue of the operator of linearisation crosses the real axis or vanishes for the critical value β_0 of the bifurcation parameter β (8.178). The associated eigenfunctions are supposed to be globally bounded in space and time, do not tend to

[3] β proportional to the Rayleigh number is chosen as the bifurcation parameter here, since the sequence of bifurcations happening on increasing the Rayleigh number was investigated by many authors, see, e.g., [227]. We could consider a bifurcation occurring on variation of any other parameter of the problem, e.g., τ proportional to the Taylor number—such bifurcations were examined in [55]. This does not affect the structure of the mean-field and amplitude equations that we derive in this section.

zero for $t \to \infty$, have zero means $\langle \mathbf{S}^v \rangle_h = \langle \mathbf{S}^h \rangle_h = 0$, $\langle S^p \rangle = 0$, and satisfy the boundary conditions defined by (8.5)–(8.7) and (8.11).

If $\beta_2 = 0$, the expansion (8.179)–(8.181) can result from an analytical dependence on ε of the source terms $\nabla \times \mathbf{F}, \mathbf{J}$ and S in the Eqs. 8.1–8.3 governing the CHM regimes. Also, for an individual regime at the point of bifurcation $\beta_2 = 0$, and only the leading-order terms (for $n = 0$) are present in series (8.179)–(8.181). Both cases are also covered by the analysis of the present section.

The reader should be warned about the following subtlety (arising, because we do not impose any periodicity or quasi-periodicity conditions in the fast time): suppose we consider a CHM regime having N independent frequencies ω_i. We can solve a Floquet problem for the linearisation in the subspace of vector fields, quasi-periodic in time, which have the same N basic frequencies ω_i. If the linearisation has an eigenvalue $\mathrm{i}\omega_{N+1}$ associated with the eigenfunction $\mathbf{S}'(\mathbf{x}, t)$, then $\mathscr{L}\mathbf{S} = 0$ for $\mathbf{S} = \mathbf{S}' \exp(\mathrm{i}\omega_{N+1} t)$, and hence we regard the field $\mathbf{S}(\mathbf{x}, t)$ as an eigenfunction from the kernel of linearisation $\mathscr{L}$. (To illustrate this, we note that it is unnatural to include the derivative in the fast time into a linearisation around a CHM steady state, this removing the ambiguity. By contrast, the general case encompassing periodic and quasi-periodic CHM regimes can only be served, if the time derivative is preserved in the linearisation.) Instead of an oscillatory eigenmode $\mathbf{S}'$ associated with imaginary eigenvalue $\mathrm{i}\omega$ we will consider the short-scale stability mode $\mathbf{S}(\mathbf{x}, t) = \mathbf{S}' \exp(\mathrm{i}\omega t) \in \ker \mathscr{L}$, which we call a *neutral oscillatory* mode (as opposed to *neutral steady* modes, not associated with a non-zero imaginary eigenvalue).

Construction of the mean-field equations is now algebraically more involved than in the problems considered so far, because the number of unknown fields—amplitudes of neutral short-scale stability modes—describing the leading terms of the power series expansions of perturbations increases. This matches the increasing number of solvability conditions for problems of the type (8.51)–(8.57) that we will have to consider: orthogonality of the r.h.s. of the equations to the kernel of $\mathscr{L}^*$, necessary for existence of a solution by the Fredholm alternative theorem, (the dimensions of $\ker \mathscr{L}^*$ and $\ker \mathscr{L}$ are equal). The number of terms in the resultant equations also significantly increases: the equations obtained from the system (8.13)–(8.15) at orders ε^n contain new terms, involving terms $\mathbf{V}_j, \mathbf{H}_j$ and Θ_j for $1 \le j \le n$ from the expansion (8.179)–(8.181) of the family of CHM regimes. This complication of algebra suggests to return to the use of linearisation of the Navier–Stokes equation for the flow instead of employing linearisation of the equation for vorticity as in the previous sections (although perhaps this makes the presentation of the present section more mathematically dry and less intuitive).

The notation $\mathscr{L} = (\mathscr{L}^v, \mathscr{L}^h, \mathscr{L}^\theta)$ is reserved in this section for the operator of linearisation around the CHM regime $\mathbf{V}_0, \mathbf{H}_0, \Theta_0$ for $\beta = \beta_0$ (fields in the domain of $\mathscr{L}$ are not restricted to have zero horizontal means). Similarly, the substitutions $\mathbf{\Omega} \to \nabla \times \mathbf{V}_0$, $\mathbf{V} \to \mathbf{V}_0, \mathbf{H} \to \mathbf{H}_0$, $\mathbf{\Theta} \to \Theta_0$ and $\beta \to \beta_0$ are assumed, when we refer in this section to any equation or quantity derived in previous sections.

8.6.1 Asymptotic Expansion

The large-scale weakly nonlinear perturbations satisfy equations (8.13)–(8.19); equation (8.13) for the flow perturbation has now an equivalent form

$$\mathscr{L}^v(\mathbf{v},\mathbf{h},\theta,p)+\varepsilon(\mathbf{v}\times(\nabla\times\mathbf{v})-\mathbf{h}\times(\nabla\times\mathbf{h}))=0, \tag{8.183}$$

where the operator of linearisation of the Navier–Stokes equation is

$$\begin{aligned}\mathscr{L}^v(\mathbf{v},\mathbf{h},\theta,p)\equiv&-\frac{\partial\mathbf{v}}{\partial t}+\nu\nabla^2\mathbf{v}-(\mathbf{V}_0\cdot\nabla)\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{V}_0\\&+(\mathbf{H}_0\cdot\nabla)\mathbf{h}+(\mathbf{h}\cdot\nabla)\mathbf{H}_0+\tau\mathbf{v}\times\mathbf{e}_3+\beta\theta\mathbf{e}_3-\nabla p.\end{aligned}$$

Solutions are sought in the form of power series (8.21)–(8.23) and

$$p=\sum_{n=-1}^{\infty}p_n(\mathbf{X},T,\mathbf{x},t)\varepsilon^n. \tag{8.184}$$

Substituting the series into the equations governing perturbations, we find at order ε^n:

$$\begin{aligned}&\mathscr{L}^v(\mathbf{v}_n,\mathbf{h}_n,\theta_n,p_n)-\frac{\partial\mathbf{v}_{n-2}}{\partial T}+\nu(2(\nabla_\mathbf{x}\cdot\nabla_\mathbf{X})\mathbf{v}_{n-1}+\nabla^2_\mathbf{X}\mathbf{v}_{n-2})\\&+\sum_{k=1}^{n}\Big(-((\mathbf{V}_k+\mathbf{v}_{k-1})\cdot\nabla_\mathbf{x})\mathbf{v}_{n-k}+((\mathbf{H}_k+\mathbf{h}_{k-1})\cdot\nabla_\mathbf{x})\mathbf{h}_{n-k}\\&-((\mathbf{V}_{k-1}+\mathbf{v}_{k-2})\cdot\nabla_\mathbf{X})\mathbf{v}_{n-k}+((\mathbf{H}_{k-1}+\mathbf{h}_{k-2})\cdot\nabla_\mathbf{X})\mathbf{h}_{n-k}\\&-(\mathbf{v}_{n-k}\cdot\nabla_\mathbf{x})\mathbf{V}_k+(\mathbf{h}_{n-k}\cdot\nabla_\mathbf{x})\mathbf{H}_k\Big)+\beta_2\theta_{n-2}\mathbf{e}_3-\nabla_\mathbf{X}p_{n-1}=0;\end{aligned} \tag{8.185}$$

$$\begin{aligned}&\mathscr{L}^h(\mathbf{v}_n,\mathbf{h}_n)-\frac{\partial\mathbf{h}_{n-2}}{\partial T}+\eta\Big(2(\nabla_\mathbf{x}\cdot\nabla_\mathbf{X})\mathbf{h}_{n-1}+\nabla^2_\mathbf{X}\mathbf{h}_{n-2}\Big)\\&+\nabla_\mathbf{x}\times\sum_{k=1}^{n}\Big(\mathbf{v}_{n-k}\times(\mathbf{H}_k+\mathbf{h}_{k-1})+\mathbf{V}_k\times\mathbf{h}_{n-k}\Big)\\&+\nabla_\mathbf{X}\times\sum_{k=0}^{n-1}\Big(\mathbf{v}_{n-1-k}\times(\mathbf{H}_k+\mathbf{h}_{k-1})+\mathbf{V}_k\times\mathbf{h}_{n-1-k}\Big)=0;\end{aligned} \tag{8.186}$$

$$\begin{aligned}&\mathscr{L}^\theta(\mathbf{v}_n,\theta_n)-\frac{\partial\theta_{n-2}}{\partial T}+\kappa\Big(2(\nabla_\mathbf{x}\cdot\nabla_\mathbf{X})\theta_{n-1}+\nabla^2_\mathbf{X}\theta_{n-2}\Big)-\sum_{k=1}^{n}\Big((\mathbf{V}_k\cdot\nabla_\mathbf{x})\theta_{n-k}\\&+(\mathbf{v}_{n-k}\cdot\nabla_\mathbf{x})(\Theta_k+\theta_{k-1})+((\mathbf{V}_{n-k}+\mathbf{v}_{n-1-k})\cdot\nabla_\mathbf{X})\theta_{k-1}\Big)=0.\end{aligned} \tag{8.187}$$

By virtue of solenoidality of $\langle \mathbf{v}_n \rangle_h$ (8.24), this mean flow perturbation can be expressed in the terms of a stream function $q_n(\mathbf{X}, T, t)$, so that $\langle \mathbf{v}_n \rangle_h \times \mathbf{e}_3 = \nabla_{\mathbf{X}} \langle q_n \rangle$. Thus we assume henceforth that equation (8.185) and the definition of the operator $\mathscr{L}^v$ are modified by changing $p_{n-1} \to p_{n-1} + \tau q_n$ in the equation and $\tau \mathbf{v} \times \mathbf{e}_3 \to \tau \{\mathbf{v}\} \times \mathbf{e}_3$ in $\mathscr{L}^v$. (This change in the operator of linearisation was also tacitly assumed in the introduction to this section.)

The means over the fast spatial variables of the horizontal components of the first two equations reduce (with relations for divergencies (8.25) and (8.26) taken into account) to

$$\begin{aligned}
&-\frac{\partial \langle \mathbf{v}_n \rangle_h}{\partial t} - \frac{\partial \langle \mathbf{v}_{n-2} \rangle_h}{\partial T} + \nu \nabla^2_{\mathbf{X}} \langle \mathbf{v}_{n-2} \rangle_h - \nabla_{\mathbf{X}} \langle p_{n-1} \rangle \\
&+ \sum_{k=0}^{n-1} \Big\langle -((\mathbf{V}_k + \mathbf{v}_{k-1}) \cdot \nabla_{\mathbf{X}}) \mathbf{v}_{n-1-k} + ((\mathbf{H}_k + \mathbf{h}_{k-1}) \cdot \nabla_{\mathbf{X}}) \mathbf{h}_{n-1-k} \\
&- (\mathbf{V}_k + \mathbf{v}_{k-1}) \nabla_{\mathbf{X}} \cdot \mathbf{v}_{n-1-k} + (\mathbf{H}_k + \mathbf{h}_{k-1}) \nabla_{\mathbf{X}} \cdot \mathbf{h}_{n-1-k} \Big\rangle_h = 0;
\end{aligned} \tag{8.188}$$

$$\begin{aligned}
&-\frac{\partial \langle \mathbf{h}_n \rangle_h}{\partial t} - \frac{\partial \langle \mathbf{h}_{n-2} \rangle_h}{\partial T} + \eta \nabla^2_{\mathbf{X}} \langle \mathbf{h}_{n-2} \rangle_h \\
&\qquad + \nabla_{\mathbf{X}} \times \sum_{k=0}^{n-1} \Big\langle \mathbf{v}_{n-1-k} \times (\mathbf{H}_k + \mathbf{h}_{k-1}) + \mathbf{V}_k \times \mathbf{h}_{n-1-k} \Big\rangle_v = 0.
\end{aligned} \tag{8.189}$$

The last equation guarantees that $\langle \mathbf{h}_n \rangle_h$ are solenoidal in the slow variables (see condition 8.24) for all $n \geq 0$ provided they are at $t = 0$.

8.6.2 The Order ε^0 Equation and Solvability Conditions for Auxiliary Problems

We enumerate the short-scale neutral eigenmodes in the kernel of linearisation as follows: $\mathbf{S}_k = (\mathbf{S}^{vv}_k + \mathbf{e}_k, \mathbf{S}^{vh}_k, S^{v\theta}_k, S^{vp}_k)$, $k = 1, 2$, and $\mathbf{S}_{k+2} = (\mathbf{S}^{hv}_k, \mathbf{S}^{hh}_k + \mathbf{e}_k, S^{h\theta}_k, S^{hp}_k)$, $k = 1, 2$, can be determined solving auxiliary problems I.1 and I.2, respectively (see Sect. 8.3.1); $\mathbf{S}_k$ for $k = 5, \ldots, K$ are the remaining zero-mean modes from $\ker \mathscr{L}$. Generically, $K = 5$ in the case of a saddle-node or pitchfork bifurcation, and $K = 6$ in the case of a Hopf bifurcation ($K = 4$ if $\beta_2 = 0$, and the expansion (8.179)–(8.181) is a consequence of a dependence of the source terms $\nabla \times \mathbf{F}, \mathbf{J}$ and s in Eqs. 8.1–8.3 on ε). We assume that there are no Jordan cells of size 2 or more associated with the eigenvalue zero (Jordan cells of size 2 exist for free CHM regimes). The flow and magnetic components of the short-scale neutral modes $\mathbf{S}_k$ are solenoidal.

Let us choose a basis in $\ker \mathscr{L}^*$ consisting of K eigenfunctions $\mathbf{S}^*_k$ biorthogonal to $\mathbf{S}_k$, i.e.

$$\langle\!\langle (\mathbf{S}^v_k, \mathbf{S}^h_k, S^\theta_k) \cdot \mathbf{S}^*_j \rangle\!\rangle = \delta^k_j \tag{8.190}$$

(here δ_j^k is the Kronecker symbol and $\cdot$ denotes the scalar product of seven-dimensional vector fields) holds for all $1 \le k, j \le K$. In particular, $\mathbf{S}_k^* = (\mathbf{e}_k, 0, 0)$ for $k = 1, 2, \mathbf{S}_k^* = (0, \mathbf{e}_{k-2}, 0)$ for $k = 3, 4,$ and $\langle\!\langle (\mathbf{S}_k^*)^v \rangle\!\rangle_h = \langle\!\langle (\mathbf{S}_k^*)^h \rangle\!\rangle_h = 0$ for $k = 5, \ldots, K$. (Here the kernel of the adjoint operator is understood in the generalised sense, as explained in the introduction to this section.)

Solutions to auxiliary problems

$$\mathscr{L}(\mathbf{v}, \mathbf{h}, \theta, p) = \mathbf{f}, \quad \nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{h} = 0$$

are supposed to be globally bounded together with their derivatives. Orthogonality of the r.h.s., $\mathbf{f}$, to vector fields $\mathbf{S}_k^*$ for all $k \le K$,

$$\langle\!\langle \mathbf{f} \cdot \mathbf{S}_k^* \rangle\!\rangle = 0, \tag{8.191}$$

is necessary for solvability of this problem; we assume that this condition is also sufficient (this holds true, for instance, when stability of a space-periodic steady or periodic in time CHM regime $\mathbf{V}_0, \mathbf{H}_0, \Theta_0$ is considered).

For $n = 0$, Eqs. 8.185–8.187 reduce to

$$\mathscr{L}(\mathbf{v}_0, \mathbf{h}_0, \theta_0, p_0) - \nabla_{\mathbf{X}} p_{-1} = 0. \tag{8.192}$$

By relation (8.188) for $n = 0$, the mean over the fast spatial variables of the horizontal flow component of this equation reduces to $\nabla_{\mathbf{X}} \langle p_{-1} \rangle = 0$, whereby we set $p_{-1} = 0$. (Recall that the operator $\mathscr{L}^v$ is modified; before this modification is implemented, it is necessary to keep the term p_{-1} in the expansion for pressure (8.184). In view of our assumptions, solutions to equation (8.192) have the following structure:

$$(\mathbf{v}_0, \mathbf{h}_0, \theta_0, p_0) = \xi_0 + \sum_{k=1}^{K} c_{k0}(\mathbf{X}, T) \mathbf{S}_k(\mathbf{x}, t). \tag{8.193}$$

Here ξ_0 is a transient, exponentially decaying in the fast time, and the first four amplitudes c_{k0} have the sense of mean fields of perturbations: $(c_{1,0}, c_{2,0}, 0) = \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h$ and $(c_{3,0}, c_{4,0}, 0) = \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h$.

8.6.3 The Combined α-Effect

At order ε^1, Eqs. 8.185–8.187 are

$$\begin{aligned}\mathscr{L}^v(\mathbf{v}_1, \mathbf{h}_1, \theta_1, p_1) = &-2\nu(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\mathbf{v}_0 + ((\mathbf{V}_1 + \mathbf{v}_0) \cdot \nabla_{\mathbf{x}})\mathbf{v}_0 - ((\mathbf{H}_1 + \mathbf{h}_0) \cdot \nabla_{\mathbf{x}})\mathbf{h}_0 \\ &+ (\mathbf{V}_0 \cdot \nabla_{\mathbf{X}})\mathbf{v}_0 - (\mathbf{H}_0 \cdot \nabla_{\mathbf{X}})\mathbf{h}_0 + (\mathbf{v}_0 \cdot \nabla_{\mathbf{x}})\mathbf{V}_1 - (\mathbf{h}_0 \cdot \nabla_{\mathbf{x}})\mathbf{H}_1 + \nabla_{\mathbf{X}} p_0;\end{aligned} \tag{8.194}$$

$$\begin{aligned}\mathscr{L}^h(\mathbf{v}_1,\mathbf{h}_1) = &-2\eta(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\mathbf{h}_0 - \nabla_{\mathbf{x}}\times(\mathbf{v}_0\times(\mathbf{H}_1+\mathbf{h}_0)+\mathbf{V}_1\times\mathbf{h}_0)\\ &-\nabla_{\mathbf{X}}\times(\mathbf{v}_0\times\mathbf{H}_0+\mathbf{V}_0\times\mathbf{h}_0);\end{aligned}\tag{8.195}$$

$$\begin{aligned}\mathscr{L}^\theta(\mathbf{v}_1,\theta_1) = &-2\kappa(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\theta_0 + (\mathbf{V}_1\cdot\nabla_{\mathbf{x}})\theta_0 + (\mathbf{v}_0\cdot\nabla_{\mathbf{x}})(\Theta_1+\theta_0)\\ &+(\mathbf{V}_0\cdot\nabla_{\mathbf{X}})\theta_0.\end{aligned}\tag{8.196}$$

We introduce new variables

$$\mathbf{v}_1 = \mathbf{v}_1' + \sum_{k=1}^{2}\sum_{m=1}^{2}\frac{\partial c_{k0}}{\partial X_m}\nabla_{\mathbf{x}} s_{km}^v,\tag{8.197}$$

$$\mathbf{h}_1 = \mathbf{h}_1' + \sum_{k=1}^{2}\sum_{m=1}^{2}\frac{\partial c_{k0}}{\partial X_m}\nabla_{\mathbf{x}} s_{km}^h,\tag{8.198}$$

where $s(\mathbf{x},t)$ are globally bounded solutions to the Neumann problems

$$\nabla_{\mathbf{x}}^2 s_{km}^v = -\{S_k^v\}_m,\qquad \left.\frac{\partial s_{km}^v}{\partial x_3}\right|_{x_3=\pm L/2} = 0;\tag{8.199}$$

$$\nabla_{\mathbf{x}}^2 s_{km}^h = -\{S_k^h\}_m,\qquad \left.\frac{\partial s_{km}^h}{\partial x_3}\right|_{x_3=\pm L/2} = 0.\tag{8.200}$$

In view of relations for divergencies (8.25) and (8.26) for $n=0$, vector fields $\mathbf{v}_1'$ and $\mathbf{h}_1'$ are solenoidal in the fast variables.

Substituting expressions (8.193) into equation (8.188) for the mean flow perturbation, taking the divergence in the slow variables and using solenoidality of the mean flow $\langle\mathbf{v}_1\rangle_h$ [see (8.13) for $n=1$], we derive a Poisson equation for the mean pressure $\langle p_0\rangle$. It has a solution

$$\langle p_0\rangle = 2\sum_{k=1}^{K}\sum_{m=1}^{2}\sum_{n=1}^{2}\langle -(V_0)_m(S_k^v)_n + (H_0)_m(S_k^h)_n\rangle(\nabla_{\mathbf{X}}^2)^{-1}\frac{\partial^2 c_{k0}}{\partial X_m\partial X_n}.\tag{8.201}$$

(Evaluation of the inverse Laplacian requires specification of the region in the smooth variables, where the perturbation is defined, and the boundary conditions.)

Substituting now expressions (8.197)–(8.201) into Eqs. 8.194–8.196, we transform the latter to the form

$$\begin{aligned}\mathscr{L}(\mathbf{v}_1',\mathbf{h}_1',\theta_1,p_1) = &\sum_{k=1}^{K}\left(\mathbf{A}_k^1 c_{k0} + \sum_{m=1}^{K}\mathbf{A}_{km}^3 c_{k0}c_{m0}\right)\\ &+\sum_{k=1}^{K}\sum_{m=1}^{2}\left(\mathbf{A}_{km}^2\frac{\partial c_{k0}}{\partial X_m} + \sum_{n=1}^{2}\sum_{j=1}^{2}\mathbf{A}_{kmnj}^4(\nabla_{\mathbf{X}}^2)^{-1}\frac{\partial^3 c_{k0}}{\partial X_m\partial X_n\partial X_j}\right).\end{aligned}\tag{8.202}$$

Here $\mathbf{A}^q(\mathbf{x},t)$ denote the following vector fields:

$$\begin{aligned}
\mathbf{A}_k^1 \equiv &\Big((\mathbf{V}_1\cdot\nabla_{\mathbf{x}})\mathbf{S}_k^v - (\mathbf{H}_1\cdot\nabla_{\mathbf{x}})\mathbf{S}_k^h + (\mathbf{S}_k^v\cdot\nabla_{\mathbf{x}})\mathbf{V}_1 - (\mathbf{S}_k^h\cdot\nabla_{\mathbf{x}})\mathbf{H}_1,\\
&-\nabla_{\mathbf{x}}\times(\mathbf{S}_k^v\times\mathbf{H}_1+\mathbf{V}_1\times\mathbf{S}_k^h),\quad (\mathbf{V}_1\cdot\nabla_{\mathbf{x}})S_k^\theta + (\mathbf{S}_k^v\cdot\nabla_{\mathbf{x}})\Theta_1\Big);\\
\mathbf{A}_{km}^2 \equiv &-\mathscr{L}(\nabla_{\mathbf{x}}s_{km}^v,\nabla_{\mathbf{x}}s_{km}^h,0,0) - \Big(2\nu\partial\mathbf{S}_k^v/\partial x_m - (V_0)_m\mathbf{S}_k^v + (H_0)_m\mathbf{S}_k^h + S_k^p\mathbf{e}_m,\\
&2\eta\partial\mathbf{S}_k^v/\partial x_m + \mathbf{e}_m\times(\mathbf{S}_k^v\times\mathbf{H}_0+\mathbf{V}_0\times\mathbf{S}_k^h),\quad 2\kappa\partial S_k^\theta/\partial x_m - (V_0)_m\Big);\\
\mathbf{A}_{km}^3 \equiv &\Big((\mathbf{S}_k^v\cdot\nabla_{\mathbf{x}})\mathbf{S}_m^v - (\mathbf{S}_k^h\cdot\nabla_{\mathbf{x}})\mathbf{S}_m^h,\quad -\nabla_{\mathbf{x}}\times(\mathbf{S}_k^v\times\mathbf{S}_m^h),\quad (\mathbf{S}_k^v\cdot\nabla_{\mathbf{x}})S_m^\theta\Big);\\
\mathbf{A}_{kmnj}^4 \equiv &\Big(2\langle -(V_0)_m(S_k^v)_n + (H_0)_m(S_k^h)_n\rangle\mathbf{e}_j,\quad 0,\quad 0\Big).
\end{aligned}$$

The system of Eqs. 8.194–8.196 transformed into equation (8.202) is ready for application of the solvability condition: a solution exists, if the r.h.s. is orthogonal to each of vector field $\mathbf{S}_k^*$ (see (8.191)). Note that the last sum in the r.h.s. can be significantly economised, using the fact that the first four coefficients c_{k0} are mean components of the flow and magnetic field perturbations, which satisfy the solenoidality conditions (8.24). Cancelling the inverse Laplacian acting on suitable combinations of terms in this sum, it is possible to reduce it to a sum, involving, for instance, only partial derivatives $\partial^3 c_{k0}/\partial X_1^2\partial X_2$ and $\partial^3 c_{k0}/\partial X_1^3$, and the first-order derivatives of c_{k0}. Also, similar products $c_{k0}c_{m0}$ can be summed up. Thus, by the solvability condition discussed in the previous subsection, it suffices for existence of solutions to the problem (8.202), that the relations

$$\langle\!\langle \mathbf{A}^q\cdot\mathbf{S}_k^*\rangle\!\rangle = 0 \tag{8.203}$$

are satisfied for all $q=1,\ldots,4$, all subscripts of $\mathbf{A}^q$, and $k=1,\ldots,K$, but these conditions can be excessive.

If at least one of the conditions (8.203) is not satisfied (even after all possible reductions are carried out), the α-effect is significant in the leading order. In this case, the appropriate scaling of the slow time is $T=\varepsilon t$, and then a new term $\partial c_{k0}/\partial T$ emerges in equation (8.202). Scalar multiplying this equation by $\mathbf{S}_j^*$, by virtue of the biorthogonality (8.190) of this basis we then obtain the system of *amplitude equations* for perturbation:

$$\begin{aligned}
-\frac{\partial c_{j0}}{\partial T} = &\sum_{k=1}^{K}\left(\langle\!\langle \mathbf{A}_k^1\cdot\mathbf{S}_j^*\rangle\!\rangle c_{k0} + \sum_{m=1}^{K}\langle\!\langle \mathbf{A}_{km}^3\cdot\mathbf{S}_j^*\rangle\!\rangle c_{k0}c_{m0}\right)\\
&+\sum_{k=1}^{K}\sum_{m=1}^{2}\left(\langle\!\langle \mathbf{A}_{km}^2\cdot\mathbf{S}_j^*\rangle\!\rangle\frac{\partial c_{k0}}{\partial X_m} + \sum_{n=1}^{2}\sum_{j=1}^{2}\langle\!\langle \mathbf{A}_{kmnj}^4\cdot\mathbf{S}_j^*\rangle\!\rangle(\nabla_{\mathbf{X}}^2)^{-1}\frac{\partial^3 c_{k0}}{\partial X_m\partial X_n\partial X_j}\right).
\end{aligned} \tag{8.204}$$

For $j = 1,\ldots,4$, they reduce to *mean-field equations* for the mean flow and magnetic field perturbations, following from (8.188) and (8.189),

$$\frac{\partial\langle\!\langle \mathbf{v}_0\rangle\!\rangle_h}{\partial T} = \sum_{k=1}^{K}\sum_{m=1}^{2}\langle\!\langle -\mathbf{S}^v_k (V_0)_m - \mathbf{V}_0 (S^v_k)_m + \mathbf{S}^h_k (H_0)_m + \mathbf{H}_0 (S^h_k)_m \rangle\!\rangle_h \frac{\partial c_{k0}}{\partial X_m} - \nabla_{\mathbf{X}} \langle\!\langle p_0\rangle\!\rangle; \tag{8.205}$$

$$\frac{\partial\langle\!\langle \mathbf{h}_0\rangle\!\rangle_h}{\partial T} = \nabla_{\mathbf{X}} c_{k0} \times \langle\!\langle \mathbf{V}_0 \times \mathbf{S}^h_k + \mathbf{S}^v_k \times \mathbf{H}_0 \rangle\!\rangle_v, \tag{8.206}$$

which must be considered together with the solenoidality conditions for mean fields of perturbation, (8.24) for $n = 0$. (We have invoked our general assumption that the fields, which are averaged, are globally bounded and all the means that we compute are well-defined.) Thus, in the present stability problem the mean-field equations remain linear in amplitudes of the short-scale neutral modes, but the α-effect equations (8.204) for other amplitudes are in general nonlinear (quadratic in amplitudes) and involve non-local operators.

8.6.4 Insignificant α-Effect: Solution of Order ε^1 Equations

In what follows we consider the case where the α-effect is insignificant in the leading order, i.e. conditions (8.203) are satisfied for all $q = 1,\ldots,4$ and all subscripts of $\mathbf{A}^q$ (after the relevant reductions, if necessary). To ensure this, we assume that the CHM regime $\mathbf{V}_0, \mathbf{H}_0, \Theta_0$ is a symmetric set (of a type considered in Sect. 8.4) and all $\mathbf{S}_k$ are antisymmetric sets. Recall that $\{\mathbf{S}_k\}$ for $k = 1,\ldots,4$ are solutions to auxiliary problems of type I and are hence antisymmetric sets, but antisymmetry is not guaranteed for the remaining $K-4$ fields $\mathbf{S}_k$. If we solve the present stability problem for a family of CHM regimes emerging in a bifurcation, we thus assume that the bifurcation is symmetry-breaking. We will also henceforth assume that the sets $\mathbf{V}_n, \mathbf{H}_n, \Theta_n$ in the expansion of the CHM regime are symmetric for even n and antisymmetric for odd n (under our assumption about antisymmetry of all eigenfunctions in the kernel of linearisation this holds in the case of a symmetry-breaking bifurcation).

Under these assumptions, all vector fields $\mathbf{A}^q$ in equation (8.202) have the symmetry opposite to that of $\mathbf{S}_k$, and hence of $\mathbf{S}^*_k$; thus, all solvability conditions (8.203) are satisfied. A solution to the system of Eqs. 8.194–8.196 has the structure implied by equation (8.202):

$$(\mathbf{v}_1, \mathbf{h}_1, \theta_1, p_1) = \boldsymbol{\xi}_1 + \sum_{k=1}^{K}\left(\mathbf{S}_k c_{k1} + \mathbf{G}^1_k c_{k0} + \sum_{m=1}^{K}\mathbf{G}^3_{km} c_{k0} c_{m0}\right) + \sum_{k=1}^{K}\sum_{m=1}^{2}\left(\widetilde{\mathbf{G}}^2_{km}\frac{\partial c_{k0}}{\partial X_m} + \sum_{n=1}^{2}\sum_{j=1}^{2}\mathbf{G}^4_{kmnj}(\nabla^2_{\mathbf{X}})^{-1}\frac{\partial^3 c_{k0}}{\partial X_m \partial X_n \partial X_j}\right). \tag{8.207}$$

Here

$$\widetilde{\mathbf{G}}^{\,2}_{km} \equiv \mathbf{G}^2_{km} + (\nabla_{\mathbf{x}} s^v_{km}, \nabla_{\mathbf{x}} s^h_{km}, 0, 0)$$

(s^v_{km} and s^h_{km} being solutions to the Poisson equations (8.199) and (8.200)), ξ_1 is an exponentially decaying transient, and $\mathbf{G}^q$ are solutions to the auxiliary problem:

$$\mathscr{L}(\mathbf{G}^q) = \mathbf{A}^q, \tag{8.208}$$

satisfying the boundary conditions, defined by (8.5)–(8.7) and (8.11). The means of the horizontal hydrodynamic $\langle\langle(\mathbf{G}^q)^v\rangle\rangle_h$ and magnetic $\langle\langle(\mathbf{G}^q)^h\rangle\rangle_h$ components, as well as of pressure $\langle(G^q)^p\rangle$ must vanish. Expressions for divergencies (8.21) and (8.22) imply that the hydrodynamic and magnetic components must be solenoidal for all $q = 1,\ldots,4$. The solvability conditions for the auxiliary problems are, evidently, satisfied. For $q \geq 2$, auxiliary problems (8.208) are analogous to the auxiliary problems of type q considered in the previous sections. Finally, ξ_1 is an exponentially decaying transient satisfying the equation $\mathscr{L}(\xi_1) = 0$.

If we consider stability of a family of CHM regimes emerging in a saddle-node or pitchfork bifurcation at $\beta = \beta_0$ and $K = 5$, the family branches along the eigenfunction $\mathbf{S}_5$:

$$(\mathbf{V}_1, \mathbf{H}_1, \Theta_1) = \chi(\mathbf{S}^v_5, \mathbf{S}^h_5, S^\theta_5).$$

Then $\mathbf{A}^1_5 = 2\chi\mathbf{A}^3_{5,5}$ implying $\mathbf{G}^1_5 = 2\chi\mathbf{G}^3_{5,5}$.

8.6.5 Mean-Field Equations Derived from Order ε^2 Equations

When the α-effect is insignificant in the leading order, amplitude equations for the leading terms in the expansions (8.21)–(8.23) and (8.184) are obtained as solvability conditions (8.191), applied to Eqs. 8.185–8.187 for $n = 2$, which are modified to involve hydrodynamic and magnetic components, solenoidal in the fast variables. For $k = 1,\ldots,4$, conditions (8.191) give rise to *mean-field equations* for the mean flow and magnetic field perturbations, which can be deduced from relations (8.188) and (8.189). These mean-field equations are supplemented by the solenoidality conditions for mean fields of perturbation, (8.24) for $n = 0$.

Substituting expressions (8.193) and (8.207) into the averaged horizontal hydrodynamic component (8.188) of the equation for perturbation at order ε^2, and noting that the transients ξ_1 and ξ_1 exponentially decay, we obtain the equation for the mean perturbation of the flow generalising (8.165):

$$-\frac{\partial}{\partial T}\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h + \nu\nabla^2_{\mathbf{X}}\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h + \sum_{k=1}^{K}\sum_{n=1}^{2}\frac{\partial}{\partial X_n}\left(\sum_{m=1}^{K}\mathbf{A}^v_{kmn}c_{k0}c_{m0} + \mathcal{A}^v_{nk}c_{k0}\right.$$
$$\left. + \sum_{m=1}^{2}\frac{\partial}{\partial X_m}\left(\mathbf{D}^v_{kmn}c_{k0} + \sum_{j=1}^{2}\sum_{i=1}^{2}\mathbf{d}^v_{kmnji}\frac{\partial^2}{\partial X_j \partial X_i}\nabla^{-2}_{\mathbf{X}}c_{k0}\right)\right) = 0. \quad (8.209)$$

Here

$$\mathbf{A}^v_{kmn} \equiv \langle\!\langle -(V_0)_n(\mathbf{G}^3_{km})^v - \mathbf{V}_0(G^3_{km})^v_n + (H_0)_n(\mathbf{G}^3_{km})^h + \mathbf{H}_0(G^3_{km})^h_n$$
$$- (S^v_k)_n\mathbf{S}^v_m + (S^h_k)_n\mathbf{S}^h_m \rangle\!\rangle_h$$

$$\mathcal{A}^v_{nk} \equiv \langle\!\langle -(V_1)_n\mathbf{S}^v_k - \mathbf{V}_1(S^v_k)_n + (H_1)_n\mathbf{S}^h_k + \mathbf{H}_1(S^h_k)_n$$
$$- (V_0)_n(\mathbf{G}^1_k)^v - \mathbf{V}_0(G^1_k)^v_n + (H_0)_n(\mathbf{G}^1_k)^h + \mathbf{H}_0(G^1_k)^h_n \rangle\!\rangle_h;$$
$$\mathbf{D}^v_{kmn} \equiv \langle\!\langle -(V_0)_n(\widetilde{\mathbf{G}}|^2_{km})^v - \mathbf{V}_0(\widetilde{G}^2_{km})^v_n + (H_0)_n(\widetilde{\mathbf{G}}^2_{km})^h + \mathbf{H}_0(\widetilde{G}^2_{km})^h_n \rangle\!\rangle_h;$$
$$\mathbf{d}^v_{kmnji} \equiv \langle\!\langle -(V_0)_i(\mathbf{G}^4_{kmnj})^v - \mathbf{V}_0(G^4_{kmnj})^v_i + (H_0)_i(\mathbf{G}^4_{kmnj})^h + \mathbf{H}_0(G^4_{kmnj})^h_i \rangle\!\rangle_h.$$

From the averaged horizontal magnetic component (8.189) of the equation for perturbation at order ε^2, we obtain similarly the equation for the mean perturbation of magnetic field generalising (8.152):

$$-\frac{\partial}{\partial T}\langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h + \eta\nabla^2_{\mathbf{X}}\langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h + \nabla_{\mathbf{X}} \times \sum_{k=1}^{K}\left(\sum_{m=1}^{K}\mathbf{A}^h_{km}c_{k0}c_{m0} + \mathcal{A}^h_k c_{k0}\right.$$
$$\left. + \sum_{m=1}^{2}\frac{\partial}{\partial X_m}\left(\mathbf{D}^h_{km}c_{k0} + \sum_{n=1}^{2}\sum_{j=1}^{2}\mathbf{d}^h_{kmj}\frac{\partial^2}{\partial X_n \partial X_j}\nabla^{-2}_{\mathbf{X}}c_{k0}\right)\right) = 0. \quad (8.210)$$

Here

$$\mathbf{A}^h_{km} \equiv \langle\!\langle \mathbf{V}_0 \times (\mathbf{G}^3_{km})^h + (\mathbf{G}^3_{km})^v \times \mathbf{H}_0 + \mathbf{S}^v_k \times \mathbf{S}^h_m \rangle\!\rangle_v;$$
$$\mathcal{A}^h_k \equiv \langle\!\langle (\mathbf{G}^1_k)^v \times \mathbf{H}_0 + \mathbf{V}_0 \times (\mathbf{G}^1_k)^h + \mathbf{S}^v_k \times \mathbf{H}_1 + \mathbf{V}_1 \times \mathbf{S}^h_k \rangle\!\rangle_v;$$
$$\mathbf{D}^h_{km} \equiv \langle\!\langle (\widetilde{\mathbf{G}}^2_k)^v \times \mathbf{H}_0 + \mathbf{V}_0 \times (\widetilde{\mathbf{G}}^2_k)^h \rangle\!\rangle_v;$$
$$\mathbf{d}^h_{kmnj} \equiv \langle\!\langle (\mathbf{G}^4_{kmnj})^v \times \mathbf{H}_0 + \mathbf{V}_0 \times (\mathbf{G}^4_{kmnj})^h \rangle\!\rangle_v.$$

The newly emerging terms involving the constant vectors $\mathcal{A}^h$ and $\mathcal{A}^v$ can be regarded as the operators of the combined magnetic α-effect and AKA-effect, respectively. If the CHM regime $\mathbf{V}_0, \mathbf{H}_0, \Theta_0$ has a symmetry considered in Sect. 8.4, and vector fields $\mathbf{V}_1, \mathbf{H}_1, \Theta_1$ do not constitute a symmetric set (e.g., if a symmetry-breaking bifurcation occurs), then solutions to the auxiliary problem $\mathbf{G}^1$ have a non-zero symmetric part, and hence generically the vector coefficients $\mathcal{A}^h$ and $\mathcal{A}^v$ do not vanish. Thus, despite the α-effect is insignificant in the leading order, combined magnetic α-effect and combined AKA-effect emerge in the mean-field equations due to a small, order ε, deviation of the CHM regime under investigation from a symmetric one, as in the models of Braginsky [33–39] and Soward [278, 279].

As in the analysis of stability of individual CHM regimes, the pseudodifferential operator describing non-standard non-local anisotropic combined eddy diffusion is present only if the regime is unsteady and does not possess the symmetry about a vertical axis or parity invariance without a time shift.

8.6.6 Amplitude Equations Derived from Order ε^2 Equations

Equations 8.185)–(8.187 for $n = 2$ are

$$\begin{aligned}
&\mathscr{L}^v(\mathbf{v}_2,\mathbf{h}_2,\theta_2,p_2) - \frac{\partial \mathbf{v}_0}{\partial T} + \nu(2(\nabla_\mathbf{x}\cdot\nabla_\mathbf{X})\mathbf{v}_1 + \nabla^2_\mathbf{X}\mathbf{v}_0) - ((\mathbf{V}_1+\mathbf{v}_0)\cdot\nabla_\mathbf{x})\mathbf{v}_1\\
&\quad + ((\mathbf{H}_1+\mathbf{h}_0)\cdot\nabla_\mathbf{x})\mathbf{h}_1 - (\mathbf{V}_0\cdot\nabla_\mathbf{X})\mathbf{v}_1 + (\mathbf{H}_0\cdot\nabla_\mathbf{X})\mathbf{h}_1 - (\mathbf{v}_1\cdot\nabla_\mathbf{x})\mathbf{V}_1 + (\mathbf{h}_1\cdot\nabla_\mathbf{x})\mathbf{H}_1\\
&\quad - ((\mathbf{V}_2+\mathbf{v}_1)\cdot\nabla_\mathbf{x})\mathbf{v}_0 + ((\mathbf{H}_2+\mathbf{h}_1)\cdot\nabla_\mathbf{x})\mathbf{h}_0 - ((\mathbf{V}_1+\mathbf{v}_0)\cdot\nabla_\mathbf{X})\mathbf{v}_0\\
&\quad + ((\mathbf{H}_1+\mathbf{h}_0)\cdot\nabla_\mathbf{X})\mathbf{h}_0 - (\mathbf{v}_0\cdot\nabla_\mathbf{x})\mathbf{V}_2 + (\mathbf{h}_0\cdot\nabla_\mathbf{x})\mathbf{H}_2 + \beta_2\theta_0\mathbf{e}_3 - \nabla_\mathbf{X}p_1 = 0;
\end{aligned} \tag{8.211}$$

$$\begin{aligned}
&\mathscr{L}^h(\mathbf{v}_2,\mathbf{h}_2) - \frac{\partial \mathbf{h}_0}{\partial T} + \eta\Big(2(\nabla_\mathbf{x}\cdot\nabla_\mathbf{X})\mathbf{h}_1 + \nabla^2_\mathbf{X}\mathbf{h}_0\Big)\\
&\quad + \nabla_\mathbf{x}\times(\mathbf{v}_1\times(\mathbf{H}_1+\mathbf{h}_0) + \mathbf{V}_1\times\mathbf{h}_1 + \mathbf{v}_0\times(\mathbf{H}_2+\mathbf{h}_1) + \mathbf{V}_2\times\mathbf{h}_0)\\
&\quad + \nabla_\mathbf{X}\times(\mathbf{v}_1\times\mathbf{H}_0 + \mathbf{V}_0\times\mathbf{h}_1 + \mathbf{v}_0\times(\mathbf{H}_1+\mathbf{h}_0) + \mathbf{V}_1\times\mathbf{h}_0) = 0;
\end{aligned} \tag{8.212}$$

$$\begin{aligned}
&\mathscr{L}^\theta(\mathbf{v}_2,\theta_2) - \frac{\partial \theta_0}{\partial T} + \kappa\Big((2\nabla_\mathbf{x}\cdot\nabla_\mathbf{X})\theta_1 + \nabla^2_\mathbf{X}\theta_0\Big)\\
&\quad - (\mathbf{V}_1\cdot\nabla_\mathbf{x})\theta_1 - (\mathbf{v}_1\cdot\nabla_\mathbf{x})(\Theta_1+\theta_0) - ((\mathbf{V}_1+\mathbf{v}_0)\cdot\nabla_\mathbf{X})\theta_0\\
&\quad - (\mathbf{V}_2\cdot\nabla_\mathbf{x})\theta_0 - (\mathbf{v}_0\cdot\nabla_\mathbf{x})(\Theta_2+\theta_1) - (\mathbf{V}_0\cdot\nabla_\mathbf{X})\theta_1\Big) = 0.
\end{aligned} \tag{8.213}$$

In order to apply to these equations the solvability condition (8.191), we introduce new variables:

$$\begin{aligned}
\mathbf{v}_2 = \mathbf{v}_2' + \sum_{k=1}^{K}\Bigg(&\sum_{m=1}^{2}\left(\nabla s^v_{km}\frac{\partial c_{k1}}{\partial X_m} + \nabla g^{1,v}_{km}\frac{\partial c_{k0}}{\partial X_m}\right) + \sum_{m=1}^{K}\sum_{n=1}^{2}\nabla g^{3,v}_{kmn}\frac{\partial}{\partial X_n}(c_{k0}c_{m0})\\
&+\sum_{m=1}^{2}\sum_{n=1}^{2}\left(\nabla g^{2,v}_{kmn}\frac{\partial^2 c_{k0}}{\partial X_m\partial X_n} + \sum_{j=1}^{2}\sum_{i=1}^{2}\nabla g^{4,v}_{kmnj}(\nabla^2_\mathbf{X})^{-1}\frac{\partial^4 c_{k0}}{\partial X_m\partial X_n\partial X_j\partial X_i}\right)\Bigg),
\end{aligned} \tag{8.214}$$

$$\begin{aligned}
\mathbf{h}_2 = \mathbf{h}_2' + \sum_{k=1}^{K}\Bigg(&\sum_{m=1}^{2}\left(\nabla s^h_{km}\frac{\partial c_{k1}}{\partial X_m} + \nabla g^{1,h}_{km}\frac{\partial c_{k0}}{\partial X_m}\right) + \sum_{m=1}^{K}\sum_{n=1}^{2}\nabla g^{3,h}_{kmn}\frac{\partial}{\partial X_n}(c_{k0}c_{m0})\\
&+\sum_{m=1}^{2}\sum_{n=1}^{2}\left(\nabla g^{2,h}_{kmn}\frac{\partial^2 c_{k0}}{\partial X_m\partial X_n} + \sum_{j=1}^{2}\sum_{i=1}^{2}\nabla g^{4,h}_{kmnji}(\nabla^2_\mathbf{X})^{-1}\frac{\partial^4 c_{k0}}{\partial X_m\partial X_n\partial X_j\partial X_i}\right)\Bigg).
\end{aligned} \tag{8.215}$$

Here $g^{q,v}(\mathbf{x},t)$ and $g^{q,h}(\mathbf{x},t)$ are globally bounded solutions to the Neumann problems

$$\nabla^2_{\mathbf{x}} g^{1,v}_{km} = -\{(\mathbf{G}^1_k)^v\}_m, \quad \partial g^{1,v}_{km}/\partial x_3\big|_{x_3=\pm L/2} = 0; \tag{8.216}$$

$$\nabla^2_{\mathbf{x}} g^{2,v}_{kmn} = -\{(\widetilde{\mathbf{G}}^2_{km})^v\}_n, \quad \partial g^{2,v}_{kmn}/\partial x_3\Big|_{x_3=\pm L/2} = 0; \tag{8.217}$$

$$\nabla^2_{\mathbf{x}} g^{3,v}_{kmn} = -\{(\mathbf{G}^3_{km})^v\}_n, \quad \partial g^{3,v}_{kmn}/\partial x_3\big|_{x_3=\pm L/2} = 0; \tag{8.218}$$

$$\nabla^2_{\mathbf{x}} g^{4,v}_{kmnji} = -\{(\mathbf{G}^4_{kmnj})^v\}_i, \quad \partial g^{4,v}_{kmnji}/\partial x_3\Big|_{x_3=\pm L/2} = 0 \tag{8.219}$$

[the problems for $g^{q,h}(\mathbf{x},t)$ are stated by replacing the superscripts v by h in Eqs. 8.216–8.219]. By virtue of relations for the divergencies (8.25) and (8.26) for $n=2$, vector fields $\mathbf{v}'_2$ and $\mathbf{h}'_2$ are solenoidal in the fast variables.

Substituting expressions for the first terms in expansion of the perturbation, (8.193), (8.207), (8.214) and (8.215), we transform Eqs. 8.211–8.213 into the following one:

$$\mathscr{L}(\mathbf{v}'_2, \mathbf{h}'_2, \theta_2, p_2) - \frac{\partial}{\partial T}(\mathbf{v}_0, \mathbf{h}_0, \theta_0) + \mathbf{C} - \nabla_{\mathbf{X}}\langle p_1\rangle + \cdots = 0, \tag{8.220}$$

where we have denoted

$$\begin{aligned}\mathbf{C} \equiv &\sum_{k=1}^{K}\left(\mathbf{B}^1_k c_{k0} + \sum_{m=1}^{2}\left(\mathbf{B}^2_{km}\frac{\partial c_{k0}}{\partial X_m} + \sum_{n=1}^{2}\left(\mathbf{B}^3_{kmn}\frac{\partial^2 c_{k0}}{\partial X_n \partial X_m} + \sum_{j=1}^{2}\left(\mathbf{B}^4_{kmnj}\frac{\partial^3 \nabla_{\mathbf{X}}^{-2} c_{k0}}{\partial X_m \partial X_n \partial X_j}\right.\right.\right.\right.\\ &+ \left.\left.\left.\left.\sum_{i=1}^{2}\mathbf{B}^5_{kmnji}\frac{\partial^4 \nabla_{\mathbf{X}}^{-2} c_{k0}}{\partial X_m \partial X_n \partial X_j \partial X_i}\right)\right)\right) + \sum_{m=1}^{K}\left(\mathbf{B}^6_{km} c_{k0} c_{m0} + \sum_{n=1}^{2}\left(\mathbf{B}^7_{kmn} c_{k0}\frac{\partial c_{m0}}{\partial X_n}\right.\right.\\ &+ \left.\left.\left.\sum_{j=1}^{2}\sum_{i=1}^{2}\mathbf{B}^8_{kmnji} c_{k0}\frac{\partial^3 \nabla_{\mathbf{X}}^{-2} c_{m0}}{\partial X_n \partial X_j \partial X_i} + \mathbf{B}^9_{kmn} c_{k0} c_{m0} c_{n0}\right)\right)\right).\end{aligned} \tag{8.221}$$

Ellipsis in the transformed equation (8.220) denote terms involving as factors the amplitudes c_{k1} or their derivatives (these amplitudes appear in the expression (8.207) for the second term in the expansion of the perturbation). Their structure is of no interest for us, since these terms are, under our assumptions, symmetric sets, and hence they do not contribute to scalar products with antisymmetric eigenfunctions $\mathbf{S}^*_k$ from the kernel of the adjoint operator. The sum $\mathbf{C}$ in equation (8.220) has the following coefficients:

$$\begin{aligned}\mathbf{B}^1_k = \Big(&-(\mathbf{V}_1\cdot\nabla)(\mathbf{G}^1_k)^v - ((\mathbf{G}^1_k)^v\cdot\nabla)\mathbf{V}_1 + (\mathbf{H}_1\cdot\nabla)(\mathbf{G}^1_k)^h + ((\mathbf{G}^1_k)^h\cdot\nabla)\mathbf{H}_1\\ &-(\mathbf{V}_2\cdot\nabla)\mathbf{S}^v_k - (\mathbf{S}^v_k\cdot\nabla)\mathbf{V}_2 + (\mathbf{H}_2\cdot\nabla)\mathbf{S}^h_k + (\mathbf{S}^h_k\cdot\nabla)\mathbf{H}_2 + \beta_2 S^\theta_k \mathbf{e}_3,\\ &\nabla\times((\mathbf{G}^1_k)^v\times\mathbf{H}_1 + \mathbf{V}_1\times(\mathbf{G}^1_k)^h + \mathbf{S}^v_k\times\mathbf{H}_2 + \mathbf{V}_2\times\mathbf{S}^h_k),\\ &-(\mathbf{V}^1\cdot\nabla)(G^1_k)^\theta - ((\mathbf{G}^1_k)^v\cdot\nabla)\Theta_1 - (\mathbf{V}_2\cdot\nabla)S^\theta_k - (\mathbf{S}^v_k\cdot\nabla)\Theta_2\Big),\end{aligned}$$

$$
\begin{aligned}
\mathbf{B}^2_{km} =& \mathscr{L}(\nabla g^{1,v}_{km}, \nabla g^{1,h}_{km}, 0, 0) + \Big(- (\mathbf{V}_1 \cdot \nabla)(\widetilde{\mathbf{G}}^2_{km})^v - ((\widetilde{\mathbf{G}}^2_{km})^v \cdot \nabla)\mathbf{V}_1 \\
& + (\mathbf{H}_1 \cdot \nabla)(\widetilde{\mathbf{G}}^2_{km})^h + ((\widetilde{\mathbf{G}}^2_{km})^h \cdot \nabla)\mathbf{H}_1 - (V_0)_m(\mathbf{G}^1_k)^v + (H_0)_m(\mathbf{G}^1_k)^h \\
& - (V_1)_m \mathbf{S}^v_k + (H_1)_m \mathbf{S}^h_k + 2\nu \frac{\partial (\mathbf{G}^1_k)^v}{\partial x_m} - (G^1_k)^p \mathbf{e}_m, \\
& + \nabla \times ((\widetilde{\mathbf{G}}^2_{km})^v \times \mathbf{H}_1 + \mathbf{V}_1 \times (\widetilde{\mathbf{G}}^2_{km})^h) + \mathbf{e}_m \times ((\mathbf{G}^1_k)^v \times \mathbf{H}_0 + \mathbf{V}_0 \times (\mathbf{G}^1_k)^h \\
& + \mathbf{S}^v_k \times \mathbf{H}_1 + \mathbf{V}_1 \times \mathbf{S}^h_k) + 2\eta \frac{\partial (\mathbf{G}^1_k)^h}{\partial x_m}, \\
& -(\mathbf{V}_1 \cdot \nabla)(\widetilde{G}^2_{km})^\theta - ((\widetilde{\mathbf{G}}^2_{km})^v \cdot \nabla)\Theta_1 - (V_0)_m (G^1_k)^\theta - (V_1)_m S^\theta_k + 2\kappa \frac{\partial (G^1_k)^\theta}{\partial x_m} \Big), \\
\mathbf{B}^3_{kmn} =& \mathscr{L}(\nabla g^{2,v}_{kmn}, \nabla g^{2,h}_{kmn}, 0, 0) \\
& + \Big(-(V_0)_n (\widetilde{\mathbf{G}}^2_{km})^v + (H_0)_n (\widetilde{\mathbf{G}}^2_{km})^h - (\widetilde{G}^2_{km})^p \mathbf{e}_n + \nu \Big(\delta^m_n \mathbf{S}^v_k + 2 \frac{\partial (\widetilde{\mathbf{G}}^2_{km})^v}{\partial x_n} \Big), \\
& \mathbf{e}_n \times ((\widetilde{\mathbf{G}}^2_{km})^v \times \mathbf{H}_0 + \mathbf{V}_0 \times (\widetilde{\mathbf{G}}^2_{km})^h) + \eta \left(\delta^m_n \mathbf{S}^h_k + 2 \frac{\partial (\widetilde{\mathbf{G}}^2_{km})^h}{\partial x_n} \right), \\
& -(V_0)_n (\widetilde{G}^2_{km})^\theta + \kappa \Big(\delta^m_n \mathbf{S}^\theta_k + 2 \frac{\partial (\widetilde{G}^2_{km})^\theta}{\partial x_n} \Big) \Big), \\
\mathbf{B}^4_{kmnj} =& \Big(-(\mathbf{V}_1 \cdot \nabla)(\mathbf{G}^4_{kmnj})^v - ((\mathbf{G}^4_{kmnj})^v \cdot \nabla)\mathbf{V}_1 \\
& + (\mathbf{H}_1 \cdot \nabla)(\mathbf{G}^4_{kmnj})^h + ((\mathbf{G}^4_{kmnj})^h \cdot \nabla)\mathbf{H}_1, \\
& \nabla \times ((\mathbf{G}^4_{kmnj})^v \times \mathbf{H}_1 + \mathbf{V}_1 \times (\mathbf{G}^4_{kmnj})^h), \\
& - (\mathbf{V}_1 \cdot \nabla)(G^4_{kmnj})^\theta - ((\mathbf{G}^4_{kmnj})^v \cdot \nabla)\Theta_1 \Big), \\
\mathbf{B}^5_{kmnji} =& \mathscr{L}(\nabla g^{4,v}_{kmnji}, \nabla g^{4,h}_{kmnji}, 0, 0) \\
& + \Bigg(-(V_0)_i (\mathbf{G}^4_{kmnj})^v + (H_0)_i (\mathbf{G}^4_{kmnj})^h + 2\nu \frac{\partial (\mathbf{G}^4_{kmnj})^v}{\partial x_i} - (G^4_{kmnj})^p \mathbf{e}_i, \\
& \mathbf{e}_i \times ((\mathbf{G}^4_{kmnj})^v \times \mathbf{H}_0 + \mathbf{V}_0 \times (\mathbf{G}^4_{kmnj})^h) + 2\eta \frac{\partial (\mathbf{G}^4_{kmnj})^h}{\partial x_i}, \\
& -(V_0)_i (G^4_{kmnj})^\theta + 2\kappa \frac{\partial (G^4_{kmnj})^\theta}{\partial x_i} \Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B}^6_{km} =\Big(&-(\mathbf{S}^v_k\cdot\nabla)(\mathbf{G}^1_m)^v-((\mathbf{G}^1_m)^v\cdot\nabla)\mathbf{S}^v_k+(\mathbf{S}^h_k\cdot\nabla)(\mathbf{G}^1_m)^h+((\mathbf{G}^1_m)^h\cdot\nabla)\mathbf{S}^h_k\\
&-(\mathbf{V}_1\cdot\nabla)(\mathbf{G}^3_{km})^v-((\mathbf{G}^3_{km})^v\cdot\nabla)\mathbf{V}_1+(\mathbf{H}_1\cdot\nabla)(\mathbf{G}^3_{km})^h+((\mathbf{G}^3_{km})^h\cdot\nabla)\mathbf{H}_1,\\
&\nabla\times((\mathbf{G}^1_m)^v\times\mathbf{S}^h_k+\mathbf{S}^v_k\times(\mathbf{G}^1_m)^h+(\mathbf{G}^3_{km})^v\times\mathbf{H}_1+\mathbf{V}_1\times(\mathbf{G}^3_{km})^h),\\
&-(\mathbf{S}^v_k\cdot\nabla)(G^1_m)^\theta-((\mathbf{G}^1_m)^v\cdot\nabla)S^\theta_k-(\mathbf{V}_1\cdot\nabla)(G^3_{km})^\theta-((\mathbf{G}^3_{km})^v\cdot\nabla)\Theta_1\Big),\\
\mathbf{B}^7_{kmn} =\ &\mathscr{L}(\nabla(g^{3,v}_{kmn}+g^{3,v}_{mkn}),\nabla(g^{3,h}_{kmn}+g^{3,h}_{mkn}),0,0)+\Big(-(S^v_k)_n\mathbf{S}^v_m+(S^h_k)_n\mathbf{S}^h_m\\
&-(\mathbf{S}^v_k\cdot\nabla)(\widetilde{\mathbf{G}}^2_{mn})^v-((\widetilde{\mathbf{G}}^2_{mn})^v\cdot\nabla)\mathbf{S}^v_k+(\mathbf{S}^h_k\cdot\nabla)(\widetilde{\mathbf{G}}^2_{mn})^h+((\widetilde{\mathbf{G}}^2_{mn})^h\cdot\nabla)\mathbf{S}^h_k\\
&-(V_0)_n((\mathbf{G}^3_{km})^v+(\mathbf{G}^3_{mk})^v)+(H_0)_n((\mathbf{G}^3_{km})^h+(\mathbf{G}^3_{mk})^h)\\
&-((G^3_{km})^p+(G^3_{mk})^p)\mathbf{e}_n+2\nu\frac{\partial}{\partial x_n}((\mathbf{G}^3_{km})^v+(\mathbf{G}^3_{mk})^v),\\
&\nabla\times((\widetilde{\mathbf{G}}^2_{mn})^v\times\mathbf{S}^h_k+\mathbf{S}^v_k\times(\widetilde{\mathbf{G}}^2_{mn})^h)+\mathbf{e}_n\times\Big(\mathbf{S}^v_k\times\mathbf{S}^h_m+\mathbf{S}^v_m\times\mathbf{S}^h_k\\
&+((\mathbf{G}^3_{km})^v+(\mathbf{G}^3_{mk})^v)\times\mathbf{H}_0+\mathbf{V}_0\times((\mathbf{G}^3_{km})^h+(\mathbf{G}^3_{mk})^h)\Big)\\
&+2\eta\frac{\partial}{\partial x_n}((\mathbf{G}^3_{km})^h+(\mathbf{G}^3_{mk})^h),\\
&-(\mathbf{S}^v_k\cdot\nabla)(\widetilde{G}^2_{mn})^\theta-((\widetilde{\mathbf{G}}^2_{mn})^v\cdot\nabla)S^\theta_k-(S^v_m)_nS^\theta_k\\
&-(V_0)_n((G^3_{km})^\theta+(G^3_{mk})^\theta)+2\kappa\frac{\partial}{\partial x_n}((G^3_{km})^\theta+(G^3_{mk})^\theta)\Big),\\
\mathbf{B}^8_{kmnji} =\Big(&-(\mathbf{S}^v_k\cdot\nabla)(\mathbf{G}^4_{mnji})^v-((\mathbf{G}^4_{mnji})^v\cdot\nabla)\mathbf{S}^v_k+(\mathbf{S}^h_k\cdot\nabla)(\mathbf{G}^4_{mnji})^h+((\mathbf{G}^4_{mnji})^h\cdot\nabla)\mathbf{S}^h_k,\\
&\nabla\times((\mathbf{G}^4_{mnji})^v\times\mathbf{S}^h_k+\mathbf{S}^v_k\times(\mathbf{G}^4_{mnji})^h),\\
&-(\mathbf{S}^v_k\cdot\nabla)(G^4_{mnji})^\theta-((\mathbf{G}^4_{mnji})^v\cdot\nabla)S^\theta_k\Big),\\
\mathbf{B}^9_{kmn} =\Big(&-(\mathbf{S}^v_k\cdot\nabla)(\mathbf{G}^3_{mn})^v-((\mathbf{G}^3_{mn})^v\cdot\nabla)\mathbf{S}^v_k+(\mathbf{S}^h_k\cdot\nabla)(\mathbf{G}^3_{mn})^h+((\mathbf{G}^3_{mn})^h\cdot\nabla)\mathbf{S}^h_k,\\
&\nabla\times((\mathbf{G}^3_{mn})^v\times\mathbf{S}^h_k+\mathbf{S}^v_k\times(\mathbf{G}^3_{mn})^h),\\
&-(\mathbf{S}^v_k\cdot\nabla)(G^3_{mn})^\theta-((\mathbf{G}^3_{mn})^v\cdot\nabla)S^\theta_k\Big).
\end{aligned}
$$

To calculate the mean pressure $\langle p_1\rangle$, we average the horizontal flow component of equation (8.220) over the fast spatial variables and take the divergence of the result over the slow spatial variables. This yields

$$\langle p_1\rangle=(\nabla^2_{\mathbf{X}})^{-1}(\nabla_{\mathbf{X}}\cdot\langle\mathbf{C}^v\rangle_h)+\cdots,$$

where ellipsis denote symmetric terms involving the amplitudes c_{k1} or their derivatives. Applying now the solvability conditions (8.191) to equation (8.220), we obtain *amplitude equations*, closing the system of mean-field equations (8.209) and (8.210):

$$\frac{\partial c_{k0}}{\partial T} = \langle\!\langle (\mathbf{C} - (\nabla_{\mathbf{X}}(\nabla^2_{\mathbf{X}})^{-1}(\nabla_{\mathbf{X}} \cdot \langle \mathbf{C}^v \rangle_h), 0, 0)) \cdot \mathbf{S}^*_k \rangle\!\rangle \tag{8.222}$$

(here $5 \le k \le K$). If the symmetry responsible for insignificance of the α-effect in the leading order is spatial (i.e., without a time shift), then $\langle p_1 \rangle$ does not contribute to the amplitude equations (also, in this case $\mathbf{B}^4_{kmnj} = \mathbf{B}^5_{kmnji} = \mathbf{B}^8_{kmnji} = 0$).

It is instructive to consider as an example the system of Eqs. 8.209, 8.210 and 8.222 obtained for a family of CHM regimes emerging in a Hopf bifurcation of a steady CHM state. In a generic case $K = 6$, neutral short-scale stability modes $\mathbf{S}_k$ do not depend on time for $k \le 4$, and are proportional to $\exp(\pm i\omega t)$ for $k = 5, 6$, where $\pm i\omega$ is the pair of eigenvalues responsible for the bifurcation. Since the fields $\mathbf{S}^*_k$ in the biorthogonal basis have the same kinds of time dependence, any non-vanishing term in the r.h.s. of amplitude equations (8.222) for $k = 5, 6$ involves as factors at least one amplitude $c_{0,5}$ or $c_{0,6}$, or their spatial derivatives. This implies $c_{0,5} = c_{0,6} = 0$ provided this is so initially at $T = 0$. Furthermore, $\mathbf{V}_1, \mathbf{H}_1, \Theta_1$ are also sums of harmonic oscillations with frequency ω, and hence linear and quadratic terms involving the amplitudes $c_{0,5}$ and $c_{0,6}$ appear in Eqs. 8.209 and 8.210. Therefore, while it is legitimate to consider just mean-field equations (8.209) and (8.210) (for $c_{0,5} = c_{0,6} = 0$) in a study of large-scale weakly nonlinear stability of regimes emerging in a Hopf bifurcation (as this is done in [330]), such a study does not reveal the complete story, because the nonlinearity gives rise to interaction of steady and oscillatory neutral modes.

8.7 Computation of Coefficients of the MHD Eddy Operators

It suffices to solve numerically 22 auxiliary problems (4, 8 and 10 problems of types I–III, respectively) to determine the vector coefficients $\mathbf{A}$ and $\mathbf{D}$ in the mean-field equations (8.152) and (8.165). (It is natural to apply the equations for vorticity in the form of their vector potentials, e.g., to apply (8.144)–(8.146) in place of (8.121), (8.125) and (8.129).)

However, we note that solutions to the auxiliary problems of types II and III enter into the means (8.153) and (8.155)–(8.157) only as factors in scalar products with the vector $(\mathbf{H} \times \mathbf{e}_3, -\mathbf{V} \times \mathbf{e}_3, 0)$, and into (8.166), (8.167) and (8.170)–(8.172) with the vectors $\mathbf{W}_{n,j} \equiv (-V_j\mathbf{e}_n - V_n\mathbf{e}_j, H_j\mathbf{e}_n + H_n\mathbf{e}_j, 0)$ (4 vectors for $n, j = 1, 2$). (We employ the scalar product[4] of seven-dimensional vector fields $\langle\!\langle \{\boldsymbol{\gamma}^v, \boldsymbol{\gamma}^h, \gamma^\theta\} \cdot \{\mathbf{W}^v, \mathbf{W}^h, W^\theta\} \rangle\!\rangle$.) Moreover, since the potential part of the field, to which the curl is applied, is inessential in (8.165), it suffices to compute the scalar products with $\mathbf{W}_{1,2}, \mathbf{W}_{2,1}$ and $\mathbf{W}_{1,1} - \mathbf{W}_{2,2}$ in (8.166), (8.167) and (8.170)–(8.172). Consequently, the number of auxiliary problems to be solved decreases to 8, if one considers, as in previous chapters, auxiliary problems for the adjoint operator

[4] See the footnote on p. 171.

(of the same numerical complexity as the auxiliary problems of types II and III) [327, 329].

We split the three-dimensional components of $\mathbf{W}$ and $\boldsymbol{\gamma}$ into the spatial mean, the solenoidal and potential components:

$$\begin{aligned} &\mathbf{W}^v = \langle \mathbf{W}^v \rangle_h + \widehat{\mathbf{W}}^v + \nabla W^v, \quad \mathbf{W}^h = \langle \mathbf{W}^h \rangle_h + \widehat{\mathbf{W}}^h + \nabla W^h, \\ &\boldsymbol{\gamma}^v = \langle \boldsymbol{\gamma}^v \rangle_h + \widehat{\boldsymbol{\gamma}}^v + \nabla \gamma^v, \quad \boldsymbol{\gamma}^h = \langle \boldsymbol{\gamma}^h \rangle_h + \widehat{\boldsymbol{\gamma}}^h + \nabla \gamma^h \end{aligned} \tag{8.223}$$

(all differential operators are in the fast variables). Suppose $\widehat{\boldsymbol{\gamma}} \equiv (\widehat{\boldsymbol{\gamma}}^v, \widehat{\boldsymbol{\gamma}}^h, \gamma^\theta)$ satisfies the equations

$$\widetilde{\mathscr{L}}(\widehat{\boldsymbol{\gamma}}, \widehat{\gamma}^p) = \mathbf{z}, \quad \nabla \cdot \widehat{\boldsymbol{\gamma}}^v = \nabla \cdot \widehat{\boldsymbol{\gamma}}^h = 0, \quad \langle \widehat{\boldsymbol{\gamma}}^v \rangle_h = \langle \widehat{\boldsymbol{\gamma}}^h \rangle_h = 0,$$

$\mathbf{Z}$ is a solution to the auxiliary problem for the adjoint operator:

$$\mathscr{L}^\dagger(\mathbf{Z}^v, \mathbf{Z}^h, Z^\theta) = (\widehat{\mathbf{W}}^v, \widehat{\mathbf{W}}^h, 0), \quad \nabla \cdot \mathbf{Z}^v = \nabla \cdot \mathbf{Z}^h = 0, \quad \langle \mathbf{Z}^v \rangle_h = \langle \mathbf{Z}^h \rangle_h = 0, \tag{8.224}$$

and the fields $\widehat{\boldsymbol{\gamma}}$ and $\mathbf{Z}$ satisfy the boundary conditions defined by (8.5)–(8.7) and (8.11). Here the operator $\mathscr{L}^\dagger$ is the adjoint to the restriction of $\widetilde{\mathscr{L}} = (\mathscr{L}^v, \mathscr{L}^h, \mathscr{L}^\theta)$ to the subspace of the domain of $\widetilde{\mathscr{L}}$ defined by the condition that the mean over the fast spatial variables of the horizontal component of the magnetic field vanishes:

$$\mathscr{L}^\dagger(\mathbf{v}, \mathbf{h}, \theta) = ((\widetilde{\mathscr{L}^*})^v(\mathbf{v}, \mathbf{h}, \theta), \{(\widetilde{\mathscr{L}^*})^h(\mathbf{v}, \mathbf{h}, \theta)\}_h, (\widetilde{\mathscr{L}^*})^\theta(\mathbf{v}, \mathbf{h}, \theta))$$

[the operator $\widetilde{\mathscr{L}^*}$ is defined by relations (8.58)–(8.60)]. Then

$$\begin{aligned} &\langle\!\langle \boldsymbol{\gamma}^v \cdot \mathbf{W}^v + \boldsymbol{\gamma}^h \cdot \mathbf{W}^h \rangle\!\rangle \\ &\quad = \langle\!\langle \langle \mathbf{W}^v \rangle_h \cdot \langle \boldsymbol{\gamma}^v \rangle_h + \langle \mathbf{W}^h \rangle_h \cdot \langle \boldsymbol{\gamma}^h \rangle_h + \widehat{\mathbf{W}}^v \cdot \widehat{\boldsymbol{\gamma}}^v + \widehat{\mathbf{W}}^h \cdot \widehat{\boldsymbol{\gamma}}^h + \nabla W^v \cdot \nabla \gamma^v + \nabla W^h \cdot \nabla \gamma^h \rangle\!\rangle \\ &\quad = \langle\!\langle \langle \mathbf{W}^v \rangle_h \cdot \langle \boldsymbol{\gamma}^v \rangle_h + \langle \mathbf{W}^h \rangle_h \cdot \langle \boldsymbol{\gamma}^h \rangle_h + \widehat{\boldsymbol{\gamma}} \cdot \widetilde{\mathscr{L}^*}(\mathbf{Z}) + \mathbf{W}^v \cdot \nabla \gamma^v + \mathbf{W}^h \cdot \nabla \gamma^h \rangle\!\rangle \\ &\quad = \langle\!\langle \langle \mathbf{W}^v \rangle_h \cdot \langle \boldsymbol{\gamma}^v \rangle_h + \langle \mathbf{W}^h \rangle_h \cdot \langle \boldsymbol{\gamma}^h \rangle_h + \mathbf{z} \cdot \mathbf{Z} - \gamma^v \nabla \cdot \mathbf{W}^v - \gamma^h \nabla \cdot \mathbf{W}^h \rangle\!\rangle. \end{aligned}$$

This formula is analogouos to (8.58)–(8.60).

So, one can avoid solving the auxiliary problems of types II and III—it suffices to know the r.h.s. in the equations in the statements of these problems. When computing the coefficients (8.153), (8.166) and (8.167) we apply the expressions (8.114) and (8.119) for the divergencies to determine the potential parts of the fields $\mathbf{G}_{mk}$ and to reformulate the problems (8.111)–(8.113) and (8.116)–(8.118) in the terms of the solenoidal parts of the unknown vector fields, and consider the vector potentials of the vorticity equations (formulated in the terms of the operator $\mathscr{L}^\omega$). As discussed in Sect. 7.5, in the case of an unsteady CHM regime, reversal of time can help to overcome the difficulties arising from non-parabolicity of the operator $\widetilde{\mathscr{L}^*}$.

Similarly, instead of solving auxiliary problems (8.208), auxiliary problems for adjoint operator can be stated and solved for computation of the coefficients $\mathbf{A}$, $\mathcal{A}$, $\mathbf{D}$ and $\mathbf{d}$ in the new terms in the mean-field equations (8.209) and (8.210) for perturbations of ε-dependent CHM regimes.

8.8 Conclusions

1. We have explored weakly nonlinear stability of individual short-scale forced CHM regimes to large-scale perturbations and derived expressions for the operator of the α-effect, generically governing perturbations involving large spatial and temporal scales. For CHM regimes, where the α-effect is insignificant in the leading order, we have derived the system of mean-field equations, (8.152) and (8.165), for the weakly nonlinear evolution of the averaged leading terms in the power series expansion of a large-scale perturbation in the ratio of spatial scales. The equations attest that the combined eddy diffusion and advection occur in such CHM regimes. Yet another eddy effect acts in unsteady forced CHM regimes lacking spatial symmetries—parity invariance or the symmetry about a vertical axis; it is described by a pseudodifferential operator of the second order, and hence can be interpreted as a generalised non-local anisotropic combined diffusion. As a result of insignificance of the α-effect in the leading order, the operator of the α-effect is absent in the mean-field equations.
2. The operator of the α-effect can emerge in the mean-field equations, even when the α-effect is insignificant in the leading order, in analysis of weakly nonlinear stability of short-scale forced CHM regimes depending on a small parameter, proportional to the ratio of the spatial scales. This happens [see (8.209) and (8.210)], e.g., if a family of regimes emerges in a symmetry-breaking bifurcation of a CHM regime, parity-invariant or symmetric about a vertical axis. In this case the responsibility for emergence of the combined α-effect acting on mean perturbations lies with the antisymmetric difference between the bifurcating CHM regime and the regime at the point of the bifurcation.
3. The leading-order part of large-scale perturbations of forced short-scale CHM regimes, emerging in a symmetry-breaking bifurcation, is composed of neutral steady and oscillatory stability modes (generically there is one zero-mean steady mode if a saddle-node or pitchfork bifurcation occurs, and two oscillatory modes if a Hopf bifurcation occurs). Nonlinearity of equations for perturbations gives rise to interaction of short-scale modes. Equations (A.3) for amplitudes of zero-mean neutral modes (but not of the mean-field equations, despite the latter are instances of amplitude equations) involve additional eddy effects, such as cubic nonlinearity and—unless the CHM regime possesses parity invariance or the symmetry about a vertical axis without a time shift—nonlinear (quadratic) pseudodifferential operators. Therefore, weakly nonlinear perturbations of CHM regimes from such families are likely to exhibit a

broader variety of patterns of nonlinear behaviour, than perturbations of individual CHM regimes. Solutions to such PDEs do not obey quadratic energy inequalities, typically satisfied by solutions to equations of hydrodynamic type. The presence of cubic nonlinearity can, in principle, result in development of a finite-time singularity. Thus, a numerical study of the behaviour of solutions to the system of the mean-field and amplitude equations is of major interest (although note that an unbounded growth violates the basic assumption about the smallness of magnitude of perturbations, making the equations that we have derived in this section inadequate for description of their subsequent evolution).

4. Derivation of mean-field and amplitude equations by homogenisation methods relies on the basic structural properties of the equations of magnetohydrodynamics. By a straightforward reformulation of equations for large-scale perturbations of CHM regimes (see Appendix), we acquire mean-field and amplitude equations for perturbations of families of MHD regimes of forced fluid flows residing in the entire space (mean-field equations for perturbations of generic individual MHD regimes were derived in the previous chapter).

Chapter 9
Weakly Nonlinear Stability of Free Thermal Hydromagnetic Convection

Following the paper [331], we consider in this chapter weakly nonlinear stability to large-scale perturbations of short-scale regimes of thermal hydromagnetic convection in a horizontal layer of a rotating electrically conducting fluid. In contrast with the previous chapter, we assume now that convection is free, i.e., no external forces act in the layer of fluid or on its boundaries (except for the buoyancy, Coriolis and Lorentz forces), and no external sources of heat or magnetic field are present in the fluid. Consequently, there are no additional terms in the equations governing convective hydromagnetic (CHM) regimes, and hence the CHM regime under consideration is translation-invariant in space and time. This implies that some short-scale neutral modes are obtained by differentiation of the governing equations in the horizontal spatial variables or in time. Our goal is, as before, to derive a closed system of amplitude equations describing the evolution of large-scale perturbations.

The type of convection—free and not forced—is the only difference between the CHM regimes examined in this chapter and the previous one. The same boundary conditions as in Chap. 8 are assumed; as in the previous chapter, the possibilities of rotation of the layer of fluid about the vertical axis and the unsteadiness of a regime are not excluded, we demand neither stability of the CHM regime under consideration to short-scale perturbations, nor its quasi-periodicity or periodicity in time or in horizontal directions. Consequently, all the difficulties that we have encountered when considering forced convection remain in the present problem. We follow the same approach to solve it. Periodicity of the regime guarantees the solvability of the auxiliary problems for non-exclusive sets of parameter values and correctness of averaging over the fast variables; we assume that the solvability and correctness persist in the absence of the periodicity.

In general, the operators of the combined MHD α-effect arise in the amplitude equations governing the perturbations, as for regimes of forced convection. If a CHM regime possesses parity invariance or the symmetry about a vertical axis with a time shift $\widetilde{T} \geq 0$, the α-effect is absent or insignificant in the leading order. Existence of short-scale CHM regimes with all desirable properties—stability to

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short-scale perturbations, periodicity in horizontal directions, steadiness, periodicity or quasi-periodicity in time, symmetry about a vertical axis or parity invariance without a time shift—was confirmed in computations [55, 353]. If the α-effect is insignificant in the leading order, the amplitude equations that we obtain constitute a closed system of equations of a mixed type: equations for the averaged leading term of the equations for the perturbation of the magnetic field are evolutionary, and the remaining ones involve neither the derivative in (the slow) time, nor the operators of molecular diffusion. As in the mean-field equations for perturbations derived in Chap. 8, the operator of eddy correction of diffusion, not necessarily sign-definite, and quadratic terms describing eddy correction of advection, are present in the amplitude equations. If the CHM regime is time-dependent and does not have a symmetry without a time shift (but nevertheless the α-effect is insignificant in the leading order), pseudodifferential operators of the second order appear in the amplitude equations; like in the case of forced convection, they can be interpreted as representing a non-standard non-local convection. As in the problems considered so far, all the eddy operators are generically anisotropic.

The so-called "small-angle instability" problem (see [228, 230] and references therein) is a hydrodynamic problem about linear stability of particular convective flows—the rolls—with respect to a class of perturbations involving large spatial scales (more precisely, the same rolls rotated about the vertical axis by an infinitesimally small angle). Note that although it is a large-scale stability problem for free convection, it does not fall into the category of problems that we study in this chapter (even if the difference between linear and weakly nonlinear stability is neglected): while in the small angle instability problem two asymptotically different large spatial scales are present, we consider perturbations with a single large spatial scale.

9.1 Statement of the Problem

In the Boussinesq approximation, the CHM regime $\boldsymbol{\Omega}, \mathbf{V}, \mathbf{H}, \mathscr{T}$ satisfies equations (8.1)–(8.4). The boundary conditions (8.5)–(8.8) are assumed on the horizontal boundaries. We use the new variable (8.9), the difference between the temperature and its linear profile in the layer, which satisfies the evolution equation (8.10) and homogeneous boundary conditions (8.11). We consider now *free* convection, i.e. $\mathbf{F} = \mathbf{J} = 0$, $S = 0$ in the governing equations (8.1), (8.2) and (8.10).

We will employ the same notation for different mean values over the fast variables and the respective fluctuating parts of scalar and vector fields, as in the previous chapter (see Sect. 8.1.5). The operator of linearisation of the governing equations (8.1), (8.2) and (8.10) around the CHM regime under consideration, $\mathscr{L} = (\mathscr{L}^\omega, \mathscr{L}^h, \mathscr{L}^\theta)$, is defined in Sect. 8.1.3. The profiles of weakly nonlinear perturbations $\boldsymbol{\omega}, \mathbf{v}, \mathbf{h}, \theta$ satisfy equations (8.13)–(8.19); their solution is sought in the form of power series (8.20)–(8.23), where $\mathbf{X} = \varepsilon(x_1, x_2)$ and $T = \varepsilon^2 t$ are the slow horizontal variables and the slow time, respectively. Solenoidality conditions (8.16)–(8.17) for the perturbation reduce to (8.24)–(8.28), and the identity (8.18) defining vorticity to (8.33) and (8.34). Substituting series (8.20)–(8.23) into equations (8.13)–(8.15) we obtain the hierarchy of systems of equations (8.29)–(8.31).

9.2 Neutral Short-Scale Linear Stability Modes

Equations (8.29)–(8.31) governing the evolution of the leading terms in the expansion of perturbation, and the supplementary equations for vorticity, (8.32), and solenoidality, (8.25)–(8.28), are for $n = 0$ as follows:

$$\mathscr{L}(\boldsymbol{\omega}_0, \mathbf{v}_0, \mathbf{h}_0, \theta_0) = 0, \tag{9.1}$$

$$\boldsymbol{\omega}_0 = \nabla_{\mathbf{x}} \times \mathbf{v}_0, \tag{9.2}$$

$$\nabla_{\mathbf{x}} \cdot \boldsymbol{\omega}_0 = \nabla_{\mathbf{x}} \cdot \mathbf{v}_0 = \nabla_{\mathbf{x}} \cdot \mathbf{h}_0 = 0. \tag{9.3}$$

Differentiation of the governing equations (8.1), (8.2) and (8.10) in $x_k,\ k = 1, 2$, or in the fast time t demonstrates that $\mathbf{S}_k^{x\cdot} \equiv (\partial\boldsymbol{\Omega}/\partial x_k, \partial\mathbf{V}/\partial x_k, \partial\mathbf{H}/\partial x_k, \partial\Theta/\partial x_k)$ and $\mathbf{S}^{t\cdot} \equiv (\partial\boldsymbol{\Omega}/\partial t, \partial\mathbf{V}/\partial t, \partial\mathbf{H}/\partial t, \partial\Theta/\partial t)$ are solutions to equations (9.1)–(9.3) satisfying the assumed boundary conditions. These neutral short-scale stability modes exist because of the spatial and temporal translation invariance of the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$. They are not necessarily linearly independent; for instance, $\mathbf{S}^{t\cdot} = 0$ for steady CHM states. Also, $\mathbf{S}_k^{x\cdot}$ are linearly dependent, if the fields are independent of a horizontal variable $a_1x_1 + a_2x_2$ for some constant a_1 and a_2 not vanishing simultaneously; we will not consider such two-dimensional regimes.

As in the previous chapter, we will employ the operator of linearisation in the form, not involving the flow velocity perturbation:

$$\mathscr{M}'(\boldsymbol{\omega}, \mathbf{h}, \theta) \equiv \mathscr{L}(\boldsymbol{\omega}, \mathscr{C}\boldsymbol{\omega}, \mathbf{h}, \theta).$$

Here operator $\mathscr{C}$ is the inverse curl, defined for the boundary conditions under consideration in Sect. 8.2.1. The domain of $\mathscr{M}'$ is the space of fields $(\boldsymbol{\omega}, \mathbf{h}, \theta)$, satisfying the boundary conditions defined by (8.6), (8.7) and (8.11) and having the vanishing mean $\langle\boldsymbol{\omega}\rangle_v = 0$ and solenoidal vorticity and magnetic field components. The operator $\mathscr{M}'^*$ adjoint to linearisation $\mathscr{M}'$ is presented by relations (8.61)–(8.63) and has the same domain.

Since, obviously, $\mathscr{M}'^*(0, \mathbf{C}^h, 0) = 0$, where $\mathbf{C}^h$ is a constant horizontal vector, there exist two linearly independent modes $\widetilde{\mathbf{S}}_k^{h\cdot}$ with non-zero means of the horizontal components of the magnetic field. The spatio-temporal means of the horizontal components of $\mathbf{S}_k^{xh}$ and $\mathbf{S}^{th}$ over the fast variables being zero, these modes are linearly independent from the modes with non-zero means $\left\langle\!\left\langle \widetilde{\mathbf{S}}_k^{hh} \right\rangle\!\right\rangle_h$.

We normalise $\widetilde{\mathbf{S}}_k^{h\cdot}$ by the conditions $\left\langle\!\left\langle \widetilde{\mathbf{S}}_k^{hh} \right\rangle\!\right\rangle_h = \mathbf{e}_k$. Averaging the horizontal magnetic component of equation (9.1) over the fast variables shows that the spatial mean of the horizontal component of the magnetic field does not depend on the fast time, and hence this normalisation is equivalent to the conditions that $\langle \widetilde{\mathbf{S}}_k^{hh} \rangle_h = \mathbf{e}_k$ at any time. In other words, $\widetilde{\mathbf{S}}_k^{h\cdot} = (\mathbf{S}_k^{h\omega}, \mathbf{S}_k^{v\omega}, \mathbf{S}_k^{hh} + \mathbf{e}_k, S_k^{h\theta})$, where $\mathbf{S}_k^{h\cdot}$ are solutions to the auxiliary problem I.2 (8.78)–(8.81). For all neutral short-scale modes introduced in this paragraph, the spatial means of the vertical components of the vorticity $\mathbf{S}_k^{\cdot\omega}$ vanish (note that by virtue of the identity (8.42) $\langle \boldsymbol{\omega}_0 \rangle_v$ is conserved in the fast time, and this conserved field must be zero to satisfy the necessary condition (8.33) for recovery of the flow velocity from vorticity for $n = 0$), as well as the horizontal components of the flow velocity $\mathbf{S}_k^{\cdot v} = \mathscr{C}\mathbf{S}_k^{\cdot\omega}$.

Therefore, we have found up to five linearly independent solutions to the problem (9.1)–(9.3), existing for all parameter values ($k = 1, 2$): $\mathbf{S}_k \equiv \mathbf{S}_k^{x\cdot}$, $\mathbf{S}_{k+2} \equiv \widetilde{\mathbf{S}}_k^{h\cdot}$ and $\mathbf{S}_5 \equiv \mathbf{S}^{t\cdot}$ for unsteady CHM regimes $\mathbf{V}, \mathbf{H}, \Theta$. In what follows we assume that the problem is investigated in a generic setup, i.e. the invariant subspace of dimension K spanned by the fields $(\mathbf{S}_k^{\omega}, \mathbf{S}_k^{h}, S_k^{\theta})$, $1 \le k \le K$, constitutes the kernel of $\mathscr{M}'$, where $K = 4$ or 5 for steady and time-dependent CHM regimes $\mathbf{V}, \mathbf{H}, \Theta$, respectively. We can choose a basis in $\ker \mathscr{M}'^*$ comprised of K eigenfunctions $\mathbf{S}_k^* = (\mathbf{S}_k^{*\omega}, \mathbf{S}_k^{*h}, S_k^{*\theta})$ biorthogonal to $\mathbf{S}_k$:

$$\left\langle\!\left\langle (\mathbf{S}_k^{\omega}, \mathbf{S}_k^{h}, S_k^{\theta}) \cdot \mathbf{S}_j^* \right\rangle\!\right\rangle = \delta_j^k \tag{9.4}$$

Here, δ_j^k is the Kronecker symbol, $1 \le k,\ j \le K$, and $\cdot$ denotes the usual scalar product of 7-dimensional vector fields. In particular, $\mathbf{S}_k^* \equiv (0, \mathbf{e}_{k-2}, 0)$ for $k = 3, 4$, and $\langle\!\langle \mathbf{S}_k^{*h} \rangle\!\rangle_h = 0$ for $k = 1, 2, 5$.

By our assumption, the general solution of the problem (9.1)–(9.3) is a linear combination

$$(\boldsymbol{\omega}_0, \mathbf{v}_0, \mathbf{h}_0, \theta_0) = \boldsymbol{\xi}_0 + \sum_{k=1}^{K} c_{0k} \mathbf{S}_k. \tag{9.5}$$

Here $\boldsymbol{\xi}_0(\mathbf{X}, T, \mathbf{x}, t) = (\boldsymbol{\xi}_0^{\omega}, \boldsymbol{\xi}_0^{v}, \boldsymbol{\xi}_0^{h}, \xi_0^{\theta})$ is a transient, also satisfying equations (9.1)–(9.3) and decaying exponentially in the fast time t: The initial conditions for $\boldsymbol{\xi}_0$ (found from relation (9.5) at $t = 0$) must belong to the stable manifold of the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$. Our goal is to derive a closed system of equations for the amplitudes $c_{0k}(\mathbf{X}, T)$ in the leading term (9.5) of the expansion of the perturbation.

If the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ is symmetric about a vertical axis or parity-invariant, possibly with a time shift $\widetilde{T} \ge 0$ (see the definition in Sect. 8.4), then symmetric and antisymmetric sets of fields are invariant subspaces of the operator of linearisation $\mathscr{M}'$. Any neutral stability mode is thus a symmetric or antisymmetric set of fields (exponentially decaying transients, not required to possess any symmetry or antisymmetry, are neglected). In particular, all $\mathbf{S}_k$ are antisymmetric modes, except for the mode $\mathbf{S}_5 = \mathbf{S}^{t\cdot}$, which is symmetric. Accordingly, all vector fields in the biorthogonal basis in $\ker \mathscr{M}'^{*}$ are antisymmetric sets, except for the symmetric set $\mathbf{S}_5^*$ (this neutral mode exists only if the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ is unsteady).

9.3 Solvability of Auxiliary Problems

Consider a system of equations

$$\mathscr{M}'(\boldsymbol{\omega}, \mathbf{h}, \theta) = (\mathbf{f}^{\omega}, \mathbf{f}^{h}, f^{\theta}), \tag{9.6}$$

$$\nabla_{\mathbf{x}} \cdot \boldsymbol{\omega} = \nabla_{\mathbf{x}} \cdot \mathbf{h} = 0, \tag{9.7}$$

$$\langle \boldsymbol{\omega} \rangle_v = 0, \tag{9.8}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{f}^{\omega} = \nabla_{\mathbf{x}} \cdot \mathbf{f}^{h} = 0. \tag{9.9}$$

It has a solution only as long as the following solvability conditions are satisfied:

$$\langle \mathbf{f}^{\omega} \rangle_v = 0, \tag{9.10}$$

$$\langle\!\langle \mathbf{f}^{h} \rangle\!\rangle_h = 0, \tag{9.11}$$

$$\langle\!\langle \mathbf{f} \cdot \mathbf{S}_k^* \rangle\!\rangle = 0 \tag{9.12}$$

for $k = 1, 2, 5$. Condition (9.11) is identical to the relations (9.12) for $k = 3, 4$. (For steady CHM regimes $\mathbf{V}, \mathbf{H}, \Theta$, the condition (9.12) for $k = 5$ is not considered). In view of the identity (8.42) for the image of the operator of linearisation, condition (9.10) guarantees that (9.8) is satisfied.

A demonstration of solvability of the problem (9.6)–(9.9), when relations (9.10)–(9.12) hold, under the additional condition of periodicity in horizontal directions and in time, is based on application of the Fredholm alternative theorem; it does not differ significantly from the proof for forced convection (see Sect. 8.2.2). If the periodicity conditions are not imposed, then, as in the case of forced convection, conditions (9.10)–(9.12) do not guarantee existence of a globally bounded solution to the problem (9.6)–(9.9). Nevertheless, we do not demand the solution to be periodic, since this would be too restrictive; instead, we assume that all auxiliary problems (we will render them in the form (9.6)–(9.9)) have globally bounded solutions provided conditions (9.10)–(9.12) are satisfied.

Following this discussion, we can formulate a plan for solving the system of equations (8.29)–(8.31) obtained at order ε^n:

1°. Considering the spatial mean of the vertical vorticity component of equation (9.6) obtained at order ε^{n+1}, derive a PDE in the slow variables in $\langle\!\langle \boldsymbol{\omega}_{n-1} \rangle\!\rangle_v$; apply relation (8.33) to calculate the fluctuating (in the fast time) part of $\langle \mathbf{v}_n \rangle_h$.

2°. Considering the spatial mean of the horizontal magnetic component of the equation (9.6) obtained at order ε^n, derive a PDE in slow variables in $\langle\!\langle \mathbf{h}_{n-2} \rangle\!\rangle_h$ and calculate the fluctuating (in the fast time) part of $\langle \mathbf{h}_n \rangle_h$.
3°. From the expressions for divergencies (8.25)–(8.28) find the potential parts of $\{\boldsymbol{\omega}_n\}_v$, $\{\mathbf{v}_n\}_h$ and $\mathbf{h}_n$, and render the system of equations (8.29)–(8.31) and the supplementary conditions in the form of the problem (9.6)–(9.9).
4°. Calculate $\langle\!\langle \mathbf{v}_n \rangle\!\rangle$, applying the solvability condition (9.12) for $k = 1, 2$.
5°. The solvability condition (9.10) is now taken care of at step 1°, (9.11) at step 2°, and (9.12) for $k = 1, 2$ at step 5°. Apply the remaining solvability condition (9.12) for $k = 5$, if $K = 5$.
6°. From the system of PDE's derived at step 3° find expressions for $\boldsymbol{\omega}_n, \mathbf{v}_n, \mathbf{h}_n$ and θ_n in the terms of solutions to auxiliary problems. This terminates solution of the system (8.29)–(8.31) at hand, and one proceeds by solving the system obtained at the next order ε^{n+1}.

For $n = 0$ we have not followed the sequence of operations prescribed by this plan, and we must still implement the remaining step:

Step 4° for $n = 0$. We have shown in the previous chapter that

$$\langle \mathbf{v}_0 \rangle_h = \langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h \tag{9.13}$$

(see the derivation of (8.69), which is the same for free and forced convection), and hence (9.1) takes the form

$$\mathscr{M}'(\boldsymbol{\omega}_0, \mathbf{h}_0, \theta_0) = (\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h \cdot \nabla_{\mathbf{x}})(\boldsymbol{\Omega}, \mathbf{H}, \Theta).$$

Scalar multiplication of this equation by $\mathbf{S}_j^*$ and averaging of the result over the fast variables yields

$$\left\langle \mathbf{S}_k^{*} \cdot \left((\langle\!\langle \mathbf{v}_0 \rangle\!\rangle_h \cdot \nabla_{\mathbf{x}})(\boldsymbol{\Omega}, \mathbf{H}, \Theta) \right) \right\rangle = 0.$$

Now biorthogonality (9.4) of the fields $\mathbf{S}_k$ and $\mathbf{S}_j^*$ for $k, j = 1, 2$, and equations (9.13) and (8.33) for $n = 1$ imply

$$\langle \mathbf{v}_0 \rangle_h = \langle \boldsymbol{\omega}_1 \rangle_v = 0. \tag{9.14}$$

This is in contrast with the case of forced convection, where a mean flow perturbation is possible in the leading order (see also a discussion in Sect. 9.8).

9.4 The Combined α-Effect

In this section, we derive the operators describing the α-effects in the free CHM regime; the derivation is identical to the one in the case of forced convection presented in Chap. 8, with the sole exception that the structure of the leading term in the expansion of the perturbation is now determined by expression (9.5).

Step 1° for $n = 1$. Equation (8.90) for the evolution of $\langle \boldsymbol{\omega}_2 \rangle_v$ in the fast time, derived in the previous chapter, remains true in the case of free convection. Substituting into it the flow velocity and magnetic field components of expressions (9.5), we obtain

$$\frac{\partial \langle \boldsymbol{\omega}_2 \rangle_v}{\partial t} = \nabla_{\mathbf{X}} \times \left(\sum_{k=1}^{K} \sum_{m=1}^{2} \boldsymbol{\alpha}^{\omega}_{mk} \frac{\partial c_{0k}}{\partial X_m} + \widetilde{\boldsymbol{\xi}}^{\omega} \right), \tag{9.15}$$

where

$$\boldsymbol{\alpha}^{\omega}_{mk} = \langle \mathbf{S}^h_k H_m + \mathbf{H}(S^h_k)_m - \mathbf{S}^v_k V_m - \mathbf{V}(S^v_k)_m \rangle_h, \tag{9.16}$$

and the evolution of the transient $\widetilde{\boldsymbol{\xi}}^{\omega}$ is governed by equation (8.98). The operator

$$\sum_{k=1}^{K} \sum_{m=1}^{2} \langle\!\langle \boldsymbol{\alpha}^{\omega}_{mk} \rangle\!\rangle \frac{\partial c_{0k}}{\partial X_m}, \tag{9.17}$$

emerging in the r.h.s. of (9.15) upon averaging over the fast time, describes the *combined AKA-effect*. Integrating (9.15) in the fast time, we find

$$\langle \boldsymbol{\omega}_2 \rangle_v = \langle\!\langle \boldsymbol{\omega}_2 \rangle\!\rangle_v + \nabla_{\mathbf{X}} \times \left(\sum_{k=1}^{K} \sum_{m=1}^{2} \left\{\!\!\left\{ \int_0^t \boldsymbol{\alpha}^{\omega}_{mk} \mathrm{d}t \right\}\!\!\right\} \frac{\partial c_{0k}}{\partial X_m} + \left\{\!\!\left\{ \int_0^t \widetilde{\boldsymbol{\xi}}^{\omega} \mathrm{d}t \right\}\!\!\right\} \right). \tag{9.18}$$

Thus, $\langle \boldsymbol{\omega}_2 \rangle_v$ is well-defined if all mean values $\langle\!\langle \int_0^t \boldsymbol{\alpha}^{\omega}_{mk} dt \rangle\!\rangle$ are. This is the condition of *insignificance of the AKA-effect* in the leading order, which implies

$$\langle\!\langle \boldsymbol{\alpha}^{\omega}_{mk} \rangle\!\rangle = 0. \tag{9.19}$$

"Uncurling" equation (8.33) for $n = 2$ for the l.h.s. (9.18) and taking into account the solenoidality condition for the flow velocity, (8.24) for $n = 1$, we obtain

$$\langle \mathbf{v}_1 \rangle_h = \langle\!\langle \mathbf{v}_1 \rangle\!\rangle_h + \sum_{k=1}^{K} \sum_{m=1}^{2} \sum_{j=1}^{2} \left\{\!\!\left\{ \int_0^t (\alpha^{\omega}_{mk})_j \mathrm{d}t \right\}\!\!\right\} \left(\frac{\partial c_{0k}}{\partial X_m} \mathbf{e}_j - \nabla_{\mathbf{X}} \frac{\partial^2 \nabla^{-2}_{\mathbf{X}} c_{0k}}{\partial X_m \partial X_j} \right) + \widetilde{\boldsymbol{\xi}}^{v}, \tag{9.20}$$

where $\widetilde{\boldsymbol{\xi}}^{v}$ is determined by (8.106).

Step 2° for $n = 1$. We average the horizontal component of the equation (8.30) for $n = 1$ over the fast spatial variables and substitute the flow velocity and magnetic field components of (9.5). This yields

$$\frac{\partial\langle\mathbf{h}_1\rangle_h}{\partial t}=\nabla_{\mathbf{X}}\times\left(\sum_{k=1}^{K}\boldsymbol{\alpha}_k^h c_{0k}+\widetilde{\boldsymbol{\xi}}^h\right),\tag{9.21}$$

where

$$\boldsymbol{\alpha}_k^h=\langle\mathbf{V}\times\mathbf{S}_k^h+\mathbf{S}_k^v\times\mathbf{H}\rangle_v,\tag{9.22}$$

and the evolution of the transient $\widetilde{\boldsymbol{\xi}}^h$ is governed by Eq. 8.87. The operator

$$\nabla_{\mathbf{X}}\times\sum_{k=1}^{K}\langle\!\langle\boldsymbol{\alpha}_k^h\rangle\!\rangle c_{0k},\tag{9.23}$$

appearing in the r.h.s. of (9.21) upon averaging over the fast time, describes the *combined magnetic* α*-effect*. Integrating (9.21) in the fast time, we obtain

$$\langle\mathbf{h}_1\rangle_h=\langle\!\langle\mathbf{h}_1\rangle\!\rangle_h+\nabla_{\mathbf{X}}\times\left(\sum_{k=1}^{K}\left\{\!\!\left\{\int_0^t\boldsymbol{\alpha}_k^h\mathrm{d}t\right\}\!\!\right\}c_{0k}+\left\{\!\!\left\{\int_0^t\widetilde{\boldsymbol{\xi}}^h\mathrm{d}t\right\}\!\!\right\}\right).\tag{9.24}$$

The *magnetic* α*-effect is insignificant* in the leading order, if the means $\left\langle\int_0^t\boldsymbol{\alpha}_k^h dt\right\rangle$ are well-defined, expression (9.24) for the spatial mean $\langle\mathbf{h}_1\rangle_h$ thus being correctly defined. Insignificance of the magnetic α-effect implies

$$\left\langle\boldsymbol{\alpha}_k^h\right\rangle=0.\tag{9.25}$$

Relations (9.19) and (9.25) hold true for $k=1,2,5$, since $\boldsymbol{\alpha}_{mk}^{\omega}=\boldsymbol{\alpha}_k^h=0$ for $k=1,2$, and

$$\boldsymbol{\alpha}_{m5}^{\omega}=\frac{\partial}{\partial t}\langle H_m\mathbf{H}-V_m\mathbf{V}\rangle_h,\quad\boldsymbol{\alpha}_5^h=\frac{\partial}{\partial t}\langle\mathbf{V}\times\mathbf{H}\rangle_v$$

(recall the definitions $\mathbf{S}_k\equiv\mathbf{S}_k^{x\cdot}$ for $k=1,2$ and $\mathbf{S}_5\equiv\mathbf{S}_k^{t\cdot}$). For this reason, in the present context it is more appropriate to call (9.17) and (9.23) the operators of magnetic contribution to the AKA-effect and magnetic α-effect, respectively. If the CHM regime $\mathbf{V},\mathbf{H},\Theta$ is steady or time-periodic, then the $4K$ scalar equations comprising (9.19) imply insignificance of the AKA-effect, and K scalar equations comprising (9.25) imply insignificance of the magnetic α-effect.

If the CHM regime $\mathbf{V},\mathbf{H},\Theta$ has a symmetry considered in Sect. 8.4, the conditions (9.19) and (9.25) of insignificance of the α-effects hold for all k. Furthermore, if the symmetry is spatial (i.e., the time shift $\widetilde{T}$ is zero), then $\boldsymbol{\alpha}=0$ for all $k\le 4$; if it is spatio-temporal, then $\boldsymbol{\alpha}(t+\widetilde{T})=-\boldsymbol{\alpha}(t)$ for $k\le 4$ (the relations hold for each set of the super- and subscripts), and $\left\{\!\!\left\{\int_0^t\boldsymbol{\alpha}\,\mathrm{d}t\right\}\!\!\right\}$ satisfies an analogous relation (see Sect. 8.4); the fields $\boldsymbol{\alpha}$ for $k=5$ are $\widetilde{T}$-periodic in the fast time, and hence $\left\{\!\!\left\{\int_0^t\boldsymbol{\alpha}\,\mathrm{d}t\right\}\!\!\right\}$ are $\widetilde{T}$-periodic as well.

The terms $\left\{\!\!\left\{\int_0^t \widetilde{\xi}\,\mathrm{d}t\right\}\!\!\right\}$ in expressions (9.18) for $\langle\boldsymbol{\omega}_2\rangle_v$ and (9.24) for $\langle\mathbf{h}_1\rangle_h$ are not problematic, because $\widetilde{\xi}^\omega$ and $\widetilde{\xi}^h$ decay exponentially, and hence the means $\left\langle\!\!\left\langle\int_0^t \widetilde{\xi}\,\mathrm{d}t\right\rangle\!\!\right\rangle$ are correctly defined. The quantities $\left\{\!\!\left\{\int_0^t \widetilde{\xi}\,\mathrm{d}t\right\}\!\!\right\}$ and $\widetilde{\xi}^v$ decay exponentially as well (see Sect. 8.5.1).

As in the case of forced convection, one expects the technique of homogenisation for multiscale systems to remain applicable, when some of the conditions of insignificance of the α-effect, (9.19) or (9.25), are not satisfied, the appropriate scaling of the slow time being $T=\varepsilon t$ [82]. For this scaling, a new term, $\partial\langle\mathbf{h}_0\rangle_h/\partial T$, emerges in the l.h.s. of equation (9.21). (Due to relation (9.14), a similar new term $\partial\langle\boldsymbol{\omega}_1\rangle_v/\partial T$ does not emerge in (9.14).) The system of *amplitude equations* then involves the modified equations (9.15) and (9.21), averaged over the fast time, where the l.h.s. are set to zero and $\partial\langle\mathbf{h}_0\rangle_h/\partial T$, respectively. The two equations are supplemented by the solenoidality conditions (8.24) for the mean magnetic field perturbation (for $n=0$) and for the mean flow velocity (9.33) (for $n=1$), and, if the CHM regime $\mathbf{V},\mathbf{H},\Theta$ is unsteady, by equation (9.34). In contrast with the case of forced convection, this system is mixed (only the equation for the mean magnetic perturbation is evolutionary) and nonlinear (the mean flow velocity (9.33) is quadratic in amplitudes c_{0k}, as is equation (9.34) ((9.33) and (9.34), derived in the next section, remain unaltered for the new scaling of the slow time). Furthermore, since relations (9.19) and (9.25) hold true for $k=1,2,5$, the modified equations (9.15) and (9.21) generically constitute an overdetermined subsystem yielding a solution $c_{03}=c_{04}=0$ (i.e., $\langle\mathbf{h}_0\rangle_h=0$). This indicates that averaging of equations (9.15) and (9.21) over the fast time cannot be performed, i.e., the large-scale dynamics of the perturbations of free CHM regimes, involving a non-zero magnetic component, in the presence of the significant α-effect cannot be separated out. The system of the remaining equations is underdetermined.

9.5 Amplitude Equations in the Absence of Significant α-Effect

In this section we construct a solution to the second system in the hierarchy (Eqs. 8.29–8.31 for $n=1$) assuming that the α-effect is insignificant in the leading order.

9.5.1 Solvability Conditions for Order ε^1 Equations

Step 3° for $n=1$. In order to apply the solvability conditions (9.10)–(9.12), the equations (8.29)–(8.31) for $n=1$ must be rendered in the form of the problem (9.6)–(9.9), separating out the potential parts of $\boldsymbol{\omega}_1$, $\mathbf{v}_1$ and $\mathbf{h}_1$, and the spatial mean $\langle\mathbf{v}_1\rangle_h$ (the mean $\langle\boldsymbol{\omega}_1\rangle_v$ vanishes by virtue of the relations (9.14). The potential parts can be calculated applying expressions (9.5) for the leading terms in the power series expansion of the perturbation, relations for divergencies (8.25), (8.26) and (8.28) for $n = 1$, and recalling the identity $\nabla \times \mathbf{S}_k^v = \mathbf{S}_k^\omega$:

$$\boldsymbol{\omega}_1 = \sum_{k=1}^{K}\sum_{m=1}^{2} \frac{\partial c_{0k}}{\partial X_m} \mathbf{e}_m \times \mathbf{S}_k^v + \boldsymbol{\omega}_1', \tag{9.26}$$

$$\mathbf{v}_1 = \langle \mathbf{v}_1 \rangle_h + \sum_{k=1}^{K}\sum_{m=1}^{2} \frac{\partial c_{0k}}{\partial X_m} \nabla_{\mathbf{x}} s_{mk}^v + \mathbf{v}_1', \tag{9.27}$$

$$\mathbf{h}_1 = \sum_{k=1}^{K}\sum_{m=1}^{2} \frac{\partial c_{0k}}{\partial X_m} \nabla_{\mathbf{x}} s_{mk}^h + \mathbf{h}'_1, \tag{9.28}$$

where

$$\nabla_{\mathbf{x}} \cdot \boldsymbol{\omega}_1' = \nabla_{\mathbf{x}} \cdot \mathbf{v}_1' = \nabla_{\mathbf{x}} \cdot \mathbf{h}_1' = 0, \quad \langle \boldsymbol{\omega}_1' \rangle_v = \langle \mathbf{v}_1' \rangle_h = 0, \quad \nabla_{\mathbf{x}} \times \mathbf{v}_1' = \boldsymbol{\omega}_1' \tag{9.29}$$

and $s(\mathbf{x}, t)$ are globally bounded solutions to the Neumann problems

$$\nabla_{\mathbf{x}}^2 s_{mk}^v = -(S_k^v)_m, \qquad \left.\frac{\partial s_{mk}^v}{\partial x_3}\right|_{x_3 = \pm L/2} = 0;$$

$$\nabla_{\mathbf{x}}^2 s_{mk}^h = -\{S_k^h\}_m, \qquad \left.\frac{\partial s_{mk}^h}{\partial x_3}\right|_{x_3 = \pm L/2} = 0.$$

The fields $\boldsymbol{\omega}_1'$, $\mathbf{v}'_1$ and $\mathbf{h}'_1$ satisfy the same boundary conditions as $\boldsymbol{\omega}_1$, $\mathbf{v}_1$ and $\mathbf{h}_1$, respectively (see (8.5)–(8.8)).

Substituting expressions (9.26)–(9.28) for the second term in the expansion of the perturbation, we transform Eqs. 8.29–8.31 for $n = 1$, which become as follows:

$$\begin{aligned} \mathscr{M}'^{\omega}(\boldsymbol{\omega}_1', \mathbf{h}'_1, \theta_1) = &- \sum_{k=1}^{K}\sum_{m=1}^{2} \mathscr{L}^{\omega}(\mathbf{e}_m \times \mathbf{S}_k^v, \nabla_{\mathbf{x}} s_{mk}^v, \nabla_{\mathbf{x}} s_{mk}^h, 0) \frac{\partial c_{0k}}{m} + (\langle \mathbf{v}_1 \rangle_h \cdot \nabla_{\mathbf{x}})\boldsymbol{\Omega} \\ &- 2\nu(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\boldsymbol{\omega}_0 - \nabla_{\mathbf{X}} \times (\mathbf{V} \times \boldsymbol{\omega}_0 + \mathbf{v}_0 \times \boldsymbol{\Omega} - \mathbf{H} \times (\nabla_{\mathbf{x}} \times \mathbf{h}_0) \\ &- \mathbf{h}_0 \times (\nabla_{\mathbf{x}} \times \mathbf{H})) - \nabla_{\mathbf{x}} \times (-\mathbf{H} \times (\nabla_{\mathbf{X}} \times \mathbf{h}_0) \\ &+ \mathbf{v}_0 \times \boldsymbol{\omega}_0 - \mathbf{h}_0 \times (\nabla_{\mathbf{x}} \times \mathbf{h}_0)) - \beta \nabla_{\mathbf{X}} \theta_0 \times \mathbf{e}_3; \end{aligned} \tag{9.30}$$

$$\begin{aligned} \mathscr{M}'^{h}(\boldsymbol{\omega}_1', \mathbf{h}'_1) = &- \sum_{k=1}^{K}\sum_{m=1}^{2} \mathscr{L}^{h}(\nabla_{\mathbf{x}} s_{mk}^v, \nabla_{\mathbf{x}} s_{mk}^h) \frac{\partial c_{0k}}{\partial X_m} + (\langle \mathbf{v}_1 \rangle_h \cdot \nabla_{\mathbf{x}})\mathbf{H} \\ &- 2\eta(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\mathbf{h}_0 - \nabla_{\mathbf{X}} \times (\mathbf{v}_0 \times \mathbf{H} + \mathbf{V} \times \mathbf{h}_0) - \nabla_{\mathbf{x}} \times (\mathbf{v}_0 \times \mathbf{h}_0); \end{aligned} \tag{9.31}$$

$$\mathscr{M}'^{\theta}(\boldsymbol{\omega}'_1,\theta_1) = -\sum_{k=1}^{K}\sum_{m=1}^{2}\mathscr{L}^{\theta}(\nabla_{\mathbf{x}}s^{v}_{mk},0)\frac{\partial c_{0k}}{\partial X_m} + (\langle\mathbf{v}_1\rangle_h\cdot\nabla_{\mathbf{x}})\Theta$$
$$- 2\kappa(\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{X}})\theta_0 + (\mathbf{V}\cdot\nabla_{\mathbf{X}})\theta_0 + (\mathbf{v}_0\cdot\nabla_{\mathbf{x}})\theta_0. \tag{9.32}$$

Step 4° for $n = 1$. Scalar multiplying equations (9.30)–(9.32) by $\mathbf{S}^*_j$, $j = 1, 2$, and using the biorthogonality and normalisation conditions (9.4), the identity

$$-\frac{\partial\mathbf{S}^v_k}{\partial t} + \nu\nabla^2\mathbf{S}^v_k + \mathbf{V}\times\mathbf{S}^{\omega}_k + \mathbf{S}^v_k\times\boldsymbol{\Omega} - \mathbf{H}\times(\nabla\times\mathbf{S}^h_k) - \mathbf{S}^h_k\times(\nabla\times\mathbf{H})$$
$$= -\tau\mathbf{S}^v_k\times\mathbf{e}_3 - \beta S^{\theta}_k\mathbf{e}_3 + \nabla S^p_k,$$

(obtained by "uncurling" the vorticity component of Eq. 9.1), and expressions (9.5) and (9.20), we obtain

$$\langle\!\langle\mathbf{v}_1\rangle\!\rangle_h = \sum_{k=1}^{K}\left(\sum_{m=1}^{2}\left(\boldsymbol{\beta}_{1mk}\frac{\partial c_{0k}}{\partial X_m} + \sum_{j=1}^{2}\sum_{i=1}^{j}\boldsymbol{\beta}_{3ijmk}\frac{\partial^3\nabla_{\mathbf{X}}^{-2}c_{0k}}{\partial X_m\partial X_j\partial X_i}\right) + \sum_{m=1}^{k}\boldsymbol{\beta}_{2mk}c_{0m}c_{0k}\right). \tag{9.33}$$

Here

$$\boldsymbol{\beta}_{\cdot} \equiv (\langle\!\langle\mathbf{E}_{\cdot}\cdot\mathbf{S}^*_1\rangle\!\rangle, \langle\!\langle\mathbf{E}_{\cdot}\cdot\mathbf{S}^*_2\rangle\!\rangle, 0)$$

are constant three-dimensional vectors,

$$\mathbf{E}_{1mk} \equiv \mathscr{L}(0,\nabla s^v_{mk},\nabla s^h_{mk},0) - \left(\left\{\!\!\left\{\int_0^t \boldsymbol{\alpha}^{\omega}_{mk}\,\mathrm{d}t\right\}\!\!\right\}\cdot\nabla\right)(\boldsymbol{\Omega},\mathbf{H},\Theta)$$
$$+\left(2\nu\frac{\partial\mathbf{S}^{\omega}_k}{\partial x_m} + \nabla\times\left(\mathbf{V}\times(\mathbf{e}_m\times\mathbf{S}^v_k) - \mathbf{H}\times(\mathbf{e}_m\times\mathbf{S}^h_k)\right)\right.$$
$$\left.+\mathbf{e}_m\times\nabla S^p_k + \tau(S^v_k)_m\mathbf{e}_3, 2\eta\frac{\partial\mathbf{S}^h_k}{\partial x_m} + \mathbf{e}_m\times(\mathbf{V}\times\mathbf{S}^h_k + \mathbf{S}^v_k\times\mathbf{H}), 2\kappa\frac{\partial S^{\theta}_k}{\partial x_m} - V_m S^{\theta}_k\right);$$
$$\mathbf{E}_{2mk} \equiv \rho_{mk}\left(\nabla\times\left(\mathbf{S}^v_k\times\mathbf{S}^{\omega}_m - \mathbf{S}^h_k\times(\nabla\times\mathbf{S}^h_m) + \mathbf{S}^v_m\times\mathbf{S}^{\omega}_k - \mathbf{S}^h_m\times(\nabla\times\mathbf{S}^h_k)\right),\right.$$
$$\left.\nabla\times(\mathbf{S}^v_k\times\mathbf{S}^h_m + \mathbf{S}^v_m\times\mathbf{S}^h_k), -(\mathbf{S}^v_k\cdot\nabla)S^{\theta}_m - (\mathbf{S}^v_m\cdot\nabla)S^{\theta}_k\right);$$
$$\mathbf{E}_{3ijmk} \equiv \rho_{ij}\left(\left(\left\{\!\!\left\{\int_0^t(\boldsymbol{\alpha}^{\omega}_{mk})_j\,\mathrm{d}t\right\}\!\!\right\}\mathbf{e}_i + \left\{\!\!\left\{\int_0^t(\boldsymbol{\alpha}^{\omega}_{mk})_i\,\mathrm{d}t\right\}\!\!\right\}\mathbf{e}_j\right)\cdot\nabla\right)(\boldsymbol{\Omega},\mathbf{H},\Theta),$$

$\rho_{mk} \equiv 1$ for $m<k$, $\rho_{mk} \equiv 1/2$ for $m = k$, and $\rho_{mk} \equiv 0$ for $m > k$. If the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ possesses a symmetry considered in Sect. 8.4, then vector fields $\mathbf{E}$ for $k \le 4$ are symmetric and $\mathbf{S}^*_k$ for $k = 1, 2$ antisymmetric sets, respectively. Consequently, $\langle\!\langle\mathbf{v}_1\rangle\!\rangle_h = 0$ for symmetric steady states $\mathbf{V}, \mathbf{H}, \Theta$.

Step 5° for $n = 1$. If the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ is unsteady, then scalar multiplying (9.30)–(9.32) by $\mathbf{S}_5^*$ and using orthogonality of this field to $\mathbf{S}_k$ for $k = 1, 2$ (9.4), we obtain a non-evolutionary amplitude equation

$$\sum_{k=1}^{K}\left(\sum_{m=1}^{2}\left(\beta'_{1mk}\frac{\partial c_{0k}}{\partial X_m}+\sum_{j=1}^{2}\sum_{i=1}^{j}\beta'_{3ijmk}\frac{\partial^3\nabla_{\mathbf{X}}^{-2}c_{0k}}{\partial X_m\partial X_j\partial X_i}\right)+\sum_{m=1}^{k}\beta'_{2mk}c_{0m}c_{0k}\right)=0, \quad (9.34)$$

where

$$\beta'_{\cdot} = \langle\!\langle \mathbf{E}_{\cdot} \cdot \mathbf{S}_5^* \rangle\!\rangle.$$

9.5.2 Solution of Order ε^1 Equations

Step 6° for $n = 1$. Let $\widehat{\mathscr{P}}(\mathbf{a}^\omega, \mathbf{a}^h, a^\theta)$ denote the projection of a field $(\mathbf{a}^\omega, \mathbf{a}^h, a^\theta)$ into the subspace, orthogonal to $\ker \mathscr{M}'^*$. (If the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ is unsteady and has a symmetry considered in Sect. 8.4, then application of $\widehat{\mathscr{P}}$ amounts to subtraction of the component, proportional to $\mathbf{S}_5^*$, since in this case the r.h.s. of equations (9.30)–(9.32) is a symmetric set, and $\mathbf{S}_5^*$ is the only symmetric basic eigenfunction in $\ker \mathscr{M}'^*$). Noting expressions (9.5) for the leading terms in the power series expansion (8.20)–(8.23) and (9.20) for $\langle \mathbf{v}_1 \rangle_h$, we find that a solution to the problem (9.30)–(9.32) has the following structure:

$$(\boldsymbol{\omega}'_1, \mathbf{v}'_1, \mathbf{h}'_1, \theta_1) = \boldsymbol{\xi}_1 + \sum_{k=1}^{K}\left(\mathbf{S}_k c_{1k} + \sum_{m=1}^{2}\left(\mathbf{G}_{mk}\frac{\partial c_{0k}}{\partial X_m} + \sum_{j=1}^{2}\sum_{i=1}^{j}\mathbf{Y}_{ijmk}\frac{\partial^3\nabla_{\mathbf{X}}^{-2}c_{0k}}{\partial X_m\partial X_j\partial X_i}\right)\right.$$
$$\left. + \sum_{m=1}^{k}\mathbf{Q}_{mk}c_{0k}c_{0m}\right). \quad (9.35)$$

Here vector fields $\mathbf{G}_{mk}(\mathbf{x}, t) = (\mathbf{G}_{mk}^\omega, \mathbf{G}_{mk}^v, \mathbf{G}_{mk}^h, G_{mk}^\theta)$ are solutions to the auxiliary problems of type II:

$$\mathscr{M}'(\mathbf{G}_{mk}) = -\widehat{\mathscr{P}}\boldsymbol{E}_{1mk}, \quad (9.36)$$

$$\nabla_{\mathbf{x}} \times \mathbf{G}_{mk}^v = \mathbf{G}_{mk}^\omega, \quad (9.37)$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{G}_{mk}^\omega = \nabla_{\mathbf{x}} \cdot \mathbf{G}_{mk}^v = \nabla_{\mathbf{x}} \cdot \mathbf{G}_{mk}^h = 0. \quad (9.38)$$

Fields $\mathbf{Q}_{mk}(\mathbf{x}, t) = (\mathbf{Q}_{mk}^\omega, \mathbf{Q}_{mk}^v, \mathbf{Q}_{mk}^h, Q_{mk}^\theta)$ are solutions to the auxiliary problems of type III:

$$\mathscr{M}'^\omega(\mathbf{Q}_{mk}) = -\widehat{\mathscr{P}}\mathbf{E}_{2mk}, \quad (9.39)$$

$$\nabla_{\mathbf{x}} \times \mathbf{Q}^v_{mk} = \mathbf{Q}^\omega_{mk}, \tag{9.40}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{Q}^\omega_{mk} = \nabla_{\mathbf{x}} \cdot \mathbf{Q}^v_{mk} = \nabla_{\mathbf{x}} \cdot \mathbf{Q}^h_{mk} = 0 \tag{9.41}$$

($\mathbf{Q}_{mk} = 0$ for $m > k$).

Fields $\mathbf{Y}_{ijmk}(\mathbf{x}, t) = (\mathbf{Y}^\omega_{ijmk}, \mathbf{Y}^v_{ijmk}, \mathbf{Y}^h_{ijmk}, Y^\theta_{ijmk})$ are solutions to the <u>auxiliary problems of type IV</u>:

$$\mathscr{M}'(\mathbf{Y}_{ijmk}) = -\widehat{\mathscr{P}}\mathbf{E}_{3ijmk}, \tag{9.42}$$

$$\nabla_{\mathbf{x}} \times \mathbf{Y}^v_{ijmk} = \mathbf{Y}^\omega_{ijmk}, \tag{9.43}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{Y}^\omega_{ijmk} = \nabla_{\mathbf{x}} \cdot \mathbf{Y}^v_{ijmk} = \nabla_{\mathbf{x}} \cdot \mathbf{Y}^h_{ijmk} = 0 \tag{9.44}$$

($\mathbf{Y}_{ijmk} = 0$ for $i > j$).

Finally, $\xi_1(\mathbf{X}, T, \mathbf{x}, t) = (\xi^\omega_1, \xi^v_1, \xi^h_1, \xi^\theta_1)$ is a solution to the problem

$$\begin{aligned}\mathscr{L}^\omega(\xi^\omega_1, \xi^v_1, \xi^h_1, \xi^\theta_1) = {} & -2\nu(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\xi^\omega_0 - \nabla_{\mathbf{X}} \times \Big(\mathbf{V} \times \xi^\omega_0 + \xi^v_0 \times \mathbf{\Omega} \\ & - \mathbf{H} \times (\nabla_{\mathbf{x}} \times \xi^h_0) - \xi^h_0 \times (\nabla_{\mathbf{x}} \times \mathbf{H})\Big) \\ & - \nabla_{\mathbf{x}} \times \Big(\mathbf{v}_0 \times \xi^\omega_0 + \xi^v_0 \times (\boldsymbol{\omega}_0 - \xi^\omega_0) - \mathbf{H} \times (\nabla_{\mathbf{X}} \times \xi^h_0) \\ & - \mathbf{h}_0 \times (\nabla_{\mathbf{x}} \times \xi^h_0) - \xi^h_0 \times (\nabla_{\mathbf{x}} \times (\mathbf{h}_0 - \xi^h_0))\Big) - \beta\nabla_{\mathbf{X}}\xi^\theta_0 \times \mathbf{e}_3,\end{aligned} \tag{9.45}$$

$$\begin{aligned}\mathscr{L}^h(\xi^v_1, \xi^h_1) = {} & -2\eta(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\xi^h_1 - \nabla_{\mathbf{X}} \times (\xi^v_0 \times \mathbf{H} + \mathbf{V} \times \xi^h_0) \\ & - \nabla_{\mathbf{x}} \times (\mathbf{v}_0 \times \xi^h_0 + \xi^v_0 \times (\mathbf{h}_0 - \xi^h_0))\end{aligned} \tag{9.46}$$

$$\mathscr{L}^\theta(\xi^v_1, \xi^\theta_1) = -2\kappa(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\xi^\theta_0 + (\mathbf{V} \cdot \nabla_{\mathbf{X}})\xi^\theta_0 + (\mathbf{v}_0 \cdot \nabla_{\mathbf{x}})\xi^\theta_0 + (\xi^v_0 \cdot \nabla_{\mathbf{x}})(\theta_0 - \xi^\theta_0), \tag{9.47}$$

$$\nabla_{\mathbf{x}} \times \xi^v_1 - \xi^\omega_1 = -\nabla_{\mathbf{X}} \times \xi^v_0, \tag{9.48}$$

$$\nabla_{\mathbf{x}} \cdot \xi^\omega_1 = -\nabla_{\mathbf{X}} \cdot \xi^\omega_0, \qquad \nabla_{\mathbf{x}} \cdot \xi^v_1 = -\nabla_{\mathbf{X}} \cdot \xi^v_0, \qquad \nabla_{\mathbf{x}} \cdot \xi^h_1 = -\nabla_{\mathbf{X}} \cdot \xi^h_0. \tag{9.49}$$

The physical components of vector fields $\mathbf{G}_{mk}, \mathbf{Q}_{mk}, \mathbf{Y}_{ijmk}$ and ξ_1 satisfy the boundary conditions (8.5)–(8.7) and (8.11). The spatial means of the horizontal components of the flow velocity and the vertical components of the vorticity, as well as the spatio-temporal means of the horizontal components of the magnetic field must be zero.

Together, the solenoidality conditions (9.38), (9.41), (9.44), relations (9.49), and expressions (9.26)–(9.28) for the second term in the expansion of the perturbation are equivalent to the relations for divergencies (8.25), (8.26) and (8.28) for $n = 1$. It suffices to satisfy the solenoidality conditions for the vorticity and magnetic field at $t = 0$; for solutions to the auxiliary problems of types II–IV this can be easily shown taking the divergence of the equations for vorticity and magnetic field. Relations for vorticity (9.37), (9.40), (9.43) and (9.48) are equivalent to the equation for determination of $\{\mathbf{v}_1\}_h$, (8.34) for $n = 1$.

The spatial means for the initial conditions of the horizontal magnetic components can be found considering the means of the respective components of equations (9.36), (9.39) and (9.42):

$$\langle \mathbf{G}^h_{mk} \rangle_h \Big|_{t=0} = -\left\langle\!\!\left\langle \mathbf{e}_m \times \int_0^t \boldsymbol{\alpha}^h_k \mathrm{d}t \right\rangle\!\!\right\rangle_h, \qquad \langle \boldsymbol{\xi}^h_1 \rangle_h \Big|_{t=0} = -\left\langle\!\!\left\langle \int_0^t \nabla_{\mathbf{X}} \times \widetilde{\boldsymbol{\xi}}^h \mathrm{d}t \right\rangle\!\!\right\rangle_h$$

(the means exist by the assumption of insignificance of the magnetic α-effect and because $\widetilde{\boldsymbol{\xi}}^h$ decays exponentially), and at any $t \geq 0$

$$\langle \mathbf{Q}^h_{mk} \rangle_h = \langle \mathbf{Y}^h_{ijmk} \rangle_h = 0.$$

Averaging the horizontal magnetic component of the expressions (9.35) and (9.28) for the second term in the expansion of the perturbation, we find

$$\langle\!\langle \mathbf{h}_1 \rangle\!\rangle_h \Big|_{T=0} = \langle \mathbf{h}_1 \rangle_h \Big|_{t=0} - \langle \boldsymbol{\xi}^h_1 \rangle_h \Big|_{t=0} - \sum_{k=1}^{K} \sum_{m=1}^{2} \langle \mathbf{G}^h_{mk} \rangle_h \Big|_{t=0} \frac{\partial \langle\!\langle c_{0k} \rangle\!\rangle}{\partial X_m} \Big|_{T=0}.$$

Initial conditions for the mean magnetic field perturbation for the problem in the slow time are determined from this equation.

The choice of initial conditions must guarantee the global boundedness of solutions to the auxiliary problems and their derivatives. As shown in Sect. 7.4.2, the boundedness is ensured, if the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ is space-periodic and stable to short-scale perturbations. Variation of initial conditions for $\mathbf{G}_{mk}$, $\mathbf{Q}_{mk}$ and $\mathbf{Y}_{ijmk}$ causes a respective modification of initial conditions for $\boldsymbol{\xi}_1$, but the resultant changes in all the fields must decay exponentially in the fast time.

Since $\boldsymbol{\xi}_0$ and its derivatives decay exponentially in the fast time, the same holds true for the r.h.s. of Eqs. 9.45–9.49. If the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ is linearly stable to short-scale perturbations, this implies an exponential decay of $\boldsymbol{\xi}_1$ and of any changes in $\mathbf{G}_{mk}$, $\mathbf{Q}_{mk}$ and $\mathbf{Y}_{ijmk}$ following a variation of initial conditions for $\mathbf{S}_k$ (see Sect. 7.4.2). Alternatively, an exponential decay of $\boldsymbol{\xi}_1$ is a constraint which we impose on initial conditions for $\boldsymbol{\xi}_1$.

If the CHM regime is steady or periodic in time, and periodic in horizontal directions, solutions to the auxiliary problems $\mathbf{S}_k$, $\mathbf{G}_{mk}$, $\mathbf{Q}_{mk}$ and $\mathbf{Y}_{ijmk}$ with the same properties and the same periods (see Sect. 8.2.2) exist for all non-exclusive sets of parameter values. If the regime possesses a symmetry considered in Sect. 8.2.2, then $\mathbf{S}_k$ are antisymmetric sets of fields, except for $\mathbf{S}_5$ for an unsteady regime. Consequently, $\mathbf{G}_{mk}$, $\mathbf{Q}_{mk}$ and $\mathbf{Y}_{ijmk}$ are then symmetric sets for $k < 5$ (an antisymmetric part can be present if the initial conditions are not symmetric, but by construction it decays exponentially for any permissible initial conditions, and hence we neglect it), and antisymmetric sets for $k = 5$. If the CHM regime is steady or possesses a symmetry without a time shift, the r.h.s. of equations in the statement of the auxiliary problems of type IV vanish, and hence $\mathbf{Y}_{ijmk} = 0$.

9.6 Solvability Conditions for Order ε^2 and ε^3 Equations

Combining expressions (9.20), (9.27) and (9.35) for the mean and fluctuating parts of the second term in the expansion of the perturbation of the flow, we find

$$\mathbf{v}_1 = \boldsymbol{\xi}_1^v + \sum_{k=1}^{K}\left(\mathbf{S}_k^v c_{1k} + \sum_{m=1}^{2}\left(\widetilde{\mathbf{G}}^v_{mk}\frac{\partial c_{0k}}{\partial X_m} + \sum_{j=1}^{2}\sum_{i=1}^{j}\widetilde{\mathbf{Y}}^v_{ijmk}\frac{\partial^3\nabla_{\mathbf{X}}^{-2}c_{0k}}{\partial X_m\partial X_j\partial X_i}\right)\right.$$
$$\left. + \sum_{m=1}^{k}\widetilde{\mathbf{Q}}^v_{mk}c_{0k}c_{0m}\right), \qquad (9.50)$$

where

$$\widetilde{\mathbf{G}}^v_{mk} = \mathbf{G}^v_{mk} + \boldsymbol{\beta}_{1mk} + \left\{\!\!\left\{\int_0^t \boldsymbol{\alpha}^{\omega}_{mk}\,\mathrm{d}t\right\}\!\!\right\} + \nabla_{\mathbf{x}}s^v_{mk},$$

$$\widetilde{\mathbf{Q}}^v_{mk} = \mathbf{Q}^v_{mk} + \boldsymbol{\beta}_{2mk},$$

$$\widetilde{\mathbf{Y}}^v_{ijmk} = \mathbf{Y}^v_{ijmk} + \rho_{ij}\left(\boldsymbol{\beta}_{3ijmk} - \left\{\!\!\left\{\int_0^t (\boldsymbol{\alpha}^{\omega}_{mk})_j\,\mathrm{d}t\right\}\!\!\right\}\mathbf{e}_i + \boldsymbol{\beta}_{3jimk} - \left\{\!\!\left\{\int_0^t (\boldsymbol{\alpha}^{\omega}_{mk})_i\,\mathrm{d}t\right\}\!\!\right\}\mathbf{e}_j\right)$$

We obtain similarly

$$\mathbf{h}_1 = \boldsymbol{\xi}_1^h + \sum_{k=1}^{K}\left(\mathbf{S}_k^h c_{1k} + \sum_{m=1}^{2}\left(\widetilde{\mathbf{G}}^h_{mk}\frac{\partial c_{0k}}{\partial X_m} + \sum_{j=1}^{2}\sum_{i=1}^{j}\mathbf{Y}^h_{ijmk}\frac{\partial^3\nabla_{\mathbf{X}}^{-2}c_{0k}}{\partial X_m\partial X_j\partial X_i}\right)\right.$$
$$\left. + \sum_{m=1}^{k}\mathbf{Q}^h_{mk}c_{0k}c_{0m}\right), \qquad (9.51)$$

where $\widetilde{\mathbf{G}}^h_{mk} = \mathbf{G}^h_{mk} + \nabla_{\mathbf{x}}s^h_{mk}$.

Step 1° for $n = 2$. Averaging the vertical component of the equation (8.29) for $n = 3$ over the fast spatial variables and noting for $n = 0, 1, 2$ the relations (8.24)– (8.26) for the divergencies and (8.32) for the vorticity, we find

$$-\frac{\partial\langle\boldsymbol{\omega}_3\rangle_v}{\partial t} - \frac{\partial\langle\boldsymbol{\omega}_1\rangle_v}{\partial T} + \nu\nabla^2_{\mathbf{X}}\langle\boldsymbol{\omega}_1\rangle_v$$
$$+\nabla_{\mathbf{X}}\times\Big\langle\mathbf{V}\times(\nabla_{\mathbf{X}}\times\mathbf{v}_1) - \mathbf{V}\nabla_{\mathbf{X}}\cdot\{\mathbf{v}_1\}_h \quad - \mathbf{H}\times(\nabla_{\mathbf{X}}\times\mathbf{h}_1) + \mathbf{H}\nabla_{\mathbf{X}}\cdot\{\mathbf{h}_1\}_h$$
$$+\mathbf{v}_0\times(\nabla_{\mathbf{X}}\times\mathbf{v}_0) - \mathbf{v}_0\nabla_{\mathbf{X}}\cdot\{\mathbf{v}_0\}_h - \mathbf{h}_0\times(\nabla_{\mathbf{X}}\times\mathbf{h}_0) + \mathbf{h}_0\nabla_{\mathbf{X}}\cdot\{\mathbf{h}_0\}_h\Big\rangle_h = 0.$$

We average this equation over the fast time, substitute $\langle \boldsymbol{\omega}_1 \rangle_v = 0$ (see (9.14)), relations (9.50) and (9.51), and the expression (9.5) for the leading term of the flow velocity and magnetic perturbation in the power series expansion, employ the relation (9.19) expressing the absence of the AKA-effect, and recall that ξ_0 and ξ_1 are exponentially decaying transients. The calculations yield a non-evolutionary amplitude equation

$$\nabla_{\mathbf{X}} \times \sum_{k=1}^{K} \sum_{n=1}^{2} \frac{\partial}{\partial X_n} \left(\sum_{m=1}^{2} \frac{\partial}{\partial X_m} \left(\mathbf{D}^v_{nmk} c_{0k} + \sum_{j=1}^{2} \sum_{i=1}^{j} \mathbf{d}^v_{nijmk} \frac{\partial^2 \nabla_{\mathbf{X}}^{-2} c_{0k}}{\partial X_i \partial X_j} \right) + \sum_{m=1}^{k} \mathbf{A}^v_{nmk} c_{0m} c_{0k} \right) = 0, \tag{9.52}$$

where

$$\begin{aligned}
\mathbf{D}^v_{nmk} &= \left\langle\!\!\left\langle -V_n \widetilde{\mathbf{G}}^v_{mk} - \mathbf{V}(\widetilde{G}^v_{mk})_n + H_n \widetilde{\mathbf{G}}^h_{mk} + \mathbf{H}(\widetilde{G}^h_{mk})_n \right\rangle\!\!\right\rangle_h, \\
\mathbf{d}^v_{nijmk} &= \left\langle\!\!\left\langle -V_n \widetilde{\mathbf{Y}}^v_{ijmk} - \mathbf{V}(\widetilde{Y}^v_{ijmk})_n + H_n \mathbf{Y}^h_{ijmk} + \mathbf{H}(Y^h_{ijmk})_n \right\rangle\!\!\right\rangle_h, \\
\mathbf{A}^v_{nmk,j} &= \left\langle\!\!\left\langle -V_n \widetilde{\mathbf{Q}}^v_{mk} - \mathbf{V}(\widetilde{Q}^v_{mk})_n + H_n \mathbf{Q}^h_{mk} + \mathbf{H}(Q^h_{mk})_n - (S^v_k)_n \mathbf{S}^v_m + (S^h_k)_n \mathbf{S}^h_m \right\rangle\!\!\right\rangle_h.
\end{aligned}$$

Step 2° for $n = 2$. We proceed by averaging the horizontal component of the equation for the evolution of magnetic field, (8.30) for $n = 2$, over the fast variables, substitute the expressions (9.5), (9.50) and (9.51) for the first two terms in the expansion of the perturbations of the flow velocity and magnetic field, and use the expression (8.43) for the mean of the image of the magnetic part of the operator of linearisation. We obtain an equation for the mean magnetic field perturbation

$$-\frac{\partial}{\partial T} \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h + \eta \nabla^2_{\mathbf{X}} \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h + \nabla_{\mathbf{X}} \times \sum_{k=1}^{K} \left(\sum_{m=1}^{2} \frac{\partial}{\partial X_m} \left(\mathbf{D}^h_{mk} c_{0k} + \sum_{j=1}^{2} \sum_{i=1}^{j} \mathbf{d}^h_{ijmk} \frac{\partial^2}{\partial X_i \partial X_j} \nabla_{\mathbf{X}}^{-2} c_{0k} \right) . + \sum_{m=1}^{k} \mathbf{A}^h_{mk} c_{0k} c_{0m} \right) = 0 \tag{9.53}$$

where

$$\begin{aligned}
\mathbf{D}^h_{mk} &= \left\langle\!\!\left\langle \mathbf{V} \times \widetilde{\mathbf{G}}^h_{mk} - \mathbf{H} \times \widetilde{\mathbf{G}}^v_{mk} \right\rangle\!\!\right\rangle_v, \\
\mathbf{d}^h_{ijmk} &= \left\langle\!\!\left\langle \mathbf{V} \times \mathbf{Y}^h_{ijmk} - \mathbf{H} \times \widetilde{\mathbf{Y}}^v_{ijmk} \right\rangle\!\!\right\rangle_v, \\
\mathbf{A}^h_{mk} &= \left\langle\!\!\left\langle \mathbf{V} \times \mathbf{Q}^h_{mk} - \mathbf{H} \times \widetilde{\mathbf{Q}}^v_{mk} + \mathbf{S}^v_k \times \mathbf{S}^h_m \right\rangle\!\!\right\rangle_v.
\end{aligned}$$

In the amplitude equations (9.52) and (9.53), constant vectors $\mathbf{D}$ are coefficients of partial differential operators of the second order, which, as before, we interpret as operators describing anisotropic *combined eddy correction of magnetic diffusion and kinematic viscosity*; $\mathbf{d}$ are coefficients of pseudodifferential second-order operators, which can be interpreted as describing a non-standard *non-local combined eddy diffusion*. All $\mathbf{d} = 0$, if the CHM regime is steady or possesses a symmetry without a time shift. Constant vectors $\mathbf{A}$ enter into quadratic terms, representing the nonlinear operators of *combined eddy correction of the fluid and magnetic field advection*. The coefficients $\mathbf{D}$, $\mathbf{d}$ and $\mathbf{A}$ that we derived here have the same structure, as the coefficients of the respective eddy operators in the mean-field equations for perturbations obtained in the previous chapter in the case of forced convection; hence the same methods (considered in Sect. 8.7) can be applied to compute them.

Amplitude equations (9.52) and (9.53), governing the leading terms in the power series expansion of perturbations, constitute a closed system, supplemented by equation (9.34) (if the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ is unsteady) and the solenoidality conditions for the mean perturbations of magnetic field, $\langle\!\langle \mathbf{h}_0 \rangle\!\rangle_h$, and flow velocity, $\langle\!\langle \mathbf{v}_1 \rangle\!\rangle_h$ (9.33) (see (8.24) for $n = 0$ and 1). Note that $c_{03} = \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_1$, $c_{04} = \langle\!\langle \mathbf{h}_0 \rangle\!\rangle_2$, since $\langle \mathbf{S}^h_{k+2} \rangle_h = \mathbf{e}_k$, $k = 1, 2$. Equation (9.53) preserves solenoidality of the mean magnetic perturbation, and hence it must only be ensured that it is solenoidal at $T = 0$. Equations (9.52) and (9.53) are lacking the horizontal and vertical components, respectively, i.e. (9.53) involves two, and (9.52) one scalar equation. In contrast with the case of forced convection, the system of amplitude equations is mixed: it involves an evolutionary equation (9.53), as well as non-evolutionary ones such as (9.52) and (9.34). Equation (9.52), stemming from the equation for the perturbation of vorticity, has changed its nature—it involves neither a derivative in the slow time, nor the molecular diffusion operator.

9.7 An Alternative Amplitude Equation for Steady Symmetric CHM Regimes

If the CHM regime $\mathbf{V}, \mathbf{H}, \Theta$ is steady, and either parity-invariant or symmetric about a vertical axis, then $\langle\!\langle \mathbf{v}_1 \rangle\!\rangle_h = 0$ (all constant vectors $\boldsymbol{\beta} = 0$ involved in equation (9.33) are zero), and the system of amplitude equations that we have derived is underdetermined. Then the missing equation is the condition of solenoidality for the leading term of the mean flow velocity perturbation $\langle \mathbf{v}_2 \rangle_h$. This section is devoted to calculation of $\langle \mathbf{v}_2 \rangle_h$ for such steady symmetric CHM regimes. In this case, by virtue of (9.14), (9.18), (9.20), (9.33) and (8.33) for $n = 0$ and 1, we obtain

$$\langle \mathbf{v}_0 \rangle_h = \langle \mathbf{v}_1 \rangle_h = \langle \boldsymbol{\omega}_0 \rangle_v = \langle \boldsymbol{\omega}_1 \rangle_v = \langle \boldsymbol{\omega}_2 \rangle_v = 0. \tag{9.54}$$

Step 3° for $n = 2$. In order to apply the solvability condition (9.12) to the system (8.29)–(8.31) for $n = 2$, we render this system in the form of the problem (9.6)–(9.9). Potential parts of $\boldsymbol{\omega}_2, \mathbf{v}_2$ and $\mathbf{h}_2$ are calculated using expressions (9.26)–(9.28) and (9.35) for the second term in the expansion of the perturbation, relations for divergencies (8.25), (8.26) and (8.28) for $n = 2$, and vorticity relations (9.37) and (9.40) in the auxiliary problems of types II and III. This yields

$$\boldsymbol{\omega}_2 = \sum_{k=1}^{K}\sum_{n=1}^{2}\frac{\partial}{\partial X_n}\left(\sum_{m=1}^{2}\frac{\partial c_{0k}}{\partial X_m}\mathbf{e}_n \times \mathbf{G}^v_{mk} + \sum_{m=1}^{K} c_{0m}c_{0k}\mathbf{e}_n \times \mathbf{Q}^v_{mk}\right) + \boldsymbol{\omega}'_2, \qquad (9.55)$$

$$\mathbf{v}_2 = \langle \mathbf{v}_2 \rangle_h + \sum_{k=1}^{K}\sum_{n=1}^{2}\frac{\partial}{\partial X_n}\left(\sum_{m=1}^{2}\frac{\partial c_{0k}}{\partial X_m}\nabla_{\mathbf{x}} g^v_{nmk} + \sum_{m=1}^{K} c_{0m}c_{0k}\nabla_{\mathbf{x}} q^v_{nmk}\right) + \mathbf{v}'_2, \qquad (9.56)$$

$$\mathbf{h}_2 = \sum_{k=1}^{K}\sum_{n=1}^{2}\frac{\partial}{\partial X_n}\left(\sum_{m=1}^{2}\frac{\partial c_{0k}}{\partial X_m}\nabla_{\mathbf{x}} g^h_{nmk} + \sum_{m=1}^{K} c_{0m}c_{0k}\nabla_{\mathbf{x}} q^h_{nmk}\right) + \mathbf{h}'_2, \qquad (9.57)$$

where

$$\nabla_{\mathbf{x}} \cdot \boldsymbol{\omega}'_2 = \nabla_{\mathbf{x}} \cdot \mathbf{v}'_2 = \nabla_{\mathbf{x}} \cdot \mathbf{h}'_2 = 0, \quad \langle \boldsymbol{\omega}'_2 \rangle_v = \langle \mathbf{v}'_2 \rangle_h = 0, \quad \nabla_{\mathbf{x}} \times \mathbf{v}'_2 = \boldsymbol{\omega}'_2,$$

$g_{nmk}(\mathbf{x}, t)$ and $q_{nmk}(\mathbf{x}, t)$ are globally bounded solutions to the Neumann problems

$$\nabla^2_{\mathbf{x}} g^v_{nmk} = -(\widetilde{G}^v_{mk})_n, \quad \left.\frac{\partial g^v_{nmk}}{\partial x_3}\right|_{x_3=\pm L/2} = 0;$$

$$\nabla^2_{\mathbf{x}} g^h_{nmk} = -(\widetilde{G}^h_{mk})_n, \quad \left.\frac{\partial g^h_{nmk}}{\partial x_3}\right|_{x_3=\pm L/2} = 0;$$

$$\nabla^2_{\mathbf{x}} q^v_{nmk} = -(Q^v_{mk})_n, \quad \left.\frac{\partial q^v_{nmk}}{\partial x_3}\right|_{x_3=\pm L/2} = 0;$$

$$\nabla^2_{\mathbf{x}} q^h_{nmk} = -(Q^h_{mk})_n, \quad \left.\frac{\partial q^h_{nmk}}{\partial x_3}\right|_{x_3=\pm L/2} = 0,$$

and, consistently with the notation that we have already used in Sect. 9.6,

$$\widetilde{\mathbf{G}}_{mk} \equiv (\mathbf{G}^{\omega}_{mk} + \mathbf{e}_m \times \mathbf{S}^v_k, \mathbf{G}^v_{mk} + \nabla_{\mathbf{x}} s^v_{mk}, \mathbf{G}^h_{mk} + \nabla_{\mathbf{x}} s^h_{mk}, G^{\theta}_{mk}).$$

The fields $\boldsymbol{\omega}'_2, \mathbf{v}'_2$ and $\mathbf{h}'_2$ satisfy the assumed boundary conditions for vorticity, flow velocity and magnetic field, respectively.

We transform equations (8.29)–(8.31) for $n = 2$, substituting expressions (9.55)–(9.57) for the third term in the expansion of the perturbation, and obtain

$$\mathscr{M}'^{\omega}(\boldsymbol{\omega}_2', \mathbf{h}_2', \theta_2) = -\sum_{k=1}^{K}\sum_{n=1}^{2}\left(\sum_{m=1}^{2}\mathscr{L}^{\omega}(\mathbf{e}_n \times \mathbf{G}^v_{mk}, \nabla_{\mathbf{x}} g^v_{nmk}, \nabla_{\mathbf{x}} g^h_{nmk}, 0)\frac{\partial^2 c_{0k}}{\partial X_m \partial X_n}\right.$$
$$\left. - \sum_{m=1}^{K}\mathscr{L}^{\omega}(\mathbf{e}_n \times \mathbf{Q}^v_{mk}, \nabla_{\mathbf{x}} q^v_{nmk}, \nabla_{\mathbf{x}} q^h_{nmk}, 0)\frac{\partial}{\partial X_n}(c_{0m}c_{0k})\right)$$
$$+ (\langle \mathbf{v}_2\rangle_h \cdot \nabla_{\mathbf{x}})\boldsymbol{\Omega} - 2\nu(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\boldsymbol{\omega}_1 - \nabla_{\mathbf{X}} \times (\mathbf{V} \times \boldsymbol{\omega}_1 + \mathbf{v}_1$$
$$\times \boldsymbol{\Omega} - \mathbf{H} \times (\nabla_{\mathbf{x}} \times \mathbf{h}_1) - \mathbf{h}_1 \times (\nabla_{\mathbf{x}} \times \mathbf{H}) - \mathbf{H} \times (\nabla_{\mathbf{X}} \times \mathbf{h}_0)$$
$$+ \mathbf{v}_0 \times \boldsymbol{\omega}_0 - \mathbf{h}_0 \times (\nabla_{\mathbf{x}} \times \mathbf{h}_0)) - \beta\nabla_{\mathbf{X}}\theta_1 \times \mathbf{e}_3$$
$$- \nabla_{\mathbf{x}} \times (-\mathbf{H} \times (\nabla_{\mathbf{X}} \times \mathbf{h}_1) + \mathbf{v}_1 \times \boldsymbol{\omega}_0 + \mathbf{v}_0 \times \boldsymbol{\omega}_1$$
$$- \mathbf{h}_1 \times (\nabla_{\mathbf{x}} \times \mathbf{h}_0) - \mathbf{h}_0 \times (\nabla_{\mathbf{x}} \times \mathbf{h}_1));$$

$$\mathscr{M}'^{h}(\boldsymbol{\omega}_2', \mathbf{h}_2') = -\sum_{k=1}^{K}\sum_{n=1}^{2}\left(\sum_{m=1}^{2}\mathscr{L}^{h}(\nabla_{\mathbf{x}} g^v_{nmk}, \nabla_{\mathbf{x}} g^h_{nmk})\frac{\partial^2 c_{0k}}{\partial X_m \partial X_n}\right.$$
$$\left. - \sum_{m=1}^{K}\mathscr{L}^{h}(\nabla_{\mathbf{x}} q^v_{nmk}, \nabla_{\mathbf{x}} q^h_{nmk})\frac{\partial}{\partial X_n}(c_{0m}c_{0k})\right) + (\langle \mathbf{v}_2\rangle_h \cdot \nabla_{\mathbf{x}})\mathbf{H}$$
$$- 2\eta(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\mathbf{h}_1 - \nabla_{\mathbf{X}} \times (\mathbf{v}_1 \times \mathbf{H} + \mathbf{V} \times \mathbf{h}_1 + \mathbf{v}_0 \times \mathbf{h}_0)$$
$$- \nabla_{\mathbf{x}} \times (\mathbf{v}_1 \times \mathbf{h}_0 + \mathbf{v}_0 \times \mathbf{h}_1);$$

$$\mathscr{M}'^{\theta}(\boldsymbol{\omega}_2', \theta_2) = -\sum_{k=1}^{K}\sum_{n=1}^{2}\left(\sum_{m=1}^{2}\mathscr{L}^{\theta}(\nabla_{\mathbf{x}} g^v_{nmk}, 0)\frac{\partial^2 c_{0k}}{\partial X_m \partial X_n}\right.$$
$$\left. - \sum_{m=1}^{K}\mathscr{L}^{\theta}(\nabla_{\mathbf{x}} q^v_{nmk}, 0)\frac{\partial}{\partial X_n}(c_{0m}c_{0k})\right) + (\langle \mathbf{v}_2\rangle_h \cdot \nabla_{\mathbf{x}})\Theta$$
$$- 2\kappa(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\theta_1 + (\mathbf{V} \cdot \nabla_{\mathbf{X}})\theta_1 + (\mathbf{v}_1 \cdot \nabla_{\mathbf{x}})\theta_0 + (\mathbf{v}_0 \cdot \nabla_{\mathbf{X}})\theta_0$$
$$+ (\mathbf{v}_0 \cdot \nabla_{\mathbf{x}})\theta_1.$$

Step 4° for $n = 2$. We substitute here expressions (9.5) for the leading term in the expansion of the perturbation and

$$(\boldsymbol{\omega}_1, \mathbf{v}_1, \mathbf{h}_1, \theta_1) = \sum_{k=1}^{K}\left(\mathbf{S}_k c_{1k} + \sum_{m=1}^{2}\widetilde{\mathbf{G}}_{mk}\frac{\partial c_{0k}}{\partial X_m} + \sum_{m=1}^{K}\mathbf{Q}_{mk}c_{0k}c_{0m}\right)$$

(a consequence of expressions (9.35) and (9.26)–(9.28) for the second term in the expansion), where the inessential exponentially decaying transients ξ_0 and ξ_1 are ignored and (9.54) is taken into account. Scalar multiplying the result by $\mathbf{S}^*_j$, $j = 1, 2$, noting that c_{1k} enter only into symmetric sets of fields, and using the biorthogonality and normalisation relations (9.4), we find

$$\langle \mathbf{v}_2 \rangle_h = \sum_{k=1}^{K} \left(\sum_{m=1}^{2} \sum_{n=1}^{2} \widetilde{\boldsymbol{\beta}}_{1nmk} \frac{\partial^2 c_{0k}}{\partial X_m \partial X_n} + \sum_{m=1}^{K} \sum_{n=1}^{2} \widetilde{\boldsymbol{\beta}}_{2nmk} c_{0m} \frac{\partial c_{0k}}{\partial X_n} \right.$$
$$\left. + \sum_{m=1}^{K} \sum_{n=1}^{K} \widetilde{\boldsymbol{\beta}}_{3nmk} c_{0k} c_{0m} c_{0n} \right). \tag{9.58}$$

Here

$$\widetilde{\boldsymbol{\beta}}_{inmk} \equiv (\langle \widetilde{\mathbf{E}}_{inmk} \cdot \mathbf{S}_1^* \rangle, \langle \widetilde{\mathbf{E}}_{inmk} \cdot \mathbf{S}_2^* \rangle, 0)$$

are constant three-dimensional vectors,

$$\widetilde{\mathbf{E}}_{1nmk} \equiv \mathscr{L}(\mathbf{e}_n \times \widetilde{\mathbf{G}}^v_{mk}, \nabla g^v_{nmk}, \nabla g^h_{nmk}, 0) + \left(2\nu \frac{\partial}{\partial x_n} \widetilde{\mathbf{G}}^\omega_{mk} \right.$$
$$+ \mathbf{e}_n \times \Big(\mathbf{V} \times \widetilde{\mathbf{G}}^\omega_{mk} + \widetilde{\mathbf{G}}^v_{mk} \times \boldsymbol{\Omega} - \mathbf{H} \times (\nabla \times \widetilde{\mathbf{G}}^h_{mk}) - \widetilde{\mathbf{G}}^h_{mk} \times (\nabla \times \mathbf{H})$$
$$- \mathbf{H} \times (\mathbf{e}_m \times \mathbf{S}^h_k) - \beta G^\theta_{mk} \mathbf{e}_3 \Big) - \nabla \times (\mathbf{H} \times (\mathbf{e}_n \times \widetilde{\mathbf{G}}^h_{mk})),$$
$$\left. 2\eta \frac{\partial}{\partial x_n} \widetilde{\mathbf{G}}^h_{mk} + \mathbf{e}_n \times (\widetilde{\mathbf{G}}^v_{mk} \times \mathbf{H} + \mathbf{V} \times \widetilde{\mathbf{G}}^h_{mk}), \qquad 2\kappa \frac{\partial}{\partial x_n} G^\theta_{mk} - V_n G^\theta_{mk} \right);$$

$$\widetilde{\mathbf{E}}_{2nmk} \equiv \widetilde{\mathbf{e}}_{nmk} + \widetilde{\mathbf{e}}_{nkm}$$
$$+ \Big(\nabla \times (\widetilde{\mathbf{G}}^v_{nk} \times \mathbf{S}^\omega_m + \mathbf{S}^v_m \times \widetilde{\mathbf{G}}^\omega_{nk} - \widetilde{\mathbf{G}}^h_{nk} \times (\nabla \times \mathbf{S}^h_m) - \mathbf{S}^h_m \times (\nabla \times \widetilde{\mathbf{G}}^h_{nk})),$$
$$\nabla \times (\widetilde{\mathbf{G}}^v_{nk} \times \mathbf{S}^h_m + \mathbf{S}^v_m \times \widetilde{\mathbf{G}}^h_{nk}), \qquad -(\widetilde{\mathbf{G}}^v_{nk} \cdot \nabla) S^\theta_m - (\mathbf{S}^v_m \cdot \nabla) G^\theta_{nk} \Big);$$

$$\widetilde{\mathbf{e}}_{nmk} \equiv \mathscr{L}(\mathbf{e}_n \times \mathbf{Q}^v_{mk}, \nabla q^v_{nmk}, \nabla q^h_{nmk}, 0) + \left(2\nu \frac{\partial}{\partial x_n} \mathbf{Q}^\omega_{mk} \right.$$
$$+ \mathbf{e}_n \times \Big(\mathbf{V} \times \mathbf{Q}^\omega_{mk} + \mathbf{Q}^v_{mk} \times \boldsymbol{\Omega} - \mathbf{H} \times (\nabla \times \mathbf{Q}^h_{mk}) - \mathbf{Q}^h_{mk} \times (\nabla \times \mathbf{H})$$
$$+ \mathbf{S}^v_m \times \mathbf{S}^\omega_k - \mathbf{S}^h_m \times (\nabla \times \mathbf{S}^h_k) - \beta Q^\theta_{mk} \mathbf{e}_3 \Big) - \nabla \times (\mathbf{H} \times (\mathbf{e}_n \times \mathbf{Q}^h_{mk})),$$
$$2\eta \frac{\partial}{\partial x_n} \mathbf{Q}^h_{mk} + \mathbf{e}_n \times (\mathbf{Q}^v_{mk} \times \mathbf{H} + \mathbf{V} \times \mathbf{Q}^h_{mk} + \mathbf{S}^v_m \times \mathbf{S}^h_k),$$
$$\left. 2\kappa \frac{\partial}{\partial x_n} Q^\theta_{mk} - V_n Q^\theta_{mk} \right);$$

$$\widetilde{\mathbf{E}}_{3nmk} \equiv \Big(\nabla \times (\mathbf{Q}^v_{mk} \times \mathbf{S}^\omega_n + \mathbf{S}^v_n \times \mathbf{Q}^\omega_{mk} - \mathbf{Q}^h_{mk} \times (\nabla \times \mathbf{S}^h_n) - \mathbf{S}^h_n \times (\nabla \times \mathbf{Q}^h_{mk})),$$
$$\nabla \times (\mathbf{Q}^v_{mk} \times \mathbf{S}^h_n + \mathbf{S}^v_n \times \mathbf{Q}^h_{mk}), \quad -(\mathbf{Q}^v_{mk} \cdot \nabla) S^\theta_n - (\mathbf{S}^v_n \cdot \nabla) Q^\theta_{mk} \Big).$$

The remaining equation in the amplitudes c_{0k} for a CHM steady state $\mathbf{V}, \mathbf{H}, \Theta$ possessing an appropriate symmetry and lacking a significant α-effect is the solenoidality condition for the leading term in the expansion of the mean flow, $\langle \mathbf{v}_2 \rangle_h$ (9.58); it is also non-evolutionary.

We have accomplished our goal to derive the amplitude equations describing the leading term in the expansion (8.20)–(8.23) of a large-scale perturbation. In principle, the procedure, outlined in Sect. 9.3, can be applied further to subsequent systems (8.29)–(8.31) in the hierarchy and thus an arbitrary number of terms in the asymptotic expansion can be determined.

9.8 Non-zero Initial Mean Flow Perturbation $\langle\langle \mathbf{v}_0 \rangle\rangle_h$

Unless the horizontal part of the mean flow perturbation vanishes, the problem (9.1)–(9.3) does not have a globally bounded solution (see Sect. 9.3). Solutions with a non-zero mean flow, which are not globally bounded in time, do exist, for instance

$$(\boldsymbol{\omega}_0, \mathbf{h}_0, \theta_0) = t(\langle\langle \mathbf{v}_0 \rangle\rangle_h \cdot \nabla_{\mathbf{x}})(\boldsymbol{\Omega}, \mathbf{H}, \Theta). \tag{9.59}$$

Formal solutions featuring linear growth in horizontal directions also exist, but we cannot interpret them as valid perturbations even initially.

If solutions to equations (8.29)–(8.31) are allowed to grow in the fast time, they can always be constructed as solutions to parabolic equations. For instance, let a CHM state $\boldsymbol{\Omega}, \mathbf{H}, \mathscr{T}$ be steady and periodic in horizontal directions. Then the operator of linearisation $\mathscr{L}$ (where the derivative in time is neglected) is elliptic and hence its eigenfunctions constitute a complete basis in the functional space, where the solution is sought, and the problem (8.29)–(8.31) can be solved, expanding the unknown fields $\boldsymbol{\omega}_n, \mathbf{h}_n, \theta_n$ in this basis. It is unnecessary any more to separate the dependence of the perturbation on time into the dependence on the slow and fast times (one can just set $\partial/\partial T = 0$ in equations (8.29)–(8.31). The perturbation is expanded in the power series in the spatial scale ratio ε, and the procedure for solution of the systems constituting the hierarchy is now more straightforward, involving for the system at order ε^n two steps:

1°. Average over the fast spatial variables the vertical component of the equation (8.29) for vorticity evolution at order ε^{n+1}, and calculate $\langle \boldsymbol{\omega}_{n+1} \rangle_v$ and $\langle \mathbf{v}_n \rangle_h$.
2°. Integrate equations (8.29)–(8.31) in time and find $\{\boldsymbol{\omega}_n\}_v, \{\mathbf{v}_n\}_h, \mathbf{h}_n$ and θ_n.

Since the zero-order terms (9.59) grow linearly in time, the quadratic nonlinearity of the equations constituting the hierarchy gives rise to the factor t^{2n+1} in the terms $\boldsymbol{\omega}_n, \mathbf{v}_n, \mathbf{h}_n, \theta_n$. Thus, the CHM state is apparently unstable to large-scale perturbations with a non-zero mean flow velocity, even if it is stable to short-scale ones. However, this may be not a genuine instability—it can rather describe the variation of the profile of the perturbed CHM state caused by advection of the state by the mean flow $\langle\langle \mathbf{v}_0 \rangle\rangle_h$. (Note that the particular solution (9.59) for the leading terms of expansion of the perturbation coincides with the second term in the Taylor expansion in time of the fields $\boldsymbol{\Omega}, \mathbf{H}, \Theta$ transported by the flow $\langle\langle \mathbf{v}_0 \rangle\rangle_h$). If the mean flow were constant, by the Galilean invariance of the equations governing CHM regimes we could remove the mean flow, considering the problem in the co-moving coordinate system, but dependence of the mean-flow on the slow spatial variables makes this impossible.

9.9 Conclusions

1. In this chapter, we have studied weakly nonlinear stability of short-scale convective hydromagnetic regimes $\mathbf{V}, \mathbf{H}, \Theta$ to large-scale perturbations. Flows in an infinite horizontal layer of incompressible electrically conducting fluid, heated from below and rotating about the vertical axis, are considered in the Boussinesq approximation, allowing for magnetic field generation. Thermal convection is supposed to be free, i.e. we assume that no source terms are present in equations (8.1)–(8.3), governing the CHM regimes, or in the boundary conditions (8.5)–(8.7) and (8.11). Such regimes are invariant with respect to translations in space and time. Applying homogenisation methods for multiscale methods, in Sect. 9.4 we have derived the tensors of the combined α-effect, (9.16) and (9.22). In the presence of the α-effect, equations for amplitudes of neutral short-scale modes, constituting the leading term in the expansion of the perturbation in the spatial scale ratio, comprise a mixed, typically overdetermined nonlinear system of PDE's. If the CHM regime is parity-invariant or symmetric about a vertical axis, possibly with a time shift, the operators of the α-effect do not contribute to the amplitude equations, i.e. the α-effect is insignificant in the leading order. If the combined α-effect is insignificant (due to the presence of the symmetries or otherwise), the system is comprised of amplitude equations (9.52) and (9.53), supplemented by equation (9.34), if the CHM regime is unsteady, and by the conditions of solenoidality in the slow variables (8.24) of the mean leading terms of perturbations of magnetic field ($n = 0$) and the flow velocity. When the combined α-effect is absent, generically the latter condition is stated for the mean flow perturbation $\langle\!\langle \mathbf{v}_1 \rangle\!\rangle_h$ (9.33); if the CHM regime is steady, and it is parity-invariant or has a symmetry about a vertical axis, then $\langle\!\langle \mathbf{v}_1 \rangle\!\rangle_h = 0$, and the solenoidality condition for the mean flow perturbation $\langle\!\langle \mathbf{v}_2 \rangle\!\rangle_h$ (9.58) is employed.
2. As in the case of forced thermal MHD convection, considered in the previous chapter, in the absence of the significant α-effect the amplitude equations involve the linear operator of the combined eddy diffusion correction and quadratic terms representing the combined eddy correction of advection. A non-standard non-local eddy diffusion is described by a pseudodifferential operator, formally of the second order; it emerges in the amplitude equations only, if the CHM regime is unsteady and does not possess a spatial (without a time shift) symmetry mentioned above. All eddy correction operators are anisotropic. The following dissimilarities between the cases of forced and free convection are notable: Due to the difference of the kernels of the operators of linearisation, the system of amplitude equations that we have derived in this chapter is mixed, and it does not include the mean-field equation for the perturbation of the flow velocity. While equations (9.53) for the mean perturbation

of magnetic field are evolutionary, the remaining amplitude equations (9.52) and (9.34) involve neither derivatives in the slow time, nor operators of molecular diffusion. If the CHM state $\mathbf{V}, \mathbf{H}, \boldsymbol{\Theta}$ is steady and has a symmetry guaranteeing insignificance of the combined α-effect, then the solenoidality condition for the mean flow perturbation (9.58) is a non-evolutionary third-order PDE with a cubic nonlinearity.

3. Because of the spatial and temporal translation invariance of free CHM regimes, $\mathbf{S}_k^{\cdot x}$ and $\mathbf{S}^{\cdot t}$ (see Sect. 9.2) are neutral modes of linear stability and solve Eqs. 9.1–9.3. They satisfy the boundary conditions, whichever physically sensible combination of boundary conditions one considers (stress-free or no-slip perfectly electrically conducting or insulating boundaries, kept at constant temperatures or letting through a fixed heat flow). Existence of other solutions to the problem (9.1)–(9.3) depends on the assumed boundary conditions and parameter values. For instance, generically the three neutral modes mentioned in the beginning of this paragraph constitute a complete basis in the kernel of the operator of linearisation for CHM regimes in a layer with no-slip isothermal boundaries between two half-spaces filled in with a dielectric substance. Existence of the neutral modes opens a possibility to perform a similar analysis of linear and weakly nonlinear stability of CHM regimes to large-scale perturbations for boundary conditions, different from the ones that we have considered here (generically this is impossible if convection is forced).
4. As we have done in Chap. 8 for forced convection, one can derive amplitude equations describing weakly nonlinear stability to large-scale perturbations for branches of CHM regimes emerging in symmetry-breaking Hopf or pitchfork bifurcations. Then, in the former case, new terms similar to the α-effect operators appear in the amplitude equations, and in the latter case, in addition a new amplitude equation emerges. Although these calculations do not require new approaches, the algebra is rather bulky, and we do not present it here.

Chapter 10
Magnetic Field Generation by a Two-Scale Flow in an Axisymmetric Volume

In this chapter we return to the kinematic dynamo problem and construct an asymptotic expansion of magnetic modes for flows of electrically conducting fluid residing in an axisymmetric region Ω surrounded by a dielectric. This geometry and boundary conditions are more appropriate for geo- and astrophysical applications than the ones employed in the previous chapters. Flows that we focus on here are significantly qualitatively different from those considered in Chaps. 3, 4, 5. We assume that they depend on the slow spatial variables, whose scale is of the order of the size of the region Ω, and on a fast variable, proportional to the azimuthal coordinate. The amplitude of the flow is supposed to grow as $\varepsilon^{-1/2}$, where $\varepsilon = 1/n$ is the scale ratio of the fast and slow variables and n is integer (in order to uphold geometric compatibility).

Kinematic dynamos for families of steady flows with the same scaling of the amplitude of the flow relative the scale ratio of spatial variables was considered in a number of papers. A complete asymptotic expansion of eigenfunctions and the associated eigenvalues of the magnetic induction operator for a family of steady space-periodic flows was constructed in [306], for flows in an axisymmetric region in [321] (the exposition of this chapter follows this paper), and in a spherical region for flows depending on three fast spatial variables proportional to the spherical coordinates in [322] (this problem is discussed in the next chapter). It was shown in these papers that for such families of flows the "limit" operator governing the leading terms of expansions of magnetic modes and their growth rates is a sum of the operators describing molecular diffusion, advection by the mean component of the flow, and the α-effect. The expansions are in power series in $\varepsilon^{1/2}$. In contrast with the expansions constructed in Chaps. 3, 4, 5, in the present case the leading term of the expansion of the magnetic field is independent of the fast variable. An analogous kinematic dynamo problem was solved in [307] under the assumptions that the flow amplitude is $O(\varepsilon^0)$, and molecular magnetic diffusivity is $O(\varepsilon)$ (similar scalings were also considered in [251, 252]). Then the limit

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operator is the operator of magnetic α-effect, and the operator of molecular diffusion is absent; in [307] this situation is called the γ-effect.

Mathematics in this chapter is more involved than in the previous ones. We construct here complete asymptotic expansions of magnetic modes and the associated eigenvalues of the magnetic induction operator in power series in $\varepsilon^{1/2}$, and, unlike in the previous chapters, prove their asymptotic convergence. This necessitates to develop additional mathematical instrumentarium, which we do in the first two sections. In particular, we apply the interpolation techniques for Hilbert spaces to construct the scale of Sobolev functional spaces of solenoidal vector fields for the boundary conditions corresponding to the presence of a dielectric outside a bounded region Ω (which can be of an arbitrary shape) with a smooth boundary. We show that for an integer index, the norm in such a space is equivalent to the respective standard Sobolev norm of the same index.

10.1 Mathematical Tools

The goal of this section is to introduce a family of functional spaces, which will be applied for derivation of bounds for solutions of elliptic problems, for the boundary conditions satisfied by magnetic field on the boundary of a volume of an electrically conducting fluid surrounded by a dielectric medium.

In this section we assume that the volume, Ω, occupied by fluid is an arbitrary bounded open region with a smooth boundary, $\partial\Omega$.

10.1.1 Functional Sobolev Spaces $\mathbb{W}^q(\Omega)$

In this subsection we recall some general facts from the functional analysis. A Sobolev space $\mathbb{W}_2^q(\Omega)$ of index $q \geq 0$ is a Hilbert space comprised of functions defined in the region Ω. We denote the norm in $\mathbb{W}_2^q(\Omega)$ by $|\cdot|_q$; for an integer $q \geq 0$,

$$|\mathbf{f}|_q^2 \equiv \sum_{m_1+m_2+m_3 \leq q} \int_\Omega \left| \frac{\partial^{m_1+m_2+m_3}\mathbf{f}}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} \right|^2 d\mathbf{x}.$$

For any pair $0 \leq q_1 < q_2$, the embedding $\mathbb{W}_2^{q_2}(\Omega) \subset \mathbb{W}_2^{q_1}(\Omega)$ is compact, and Sobolev spaces of intermediate indices can be obtained by interpolation: for any $\theta \in [0, 1]$,

$$\mathbb{W}_2^{(1-\theta)q_2+\theta q_1}(\Omega) = [\mathbb{W}_2^{q_2}(\Omega), \mathbb{W}_2^{q_1}(\Omega)]_\theta.$$

This is understood as follows: Let $\{\mathbb{X}_0, \mathbb{X}_1\}$ be an interpolation pair of separable Hilbert spaces, i.e. $\mathbb{X}_0$ is dense and continuously embedded into $\mathbb{X}_1$. Then $\mathbb{X}_0$ can be regarded as the domain in $\mathbb{X}_1$ of a self-adjoint positive-definite unbounded operator $\mathcal{Q}$, equipped with the graph norm

$$|\mathbf{f}|^2_{\mathbb{X}_0} = |\mathbf{f}|^2_{\mathbb{X}_1} + |\mathscr{Q}\mathbf{f}|^2_{\mathbb{X}_1}$$

(here $|\cdot|_{\mathbb{X}_i}$ denotes the norm in $\mathbb{X}_i$). For a $\theta \in [0, 1]$, the intermediate interpolation space $[\mathbb{X}_0, \mathbb{X}_1]_\theta$ is defined as the domain of the operator $\mathscr{Q}^{1-\theta}$ with the norm

$$|\mathbf{f}|^2_\theta = |\mathbf{f}|^2_{\mathbb{X}_1} + |\mathscr{Q}^{1-\theta}\mathbf{f}|^2_{\mathbb{X}_1}.$$

A differential operator $\mathscr{D} : \mathbb{W}^q_2(\Omega) \to \mathbb{W}^{q-m}_2(\Omega)$ of order $m \leq q$ with coefficients, smooth in $\overline{\Omega}$ (the bar denotes the closure of a set), is continuous.

For $q < 0$, the Sobolev space $\mathbb{W}^q(\Omega)$ is defined as the space, dual to $\overset{\circ}{\mathbb{W}}{}^{|q|}_2(\Omega)$; the latter is the closure in the norm of $\mathbb{W}^{|q|}(\Omega)$ of the set of infinitely continuous functions, whose support belongs to Ω.

Using an atlas of charts, one can define in a similar way Sobolev spaces of functions defined on the smooth manifold $\partial\Omega$. We denote a norm in $\mathbb{W}^s_2(\partial\Omega)$ by $|\cdot|_{s,\partial\Omega}$ and the outward unit normal to the boundary $\partial\Omega$ by $\boldsymbol{\nu}$. Relations between the Sobolev spaces of functions in Ω and their restrictions to the boundary $\partial\Omega$ (*traces*) are described by the following *trace theorem* [170]:

Theorem 10.1 *The mapping* $u \to \{(\boldsymbol{\nu}\cdot\nabla)^j u | j = 0, \ldots, m\}$ *acting from* $\mathbb{C}^\infty(\overline{\Omega})$ *to* $(\mathbb{C}^\infty(\partial\Omega))^{m+1}$ *can be extended by continuity to a surjective mapping*

$$\mathbb{W}^q(\Omega) \to \prod_{j=0}^{m} \mathbb{W}^{q-j-1/2}(\partial\Omega),$$

where m *is the maximum integer such that* $m < q - 1/2$. *There exists a continuous linear operator* $\{g_j\} \to u$ *acting from* $\prod_{j=0}^m \mathbb{W}^{q-j-1/2}(\partial\Omega)$ *to* $\mathbb{W}^q(\Omega)$ *such that* $g_j = (\boldsymbol{\nu}\cdot\nabla)^j u|_{\partial\Omega}$ *for all* $j \leq m$.

By the trace theorem, any linear differential operator $\mathscr{D} : \mathbb{W}^q_2(\Omega) \to \mathbb{W}^s_2(\partial\Omega)$ of order $m \geq 0$, whose image is restricted to the boundary $\partial\Omega$ and whose coefficients are smooth in $\partial\Omega$, is continuous for $0 \leq s < q - m - 1/2$: $|\mathscr{D}\mathbf{f}|_{s,\partial\Omega} \leq C_{\mathscr{D},\Omega,q,s}|\mathbf{f}|_q$.

The above-mentioned facts hold true for any bounded region Ω with a smooth boundary [1, 16, 170, 182, 299]. (The notation $\mathbb{W}^q_2(\Omega)$ agrees with the one in [299]; in [170], $\mathbb{W}^q_2(\Omega)$ is denoted by $H^q(\Omega)$.)

10.1.2 *The Scale of Functional Sobolev Spaces* $\mathbb{H}^q(\Omega)$ *for a Fluid Inside a Dielectric*

The boundary conditions for magnetic field at the boundary between an electrically conducting fluid and the surrounding dielectric are non-local: magnetic field permeates outside Ω and it is required to be continuous across the fluid boundary $\partial\Omega$. In mathematical terms, the conditions are as follows:

$$\nabla \times \mathbf{h} = 0 \text{ and } \nabla \cdot \mathbf{h} = 0 \text{ in } \Omega' \equiv \mathbb{R}^3 \backslash \overline{\Omega}; \quad \mathbf{h} \to 0 \text{ for } |\mathbf{x}| \to \infty; \quad [\mathbf{h}]_{\partial\Omega} = 0, \qquad (10.1)$$

where $[\cdot]$ denotes the jump of a function across $\partial\Omega$ (i.e., $[\mathbf{f}]_{\partial\Omega} \equiv (\mathbf{f}_{\text{int}} - \mathbf{f}_{\text{ext}})|_{\partial\Omega}$ is the difference between the limit internal, $\mathbf{f}_{\text{int}}$, and external, $\mathbf{f}_{\text{ext}}$, values of $\mathbf{f}$ at the boundary $\partial\Omega$). The goal of this subsection is to construct a scale of functional Hilbert spaces, $\mathbb{H}^q(\Omega)$, of solenoidal vector fields satisfying the above boundary conditions.

We denote by $\hat{\mathbf{f}}$ a continuation of a three-dimensional vector field $\mathbf{f} \in \mathbb{C}^\infty(\overline{\Omega})$ to Ω'; $\hat{\mathbf{f}} = \nabla F$, where a harmonic function F is a solution to the Neumann problem

$$\nabla^2 F = 0, \quad (\boldsymbol{\nu} \cdot \nabla)F|_{\partial\Omega} = (\boldsymbol{\nu} \cdot \nabla)\mathbf{f}|_{\partial\Omega}, \quad F \to 0 \text{ for } |\mathbf{x}| \to \infty.$$

For $\mathbf{f}, \mathbf{g} \in \mathbb{C}^\infty(\overline{\Omega})$ we define a scalar product

$$((\mathbf{f}, \mathbf{g})) \equiv \int_\Omega \mathbf{f} \cdot \bar{\mathbf{g}} \, d\mathbf{x} + \int_{\Omega'} \hat{\mathbf{f}} \cdot \bar{\hat{\mathbf{g}}} \, d\mathbf{x}$$

(here $\hat{\mathbf{f}}$ and $\hat{\mathbf{g}}$ are the continuations of $\mathbf{f}$ and $\mathbf{g}$, respectively). The integral over Ω' converges, because for any harmonic function $F(\mathbf{x})$, tending to zero for $\mathbf{x} \to \infty$, the gradient satisfies the inequality $|\nabla F| \le M_F |\mathbf{x}|^{-2}$, where the constant M_F is independent of the point $\mathbf{x}$ [273]. We define

$$\widetilde{\mathbb{H}}(\Omega) \equiv \left\{ \mathbf{f} \in \mathbb{C}^\infty(\overline{\Omega}) | \nabla \cdot \mathbf{f} = 0; \mathbf{f}|_{\partial\Omega} = \hat{\mathbf{f}}|_{\partial\Omega} \right\},$$
$$\mathbb{H}(\Omega) \equiv \left\{ \mathbf{f} | (\nabla^2)^m \mathbf{f} \in \widetilde{\mathbb{H}}(\Omega) \text{ for any } m \ge 0 \right\}.$$

Lemma 10.1 *For any* $\mathbf{g} \in \widetilde{\mathbb{H}}(\Omega)$ *and* $\mathbf{f} \in \mathbb{C}^\infty(\overline{\Omega})$,

$$((\nabla \times \mathbf{f}, \mathbf{g})) = \int_\Omega \mathbf{f} \cdot (\nabla \times \bar{\mathbf{g}}) \, d\mathbf{x}. \qquad (10.2)$$

Proof By definition,

$$((\nabla \times \mathbf{f}, \mathbf{g})) = \int_\Omega (\nabla \times \mathbf{f}) \cdot \bar{\mathbf{g}} \, d\mathbf{x} + \int_{\Omega'} \nabla F \cdot \nabla \bar{G} \, d\mathbf{x},$$

where F and G are solutions to the respective Neumann problems. By formulae of vector analysis,

$$\int_\Omega (\nabla \times \mathbf{f}) \cdot \bar{\mathbf{g}} \, d\mathbf{x} + \int_{\Omega'} \nabla F \cdot \nabla \bar{G} \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot (\nabla \times \bar{\mathbf{g}}) \, d\mathbf{x} + \int_{\partial\Omega} (\mathbf{f} \times \bar{\mathbf{g}}) \cdot \boldsymbol{\nu} \, d\sigma$$
$$- \int_{\Omega'} \nabla^2 F \bar{G} \, d\mathbf{x} - \int_{\partial\Omega} (\boldsymbol{\nu} \cdot \nabla) F \bar{G} \, d\sigma$$

(here $\mathrm{d}\sigma$ denotes an area element on the boundary $\partial\Omega$). By the Stokes theorem,

$$\int\limits_{\partial\Omega} ((\mathbf{f}\times\bar{\mathbf{g}})\cdot\boldsymbol{\nu} - (\boldsymbol{\nu}\cdot\nabla)F\bar{G})\,\mathrm{d}\sigma = \int\limits_{\partial\Omega} ((\mathbf{f}\times\nabla\bar{G})\cdot\boldsymbol{\nu} - \boldsymbol{\nu}\cdot(\nabla\times\mathbf{f})\bar{G})\,\mathrm{d}\sigma$$
$$= -\int\limits_{\partial\Omega} (\nabla\times(\mathbf{f}\bar{G}))\cdot\boldsymbol{\nu}\,\mathrm{d}\sigma = 0.$$

This proves (10.2). □

By Lemma 10.1, the operator $-\nabla^2 = (\nabla\times)^2$ acting in $\mathbb{H}(\Omega)$ is positive-definite and symmetric with respect to the scalar product $((\cdot,\cdot))$; hence, one can define real powers $(-\nabla^2)^q$ in the usual way. We define $\mathbb{H}^q(\Omega)$ as a closure of the functional set $\mathbb{H}(\Omega)$ in the norm

$$\|\mathbf{f}\|_q \equiv (((-\nabla^2)^{q/2}\mathbf{f}, (-\nabla^2)^{q/2}\bar{\mathbf{f}}))^{1/2}.$$

Lemma 10.2 1°. *For any* $q \geq 0$ *and* $\mathbf{f} \in \mathbb{H}^q(\Omega)$,

$$|\mathbf{f}|_q \leq C'_q\|\mathbf{f}\|_q. \tag{10.3}$$

2°. *For integer* $q \geq 0$, *the opposite inequality,*

$$\|\mathbf{f}\|_q \leq C''_q|\mathbf{f}|_q, \tag{10.4}$$

is also satisfied, and thus the norm $\|\cdot\|_q$ *is equivalent to* $|\cdot|_q$. *(Here constants* C'_q *and* C''_q *depend only on the index* q *and the region* Ω.)

Proof 1°. Let us establish the inequality

$$|\mathbf{g}|_{q+2} \leq C_q|\nabla^2\mathbf{g}|_q, \tag{10.5}$$

where C_q is a constant, independent of $\mathbf{g} \in \mathbb{H}(\Omega)$ and $q \geq 0$. Here and in what follows constants C^i_q and $C^{i,j}_q$ depend only on the index q and the region Ω, and constants C^i only on Ω.

Consider a finite (due to compactness of the boundary) atlas of charts of a spatial neighbourhood $U \subset \mathbb{R}^3$ of the boundary $\partial\Omega$. We suppose that each chart introduces a local curvilinear coordinate system (τ_1, τ_2, τ_3), in which the boundary is described by the equation $\tau_3 = 0$. It suffices to prove inequality (10.5) for a vector field $\mathbf{g} \in \mathbb{H}$, such that the intersection of the support of $\mathbf{g}$ with $U \cap \Omega$ is covered by a chart from the atlas. We denote an element of area on the surface $\tau_3 =$ constant for this chart by $\kappa(\tau_1, \tau_2\tau_3)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2$.

A solution to the problem

$$\nabla^2\mathbf{g}' = \mathbf{f} \text{ in } \Omega, \quad (\boldsymbol{\nu}\cdot\nabla)(\mathbf{g}'\cdot\kappa\boldsymbol{\nu})|_{\partial\Omega} = 0, \quad \mathbf{g}'\times\boldsymbol{\nu}|_{\partial\Omega} = 0$$

satisfies the inequality $|\mathbf{g}'|_{q+2} \le C^1_q |\mathbf{f}|_q$ for any $q \ge 0$ [137, 170]. The field $\mathbf{g}'$ is solenoidal (since from the first equation $\nabla^2(\nabla \cdot \mathbf{g}') = 0$, and the divergence is zero on the boundary).

The single layer potential

$$h_F(\mathbf{x}) \equiv \frac{1}{4\pi} \int\limits_{\partial\Omega} \frac{F(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, \mathrm{d}\sigma \tag{10.6}$$

is a harmonic function in Ω and Ω', which is continuous in $\mathbb{R}^3$ provided F is continuous; its gradient experiences a jump across the boundary of Ω: $[\nabla h_F]_{\partial\Omega} = F\mathbf{v}$ [273]. Employing an atlas of charts of $\partial\Omega$ one can establish that

$$|h_F|_{q+1,\partial\Omega} \le C^2_q |F|_{q,\partial\Omega} \tag{10.7}$$

for any $q \ge 0$ (this follows from properties of Riesz transforms, see [285]); bounds for a solution to Dirichlet problems [137, 170] applied to the problem defining the harmonic function $h_{\mathbf{v}\cdot\mathbf{g}'}$, and the trace theorem then imply

$$|h_{\mathbf{v}\cdot\mathbf{g}'}|_{q+3} \le C^{3,1}_q |h_{\mathbf{v}\cdot\mathbf{g}'}|_{q+5/2,\partial\Omega} \le C^{3,2}_q |\mathbf{v}\cdot\mathbf{g}'|_{q+3/2,\partial\Omega} \le C^{3,3}_q |\mathbf{g}'|_{q+2} \le C^3_q |\mathbf{f}|_q. \tag{10.8}$$

Thus, for any $\mathbf{f} \in \mathbb{H}(\Omega)$ the field $\mathbf{g} = \mathbf{g}' - \nabla h_{\mathbf{v}\cdot\mathbf{g}'} \in \mathbb{H}(\Omega)$ is a solution to the Poisson equation $\nabla^2\mathbf{g} = \mathbf{f}$ in Ω, which obeys inequality (10.5) and satisfies the required boundary condition (it can be continued to Ω' by a gradient of a harmonic function decaying at infinity). Closure of the graph of the inverse Laplacian that we have just constructed establishes inequality (10.5) for any $\mathbf{g} \in \mathbb{H}^{q+2}(\Omega)$ and any $q \ge 0$ (note that the kernel of the Laplacian ∇^2 is trivial).

We will apply now the *interpolation theorem* [16, 170]:

Theorem 10.2 *Let* $\{\mathbb{X}_0, \mathbb{X}_1\}, \{\mathbb{Y}_0, \mathbb{Y}_1\}$ *be interpolation pairs of separable Hilbert spaces. Suppose* $\mathscr{L} : \mathbb{X}_i \to \mathbb{Y}_i$ *is a continuous linear operator for* $i = 0$ *and* 1. *Then* $\mathscr{L} : [\mathbb{X}_0, \mathbb{X}_1]_\theta \to [\mathbb{Y}_0, \mathbb{Y}_1]_\theta$ *is a continuous linear operator for any* $\theta \in [0, 1]$.

We set $\mathbb{X}_1 = \mathbb{H}^0(\Omega), \mathbb{X}_0 = \mathbb{H}^{2m}(\Omega), \mathbb{Y}_1 = \mathbb{L}_2(\Omega),\ \mathbb{Y}_0 = \mathbb{W}^{2m}_2(\Omega)$, where $m > 0$ is an arbitrary integer and $\mathscr{L}\mathbf{f} = \mathbf{f}|_\Omega$. Clearly, the mapping $\mathscr{L} : \mathbb{X}_1 \to \mathbb{Y}_1$ is continuous. Employing (10.5), we establish the continuity of the mapping $\mathscr{L} : \mathbb{X}_0 \to \mathbb{Y}_0$ by induction in m. By Theorem 10.2, inequality (10.3) holds true for all $q = 2m(1-\theta)$.

2°. It is easy to establish the equivalence of norms for integer indices $q \ge 0$. First let us consider the simpler case of an odd q. It suffices to establish inequality (10.4) for smooth vector fields $\mathbf{f} \in \mathbb{H}(\Omega)$. We find

$$\|\mathbf{f}\|^2_q = (((-\nabla^2)^{q/2}\mathbf{f}, (-\nabla^2)^{q/2}\bar{\mathbf{f}})) = (((-\nabla^2)^{(q-1)/2}\mathbf{f}, (-\nabla^2)^{(q+1)/2}\bar{\mathbf{f}}))$$

(by self-adjointness of the Laplacian)

$$= (((-\nabla^2)^{(q-1)/2}\mathbf{f}, (\nabla\times)^2(-\nabla^2)^{(q-1)/2}\bar{\mathbf{f}}))$$

(since $\mathbf{f}$ is solenoidal)

$$= \int_{\Omega} |\nabla \times (-\nabla^2)^{(q-1)/2} \mathbf{f}|^2 \, d\mathbf{x}$$

(by (10.2), since $(-\nabla^2)^{(q-1)/2}\mathbf{f} \in \widetilde{\mathbb{H}}(\Omega)$). This identity demonstrates inequality (10.4) for odd q.

Let now $q > 0$ be an even integer. To prove (10.4), we must show, that for any $\mathbf{f} \in \mathbb{H}(\Omega)$ the $\mathbb{L}_2(\Omega')$ norm of the continuation of $(\nabla^2)^{q/2}\mathbf{f}$ to Ω' is bounded by $|\mathbf{f}|_q$.

By the trace theorem, for any $q > 1/2$ and any field $\mathbf{f} \in \mathbb{W}_2^q(\Omega)$ there exists a trace $\mathbf{f}|_{\partial\Omega} \in \mathbb{W}_2^{q-1/2}(\partial\Omega)$, and this mapping $\mathbb{W}_2^q(\Omega) \to \mathbb{W}_2^{q-1/2}(\partial\Omega)$ is continuous. Moreover, if the field $\mathbf{f} \in \mathbb{W}_2^q(\Omega)$ is solenoidal, then the field $\boldsymbol{v} \cdot \mathbf{f}|_{\partial\Omega}$ is also well-defined for any $q \geq 0$ and satisfies the bound

$$|\boldsymbol{v} \cdot \mathbf{f}|_{q-1/2,\partial\Omega} \leq C_q^4 |\mathbf{f}|_q. \tag{10.9}$$

For $0 \leq q \leq 1/2$, the function $\boldsymbol{v} \cdot \mathbf{f}$ is defined for solenoidal $\mathbf{f}$ by the relation

$$\int_{\partial\Omega} h\boldsymbol{v} \cdot \mathbf{f} \, d\sigma = \int_{\Omega} (\mathbf{f} \cdot \nabla)\tilde{h} \, d\mathbf{x}$$

satisfied for all test functions $h \in \mathbb{W}_2^{1/2-q}(\partial\Omega)$, where $\tilde{h} \in \mathbb{W}_2^{1-q}(\Omega)$ is an arbitrary continuation of h into Ω (the continuation is guaranteed to exist by the trace theorem).

Consider the identity

$$\int_{\Omega'} (2|\nabla F|^2 \xi - |F|^2 \nabla^2 \xi) \, d\mathbf{x} - \int_{\partial\Omega} |F|^2 (\boldsymbol{v} \cdot \nabla)\xi \, d\sigma = -2\mathrm{Re} \int_{\partial\Omega} \xi F (\boldsymbol{v} \cdot \nabla) F \, d\sigma, \tag{10.10}$$

where the potential F of the continuation of $(\nabla^2)^{q/2}\mathbf{f}$ to Ω' is a solution to the Neumann problem

$$\nabla^2 F = 0 \text{ in } \Omega', \quad (\boldsymbol{v} \cdot \nabla)F|_{\partial\Omega} = \boldsymbol{v} \cdot (\nabla^2)^{q/2}\mathbf{f}|_{\partial\Omega}, \quad F \to 0 \text{ for } |\mathbf{x}| \to \infty,$$

and $\xi \in \mathbb{C}^\infty(\overline{\Omega'})$ is an arbitrary function satisfying in Ω' the inequalities $|\xi| < M, |\nabla\xi| < M/|\mathbf{x}|, |\nabla^2\xi| < M/|\mathbf{x}|$ (they guarantee existence of the first integral in (10.10) and vanishing of the surface integral emerging at infinity, when the integral of divergence is evaluated in the course of derivation of (10.10)). Suppose, in addition, that the following inequalities hold true:

$$\nabla^2 \xi < 0 \text{ in } \Omega', \quad \inf_{\Omega'} \xi > 0, \quad (\boldsymbol{v} \cdot \nabla)\xi|_{\partial\Omega} \leq 0,$$

for instance, $\xi = (M' - \exp(-M''|\mathbf{x}|))(\xi' + M''')$, where ξ' is a solution to the Neumann problem

$$\nabla^2\xi' = 0 \text{ in } \Omega', \quad (\boldsymbol{v}\cdot\nabla)\xi'|_{\partial\Omega} = -1, \quad \xi' \to 0 \text{ for } |\mathbf{x}| \to \infty,$$

and the positive constants M', M'' and M''' are sufficiently large (we assume that the origin is inside Ω).

Let $\Omega'' \supset \overline{\Omega}$ be a bounded open region and $\Omega''' = \Omega''\backslash\overline{\Omega}$. Applying the trace theorem in Ω''' and inequality (10.9), we obtain a bound for the r.h.s. of (10.10):

$$\begin{aligned}\left|\int_{\partial\Omega} \xi F(\boldsymbol{v}\cdot\nabla)F\,\mathrm{d}\sigma\right| &\le \max_{\mathbf{x}\in\partial\Omega}\xi|F|_{1/2,\partial\Omega}|(\boldsymbol{v}\cdot\nabla)F|_{-1/2,\partial\Omega}\\ &\le C^{5,1}|F|_{1,\Omega'''}|\boldsymbol{v}\cdot(\nabla^2)^{q/2}\mathbf{f}|_{-1/2,\partial\Omega}\\ &\le C^{5,2}(C^{5,3}|F|^2_{1,\Omega'''}+|(\nabla^2)^{q/2}\mathbf{f}|_0^2/C^{5,3}),\end{aligned} \qquad (10.11)$$

where $|\cdot|_{1,\Omega'''}$ denotes the norm in the Sobolev space $\mathbb{W}_2^1(\Omega''')$ and $C^{5,3}$ is any positive constant. Since $\overline{\Omega}'''$ is compact, by the imposed inequalities for ξ, this function and $-\nabla^2\xi$ have strictly positive minima in $\overline{\Omega}'''$. Consequently, choosing a sufficiently small $C^{5,3}$, we infer from (10.10) and (10.11) that

$$\int_{\Omega'}(2|\nabla F|^2\xi - |F|^2\nabla^2\xi)\,\mathrm{d}\mathbf{x} \le C_q^5|\mathbf{f}|_q^2 + \int_{\Omega'''}\left(|\nabla F|^2\xi - \frac{1}{2}|F|^2\nabla^2\xi\right)\mathrm{d}\mathbf{x},$$

whereby $|\nabla F|_{0,\Omega'} \le C_q^6|\mathbf{f}|_q$ (recall that ξ is bounded from below by a positive constant in Ω'). This demonstrates inequality (10.4) for even $q \ge 0$. □

10.1.3 Bounds in the Norms of Sobolev Spaces

We consider henceforth in this chapter a fluid volume Ω obtained by rotation of a bounded open region $\omega \subset \{(r,z)|r \ge 0\} \subset \mathbb{R}^2$, with a smooth boundary, about the axis $r = 0$ (here $\mathbf{x} = (r,\varphi,z)$ are cylindrical coordinates in space). The complementary set $\Omega' = \mathbb{R}^3\backslash\overline{\Omega}$ is supposed to be connected.

We will use Lemma 10.4 to derive uniform bounds in the proof of asymptotic convergence of power series representing solutions to the dynamo problem that we consider. Demonstration of Lemma 10.4 relies on the following lemma:

Lemma 10.3 *Suppose* $\mathbf{f} = \sum_j \mathbf{f}_j(r,z)\mathrm{e}^{\mathrm{i}j\varphi} \in \mathbb{W}_2^q(\Omega)$ *for* $q \ge 0$. *Then*

$$\sum_j |j|^{2q}\int_\omega |\mathbf{f}_j|^2 r\,\mathrm{d}r\,\mathrm{d}z \le C_q|\mathbf{f}|_q^2. \qquad (10.12)$$

(*The constants* C_q *are independent of* $\mathbf{f}$.)

Proof We intend to use the interpolation theorem and set $\mathbb{X}_0 = \mathbb{W}_2^m(\Omega)$, where $m > 0$ is an integer, and $\mathbb{X}_1 = \mathbb{L}_2(\Omega)$.

Arbitrary vector fields $\mathbf{f}, \mathbf{g} \in \mathbb{C}^\infty(\Omega)$ defined in Ω can be expanded in the Fourier series in φ:

$$\mathbf{f} = \sum_j \mathbf{f}_j(r, z)\mathrm{e}^{\mathrm{i}j\varphi}, \quad \mathbf{g} = \sum_j \mathbf{g}_j(r, z)\mathrm{e}^{\mathrm{i}j\varphi}.$$

We define a scalar product

$$(\mathbf{f}, \mathbf{g})_{m,\theta} \equiv 2\pi \sum_j (1 + |j|^{2(1-\theta)m}) \int_\omega \mathbf{f}_j \cdot \bar{\mathbf{g}}_j r \,\mathrm{d}r \,\mathrm{d}z,$$

and denote by $\mathbb{Y}_\theta$ the closure of $\mathbb{C}^\infty(\overline{\Omega})$ in the norm induced by the scalar product $(\cdot, \cdot)_{m,\theta}$. Clearly, $\mathbb{Y}_\theta = [\mathbb{Y}_0, \mathbb{Y}_1]_\theta$. We set $\mathscr{L}\mathbf{f} = \mathbf{f}$ for any $\mathbf{f} \in \mathbb{L}_2(\Omega)$. The linear operator $\mathscr{L} : \mathbb{X}_1 \to \mathbb{Y}_1$ is continuous, being the identity. Since in the Cartesian coordinate system $\partial/\partial\varphi = -y\partial/\partial x + x\partial/\partial y$ and $(\mathbf{f}, \mathbf{f})_{m,0} = |\mathbf{f}|_0^2 + |\partial^m \mathbf{f}/\partial\varphi^m|_0^2$, the restriction $\mathscr{L} : \mathbb{X}_0 \to \mathbb{Y}_0$ is also continuous. By the interpolation theorem, inequality (10.12) with an appropriate constant C_q holds true for all $0 \le q = m(1 - \theta) \le m$. □

In this chapter, $\langle\cdot\rangle$ denotes the mean over the fast azimuthal variable:

$$\langle \mathbf{g}(\mathbf{x}, \Phi) \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} \mathbf{g}(\mathbf{x}, \Phi) \,\mathrm{d}\Phi;$$

the respective fluctuating part of the field is denoted by $\{\mathbf{g}(\mathbf{x}, \Phi)\} \equiv \mathbf{g} - \langle \mathbf{g} \rangle$.

Lemma 10.4 *Suppose* $\beta \ge 0$, *and a matrix* $\mathscr{W}(\mathbf{x}, \Phi) \in \mathbb{C}^\infty(\overline{\Omega} \times [0, 2\pi])$ *of size* 3×3 *is* 2π*-periodic in* Φ,

$$\mathscr{W} \text{ and all its partial derivatives vanish for } r = 0 \tag{10.13}$$

if the region Ω *intersects with the axis* $r = 0$, *and*

$$\langle \mathscr{W} \rangle = 0. \tag{10.14}$$

Then

$$\left| \int_\Omega \mathbf{h} \cdot \mathscr{W}(\mathbf{x}, n\varphi)\mathbf{f} \,\mathrm{d}\mathbf{x} \right| \le C_{\beta,\mathscr{W}} n^{-\beta} (|\mathbf{h}|_0 |\mathbf{f}|_\beta + |\mathbf{h}|_\beta |\mathbf{f}|_0). \tag{10.15}$$

Here the constant $C_{\beta,\mathscr{W}}$ *is independent of* $\mathbf{h}$ *and* $\mathbf{f}$.

Proof We apply the Fourier series

$$\mathscr{W}(\mathbf{x}, \Phi) = \sum_{p,q} \mathscr{W}_{pq}(r, z)\mathrm{e}^{\mathrm{i}(p\varphi + q\Phi)},$$

$$\mathbf{h}(\mathbf{x}) = \sum_j \mathbf{h}_j(r,z)\mathrm{e}^{\mathrm{i}j\varphi} \in \mathbb{W}_2^\beta(\boldsymbol{\Omega}), \quad \mathbf{f}(\mathbf{x}) = \sum_j \mathbf{f}_j(r,z)\mathrm{e}^{\mathrm{i}j\varphi} \in \mathbb{W}_2^\beta(\boldsymbol{\Omega})$$

to calculate the integral in the l.h.s. of (10.15):

$$\left|\int_\Omega \mathbf{h}\cdot\mathscr{W}(\mathbf{x},n\varphi)\mathbf{f}\,\mathrm{d}\mathbf{x}\right| = 2\pi\left|\sum_{p,q,j}\int_\omega \mathbf{h}_j\cdot\mathscr{W}_{pq}\mathbf{f}_{-j-p-nq}r\,\mathrm{d}r\,\mathrm{d}z\right|.$$

For every pair p, q, we split the sum in j in the r.h.s. into three ones: a term is assigned to the first sum, if $|j|\geq n/3$; otherwise, if $|j+p+nq|\geq n/3$, to the second sum; otherwise, to the third one. Applying the Cauchy–Schwarz–Buniakowski inequality to each sum, we obtain an upper bound for the r.h.s. of the above identity:

$$\begin{aligned}2\pi\sum_{p,q}\sup_\omega|\mathscr{W}_{pq}|\Bigg[&\left(\int_\omega\sum_{|j|\geq n/3}|\mathbf{h}_j|^2\left|\frac{j}{n/3}\right|^{2\beta}r\,\mathrm{d}r\,\mathrm{d}z\right)^{1/2}\left(\int_\omega\sum_j|\mathbf{f}_{-j-p-nq}|^2r\,\mathrm{d}r\,\mathrm{d}z\right)^{1/2}\\&+\left(\int_\omega\sum_j|\mathbf{h}_j|^2r\,\mathrm{d}r\,\mathrm{d}z\right)^{1/2}\left(\int_\omega\sum_{|j+p+nq|\geq n/3}|\mathbf{f}_{-j-p-nq}|^2\left|\frac{j+p+nq}{n/3}\right|^{2\beta}r\,\mathrm{d}r\,\mathrm{d}z\right)^{1/2}\\&+\left(\int_\omega\sum_{\substack{|j|<n/3,\\|j+p+nq|<n/3}}|\mathbf{h}_j|^2r\,\mathrm{d}r\,\mathrm{d}z\right)^{1/2}\left(\int_\omega\sum_{\substack{|j|<n/3,\\|j+p+nq|<n/3}}|\mathbf{f}_{-j-p-nq}|^2r\,\mathrm{d}r\,\mathrm{d}z\right)^{1/2}\Bigg].\end{aligned} \tag{10.16}$$

Here we have denoted

$$|\mathscr{W}|\equiv\sup_{|\mathbf{e}_1|=|\mathbf{e}_2|=1}|(\mathscr{W}\mathbf{e}_1,\mathbf{e}_2)|.$$

Since $\mathscr{W}$ is a smooth (matrix-valued) periodic function of the azimuthal variables φ and Φ (in view of (10.13), no singularities arise near the axis $r=0$), it satisfies the following inequality (see, e.g., [85]): for any $E\geq 1$ and $\beta\geq 0$,

$$2\pi\sum_{|p|+|q|\geq E}\sup_\omega|\mathscr{W}_{pq}|\leq C_\beta E^{-\beta}. \tag{10.17}$$

Since $\mathscr{W}$ is zero-mean (10.14), $q\neq 0$ in the sum (10.16), and hence the last term is non-zero only for $|p|\geq n/3$. We find a bound for (10.16),

$$\left|\int_\Omega \mathbf{h}\cdot\mathscr{W}(\mathbf{x},n\varphi)\mathbf{f}\,\mathrm{d}\mathbf{x}\right|\leq C'_\beta(3/n)^\beta(|\mathbf{h}|_0|\mathbf{f}|_\beta+|\mathbf{h}|_\beta|\mathbf{f}|_0+|\mathbf{h}|_0|\mathbf{f}|_0),$$

applying inequality (10.17) and Lemma 10.3. This proves (10.15). □

Corollary *Suppose* $\delta \geq \beta \geq 0$ *and* $\mathscr{D}$ *is a differential operator of order* $m \geq 0$ *with smooth coefficients in* $\overline{\Omega}$. *Under the conditions of Lemma* 10.4,

$$|\mathscr{W}(\mathbf{x}, n\varphi)\mathscr{D}\mathbf{h}|_{-\delta} \leq C_{\delta,\beta,\mathscr{W}} n^{-\beta} |\mathbf{h}|_{\beta+m}.$$

10.2 The Kinematic Dynamo Problem

The problem of kinematic generation of a large-scale magnetic field that we consider in this chapter is analogous to the one that we studied in Chaps. 3, 4, 5. The difference is in that now the flow amplitude depends on the scale ratio, and the flow depends on both fast and slow spatial variables; this gives rise to a different asymptotics of magnetic modes and their growth rates.

We assume that the fluid occupies a bounded axisymmetric volume Ω, which has a cross-section $\omega \subset \{(r,z)|r \geq 0\}$ at each half-plane $\varphi =$ constant. The boundary of Ω is smooth, and the complement $\Omega' = \mathbb{R}^3 \backslash \overline{\Omega}$ is connected. Cylindrical coordinates (r, φ, z) of a point $\mathbf{x} \in \Omega$ have now the sense of the *slow* spatial variables, and $\Phi = n\varphi$ is the *fast* spatial variable.[1] To maintain geometric consistency, $n \to \infty$ is supposed to be integer. The case of three spherical fast variables will be considered in the next section.

10.2.1 Statement of the Problem

Suppose a smooth vector field $\mathbf{W}(\mathbf{x}, \Phi)$ is defined in Ω, which is 2π-periodic in the azimuthal variables Φ and φ, has a zero mean over the fast variable Φ,

$$\langle \mathbf{W} \rangle = 0, \tag{10.18}$$

and satisfies the condition of smoothness near the axis $r = 0$:

$$\mathbf{W} \text{ and all its partial derivatives vanish for } r = 0. \tag{10.19}$$

[1] Note that fast variables are denoted in this and the next chapter, unlike in the previous ones, by capital letters. This reflects the level in the hierarchy of scales, at which we observe the MHD system: while previously we focused on an elementary spatial periodicity cell, assuming large-scale effects as perturbations of the small-scale dynamo located there, now we consider the MHD processes in the entire fluid volume affected by small-scale perturbations. In other words, while the mathematical approach remains, at large, unaltered, the physics of the MHD system suggests that now large scales are the most important characteristic scales, and not the small ones; we continue to denote coordinates and time at this physically fundamental scale level by small letters.

Our goal is to determine the spectrum of the operator of magnetic induction

$$\mathscr{L}^n\mathbf{h} \equiv \eta\nabla^2\mathbf{h} + \nabla \times (\mathbf{v}^n \times \mathbf{h}) \tag{10.20}$$

for the flow

$$\mathbf{v}^n = \mathbf{U}(\mathbf{x}) + n^{-1/2}\nabla \times \left(\mathbf{W}(\mathbf{x}, \Phi)|_{\Phi=n\varphi}\right) \tag{10.21}$$

in the space of solenoidal vector fields

$$\nabla \cdot \mathbf{h} = 0. \tag{10.22}$$

The magnetic modes satisfy the boundary conditions (10.1), i.e. we assume that the fluid volume Ω is surrounded by a dielectric. Unlike in the previous chapters, incompressibility of the fluid (i.e., solenoidality of flow (10.21)) is not required (except for Sect. 10.5.3, where we consider the helicity of the flow vector potential).

10.2.2 Formal Asymptotic Expansion of Large-Scale Magnetic Modes

Elliptic operators $\mathscr{L}^n$ have discrete spectra; every eigenvalue Λ^n is associated with a finite-dimensional subspace of eigenfunctions $\mathbf{h}^n$ (ordinary or generalised), belonging to $\mathbb{H}^q(\Omega)$ for all $q \geq 0$. We will construct asymptotic expansions for solutions to the eigenvalue problem

$$\mathscr{L}^n\mathbf{h}^n = \Lambda^n\mathbf{h}^n \tag{10.23}$$

as power series in $n^{-1/2}$:

$$\Lambda^n = \sum_{j\geq 0} n^{-j/2}\lambda_j, \tag{10.24}$$

$$\mathbf{h}^n = \sum_{j\geq 0} n^{-j/2}\mathbf{h}_j(\mathbf{x}, \Phi)|_{\Phi=n\varphi}. \tag{10.25}$$

Terms in the expansion (10.25) of a magnetic mode are required to be 2π-periodic in φ and Φ, and to satisfy the boundary conditions (10.1).

10.2.3 The Hierarchy of Equations for Large-Scale Magnetic Modes

By the chain rule,

$$\nabla \times (\mathbf{g}(\mathbf{x}, \Phi)|_{\Phi=n\varphi}) = \nabla_{\mathbf{x}} \times \mathbf{g} + n\mathscr{N}\mathbf{g}, \quad \mathscr{N}\mathbf{g} \equiv r^{-1}\frac{\partial}{\partial\Phi}(g^z\mathbf{i}_r - g^r\mathbf{i}_z);$$

partial differential operators with the subscript $\mathbf{x}$ denote the operators with differentiation in the slow variables, $\mathbf{i}_r, \mathbf{i}_\varphi$ and $\mathbf{i}_z$ are unit vectors of the cylindrical coordinate system, and the superscripts r, φ and z denote the respective components of a three-dimensional vector field in cylindrical coordinates.

Substitution of expressions (10.20) and (10.21) for the magnetic induction operator, and the series (10.24) and (10.25) into the eigenvalue equation (10.23) yields

$$\sum_{j\ge -4} n^{-j/2}\left(\sum_{m=-1}^{4}(\mathscr{A}_m\langle \mathbf{h}_{j+m}\rangle + \mathscr{B}_m\{\mathbf{h}_{j+m}\}) - \sum_{m=0}^{j}\lambda_{j-m}\mathbf{h}_m\right) = 0, \qquad (10.26)$$

where we have denoted

$$\begin{aligned}
\mathbf{Q} &\equiv \nabla_{\mathbf{x}} \times \mathbf{W}, & \mathbf{W}' &\equiv -W^z\mathbf{i}_r + W^r\mathbf{i}_z,\\
\mathscr{A}_{-1}\mathbf{h} &\equiv \nabla_{\mathbf{x}} \times (\mathbf{Q}\times\mathbf{h}), & \mathscr{B}_{-1}\mathbf{g} &\equiv \nabla_{\mathbf{x}} \times (\mathbf{Q}\times\mathbf{g}),\\
\mathscr{A}_0\mathbf{h} &\equiv \eta\nabla^2\mathbf{h} + \nabla\times(\mathbf{U}\times\mathbf{h}), & \mathscr{B}_0\mathbf{g} &\equiv \eta\nabla^2_{\mathbf{x}}\mathbf{g} + \nabla_{\mathbf{x}}\times(\mathbf{U}\times\mathbf{g}),\\
\mathscr{A}_1\mathbf{h} &\equiv \nabla_{\mathbf{x}}\times(\mathscr{N}\mathbf{W}\times\mathbf{h}) + \mathscr{N}(\mathbf{Q}\times\mathbf{h}), & \mathscr{B}_1\mathbf{g} &\equiv \nabla_{\mathbf{x}}\times(\mathscr{N}\mathbf{W}\times\mathbf{g}) + \mathscr{N}(\mathbf{Q}\times\mathbf{g}),\\
\mathscr{A}_2\mathbf{h} &\equiv 0, & \mathscr{B}_2\mathbf{g} &\equiv 2\eta r^{-2}\frac{\partial^2\mathbf{g}}{\partial\Phi\partial\varphi} + \mathscr{N}(\mathbf{U}\times\mathbf{g}),\\
\mathscr{A}_3\mathbf{h} &\equiv -r^{-2}h^\varphi\frac{\partial^2\mathbf{W}'}{\partial\Phi^2}, & \mathscr{B}_3\mathbf{g} &\equiv r^{-1}\frac{\partial}{\partial\Phi}(\mathscr{N}\mathbf{W}g^\varphi),\\
\mathscr{A}_4\mathbf{h} &\equiv 0, & \mathscr{B}_4\mathbf{g} &\equiv \eta r^{-2}\frac{\partial^2\mathbf{g}}{\partial\Phi^2}.
\end{aligned}$$

(For $j<0, \mathbf{h}_j = 0$ in (10.26) and the sum $\sum_{m=0}^{j}$ is not present.)

The solenoidality condition (10.22) for large-scale magnetic modes translates into the following relations for the mean and fluctuating parts of terms in the expansion (10.25):

$$\nabla_{\mathbf{x}} \cdot \langle \mathbf{h}_j\rangle = 0, \qquad (10.27)$$

$$r^{-1}\frac{\partial}{\partial\Phi}\left\{h^\varphi_{j+2}\right\} + \nabla_{\mathbf{x}}\cdot\{\mathbf{h}_j\} = 0. \qquad (10.28)$$

10.2.4 Solvability Condition for an Equation $\partial^2\{\mathbf{h}\}/\partial\mathbf{\Phi}^2 = \mathbf{g}$

In this subsection we discuss, when there exists a solution, 2π-periodic in φ and Φ, to the equation

$$\frac{\partial^2}{\partial\Phi^2}\{\mathbf{h}\} = \mathbf{g}(\mathbf{x}, \Phi). \qquad (10.29)$$

One could rely on the general theory (the Fredholm alternative theorem), as we did in previous chapters, but this approach is too sophisticated for (10.29).

Integrating (10.29) in Φ once, we find

$$\frac{\partial}{\partial\Phi}\{\mathbf{h}\} = \frac{\partial}{\partial\Phi}\{\mathbf{h}\}\bigg|_{\Phi=0} + \int_0^{\Phi} \mathbf{g}(\mathbf{x},\Phi_1)\,\mathrm{d}\Phi_1; \tag{10.30}$$

evidently, this function is 2π-periodic in Φ whenever

$$\langle \mathbf{g} \rangle = 0. \tag{10.31}$$

If this solvability condition is satisfied, we find the unique solution, integrating (10.30) in Φ the second time:

$$\{\mathbf{h}\} = \left\{ \int^{\Phi} \left\{ \int^{\Phi_2} \mathbf{g}(\mathbf{x},\Phi_1)\,\mathrm{d}\Phi_1 \right\} \mathrm{d}\Phi_2 \right\}.$$

Clearly, it is 2π-periodic in φ provided $\mathbf{g}$ is.

10.2.5 Derivation of the Limit Operator

We equate successively terms in the series in the l.h.s. of (10.26) to zero.

$j = -4$. The equation obtained from (10.26) at order n^2 is $r^{-2}\partial^2/\partial\Phi^2\{\mathbf{h}_0\} = 0$. Twice integrating it in Φ and employing the periodicity condition, we find $\{\mathbf{h}_0\} = 0$.

$j = -3$. At order $n^{3/2}$, Eq. 10.26 yields

$$r^{-2}\frac{\partial^2}{\partial\Phi^2}(\eta\{\mathbf{h}_1\} - \langle h_0^{\varphi}\rangle \mathbf{W}') = 0.$$

Twice integrating it in Φ, we determine

$$\{\mathbf{h}_1\} = \langle h_0^{\varphi}\rangle \mathbf{W}'/\eta. \tag{10.32}$$

$j = -2$. At order n^1 we obtain from (10.26) the equation

$$r^{-2}\frac{\partial^2}{\partial\Phi^2}(\eta\{\mathbf{h}_2\} - \langle h_1^{\varphi}\rangle \mathbf{W}') = 0.$$

Double integration in Φ yields

$$\{\mathbf{h}_2\} = \langle h_1^{\varphi}\rangle \mathbf{W}'/\eta. \tag{10.33}$$

$j = -1$. The equation obtained from (10.26) at order $n^{1/2}$ reduces by virtue of relation (10.32) to

$$r^{-2}\frac{\partial^2}{\partial\Phi^2}\left(\eta\{\mathbf{h}_3\} - \langle h_2^\varphi\rangle\mathbf{W}'\right) + \nabla_{\mathbf{x}}\times(\mathcal{N}\mathbf{W}\times\langle\mathbf{h}_0\rangle) + \mathcal{N}(\mathbf{Q}\times\langle\mathbf{h}_0\rangle$$
$$+\mathbf{U}\times\langle h_0^\varphi\rangle\mathbf{W}'/\eta) - 2r^{-2}\frac{\partial}{\partial\Phi}\left(\frac{\partial}{\partial\varphi}(\langle h_0^\varphi\rangle W^z)\mathbf{i}_r + \langle h_0^\varphi\rangle W^z\mathbf{i}_\varphi - \frac{\partial}{\partial\varphi}(\langle h_0^\varphi\rangle W^r)\mathbf{i}_z\right) = 0. \tag{10.34}$$

Evidently, the solvability condition (10.31) is satisfied (because the operator $\mathcal{N}$ involves differentiation in Φ). Twice integrating this equation in Φ, we obtain an expression for $\eta\{\mathbf{h}_3\} - \langle h_2^\varphi\rangle\mathbf{W}'$ in the terms of $\mathbf{h}_0$. The azimuthal component of (10.34) is equivalent to the condition for the divergence (10.28) for $j = 1$.

$j = 0$. At order n^0, Eq. 10.26 yields, upon substitution of expressions (10.32) and (10.33),

$$r^{-2}\frac{\partial^2}{\partial\Phi^2}(\eta\{\mathbf{h}_4\} - \langle h_3^\varphi\rangle\mathbf{W}') + \nabla_{\mathbf{x}}\times(\mathcal{N}\mathbf{W}\times(\mathbf{h}_1 + \langle h_0^\varphi\rangle\mathbf{W}'/\eta))$$
$$+\mathcal{N}\left(\mathbf{Q}\times(\mathbf{h}_1 + \langle h_0^\varphi\rangle\mathbf{W}'/\eta) + \mathbf{U}\times\langle h_1^\varphi\rangle\mathbf{W}'/\eta\right)$$
$$-2r^{-2}\frac{\partial}{\partial\Phi}\left(\frac{\partial}{\partial\varphi}(\langle h_1^\varphi\rangle W^z)\mathbf{i}_r + \langle h_1^\varphi\rangle W^z\mathbf{i}_\varphi - \frac{\partial}{\partial\varphi}(\langle h_1^\varphi\rangle W^r)\mathbf{i}_z\right)$$
$$+r^{-1}\frac{\partial}{\partial\Phi}\left(\{h_3^\varphi\}\mathcal{N}\mathbf{W}\right) + \eta\nabla^2\langle\mathbf{h}_0\rangle + \nabla\times(\mathbf{U}\times\langle\mathbf{h}_0\rangle) = \lambda_0\langle\mathbf{h}_0\rangle. \tag{10.35}$$

Averaging this equation over Φ, we find

$$\mathcal{L}^\infty\langle\mathbf{h}_0\rangle = \lambda_0\langle\mathbf{h}_0\rangle, \tag{10.36}$$

where

$$\mathcal{L}^\infty\mathbf{h} \equiv \eta\nabla^2\mathbf{h} + \nabla\times(\mathbf{U}\times\mathbf{h} + \alpha h^\varphi\mathbf{i}_\varphi), \quad \alpha(\mathbf{x}) \equiv \frac{2}{\eta r}\left\langle W^z\frac{\partial W^r}{\partial\Phi}\right\rangle. \tag{10.37}$$

Thus, λ_0 is an eigenvalue of the "limit" operator $\mathcal{L}^\infty$, and $\mathbf{h}_0 = \langle\mathbf{h}_0\rangle$ the associated eigenfunction, which is solenoidal (10.27) and satisfies the boundary condition (10.1). We normalise the magnetic mode $\mathbf{h}_0$ in $\mathbb{L}_2(\Omega)$. Subtraction of the eigenvalue equation (10.36) for the limit operator from (10.35) yields

$$r^{-2}\frac{\partial^2}{\partial\Phi^2}(\eta\{\mathbf{h}_4\} - \langle h_3^\varphi\rangle\mathbf{W}') + \nabla_{\mathbf{x}}\times\left(\mathcal{N}\mathbf{W}\times\mathbf{h}_1 + (\mathcal{N}\mathbf{W}\times\mathbf{W}'/\eta - \alpha\mathbf{i}_\varphi)\langle h_0^\varphi\rangle\right)$$
$$+\mathcal{N}\left(\mathbf{Q}\times(\mathbf{h}_1 + \langle h_0^\varphi\rangle\mathbf{W}'/\eta) + \mathbf{U}\times\langle h_1^\varphi\rangle\mathbf{W}'/\eta + r^{-1}\frac{\partial}{\partial\Phi}(\mathbf{W}\{h_3^\varphi\})\right)$$
$$-2r^{-2}\frac{\partial}{\partial\Phi}\left(\frac{\partial}{\partial\varphi}(\langle h_1^\varphi\rangle W^z)\mathbf{i}_r + \langle h_1^\varphi\rangle W^z\mathbf{i}_\varphi - \frac{\partial}{\partial\varphi}(\langle h_1^\varphi\rangle W^r)\mathbf{i}_z\right) = 0, \tag{10.38}$$

for which the solvability condition (10.31) is now satisfied, and from which we determine $\langle \mathbf{h}_4 \rangle - \{h_3^\varphi\}\mathbf{W}'$ in the terms of $\mathbf{h}_j$, $j = 0, 1, 2$. The azimuthal component of Eq. 10.38 is equivalent to the relation for the divergence (10.28) for $j = 2$.

Solution of equations for $j > 0$ is discussed below in Sect. 10.4.1.

10.3 Uniform Boundedness and Convergence of Resolvents

We assume here and in the next section that $\mathbf{U} \in \mathbb{C}^\infty(\overline{\Omega})$, $\mathbf{W}(\mathbf{x}, \Phi) \in \mathbb{C}^\infty(\overline{\Omega} \times [0, 2\pi])$ satisfies the condition (10.19) of smoothness at the axis of symmetry of the region Ω, and

$$(\boldsymbol{v} \cdot \nabla)^k \mathbf{W}|_{\partial\Omega} = 0 \quad \text{for all } k \geq 0. \tag{10.39}$$

Condition (10.19) implies that $r^\delta \mathbf{W}(\mathbf{x}, n\varphi) \in \mathbb{C}^\infty(\overline{\Omega})$ for any $\delta \in \mathbb{R}^1$; we demand this to overcome the difficulties related to the singularity of metric coefficients at the axis $r = 0$. Relations (10.39) guarantee that the boundary condition (10.1) is trivially satisfied for fluctuating components of terms of the asymptotic expansion (10.25) for magnetic modes (the fluctuating components are continued by a zero field into the region occupied by the dielectric).

10.3.1 Uniform Boundedness of Resolvents

Lemma 10.5 *There exists constants C' and $C > 0$, independent of $\mathbf{h} \in \mathbb{H}^{1/2}(\Omega)$ and n, such that for any $\lambda > C'$*

$$\|\mathbf{h}\|_{1/2} \leq C|(\mathscr{L}^n - \lambda)\mathbf{h}|_{-3/2}, \tag{10.40}$$

$$\|\mathbf{h}\|_{1/2} \leq C|(\mathscr{L}^\infty - \lambda)\mathbf{h}|_{-3/2}. \tag{10.41}$$

Proof We scalar multiply (using the scalar product $(\!(\cdot, \cdot)\!)$) the equation

$$\mathbf{f} = (\mathscr{L}^n - \lambda)\mathbf{h} \equiv \eta\nabla^2\mathbf{h} - \lambda\mathbf{h} + \nabla \times (\mathbf{v}^n \times \mathbf{h})$$

by $(-\nabla^2)^{-1/2}\mathbf{h}$ and employ the identity (10.2):

$$- \eta\|\mathbf{h}\|_{1/2}^2 - \lambda\|\mathbf{h}\|_{-1/2}^2 + \int_\Omega (\mathbf{U} \times \mathbf{h}) \cdot (\nabla \times (-\nabla^2)^{-1/2}\mathbf{h})\,\mathrm{d}\mathbf{x} + n^{-1/2}$$
$$\times \int_\Omega ((\nabla \times \mathbf{W}) \times \mathbf{h}) \cdot (\nabla \times (-\nabla^2)^{-1/2}\mathbf{h})\,\mathrm{d}\mathbf{x} = (\!(\mathbf{f}, (-\nabla^2)^{-1/2}\mathbf{h})\!). \tag{10.42}$$

The first integral in (10.42) can be bounded applying inequality (10.3):

$$\left|\int\limits_{\Omega}(\mathbf{U}\times\mathbf{h})\cdot(\nabla\times(-\nabla^2)^{-1/2}\mathbf{h})\,\mathrm{d}\mathbf{x}\right|\le\max_{\Omega}|\mathbf{U}||\mathbf{h}|_0|(-\nabla^2)^{-1/2}\mathbf{h}|_1\le C_u\|\mathbf{h}\|_0^2, \tag{10.43}$$

and the second one applying Lemma 10.4:

$$\begin{aligned}&n^{-1/2}\left|\int\limits_{\Omega}((\nabla\times\mathbf{W})\times\mathbf{h})\cdot(\nabla\times(-\nabla^2)^{-1/2}\mathbf{h})\,\mathrm{d}\mathbf{x}\right|\\&\le C_1\left(|\mathbf{h}|_0|\nabla\times(-\nabla^2)^{-1/2}\mathbf{h}|_{1/2}+|\mathbf{h}|_{1/2}|\nabla\times(-\nabla^2)^{-1/2}\mathbf{h}|_0\right)\le C_2\|\mathbf{h}\|_{1/2}\|\mathbf{h}\|_0.\end{aligned} \tag{10.44}$$

Here and in what follows C_n denote appropriate constants depending only on Ω and the data (η, $\mathbf{U}$ and $\mathbf{W}$) of the problem, but not on n or $\mathbf{h}$.

In order to estimate $((\mathbf{f},(-\nabla^2)^{-1/2}\mathbf{h}))$, for $\mathbf{g}=(-\nabla^2)^{-1/2}\mathbf{h}\in\mathbb{H}^{3/2}(\Omega)$ with a continuation $\hat{\mathbf{g}}=\nabla p$ in Ω' we construct such $\mathbf{a}\in\mathbb{W}^{3/2}(\Omega)$ and p' in Ω that

$$\mathbf{g}|_{\Omega}=\mathbf{a}+\nabla p',\quad \mathbf{a}|_{\partial\Omega}=0, \tag{10.45}$$

$$|\mathbf{a}|_{3/2}\le C_3\|\mathbf{g}\|_{3/2}. \tag{10.46}$$

By part 1° of Lemma 10.2, $\mathbf{g}\in\mathbb{W}^{3/2}(\Omega)$; hence by the trace theorem $\nabla p\in\mathbb{W}^1(\partial\Omega)$ implying $p\in\mathbb{W}^2(\partial\Omega)$, and, by the same theorem, there exists a continuation $p'\in\mathbb{W}^{5/2}(\Omega)$ of p to Ω such that $p'|_{\partial\Omega}=p|_{\partial\Omega}$ and $(\boldsymbol{\nu}\cdot\nabla)p'|_{\partial\Omega}=(\boldsymbol{\nu}\cdot\nabla)p|_{\partial\Omega}$. Therefore, the vector field $\mathbf{a}\equiv\mathbf{g}-\nabla p'$ vanishes (has zero trace) at the boundary $\partial\Omega$, as required by (10.45). The continuation satisfies

$$|p'|_{5/2}\le C_4|p|_{2,\partial\Omega}\le C_5|\mathbf{g}|_{3/2}\le C_6\|\mathbf{g}\|_{3/2},$$

and thus the required inequality (10.46) for $|\mathbf{a}|_{3/2}$ holds true.

For any $\mathbf{f}\in\mathbb{H}(\Omega)$, $\mathbf{f}$ in the region Ω and its continuation $\hat{\mathbf{f}}$ into Ω' are solenoidal. Since

$$\int\limits_{\Omega}\mathbf{f}\cdot\nabla p'\,\mathrm{d}\mathbf{x}+\int\limits_{\Omega'}\hat{\mathbf{f}}\cdot\nabla p\,\mathrm{d}\mathbf{x}=\int\limits_{\Omega}\nabla\cdot(p'\mathbf{f})\,\mathrm{d}\mathbf{x}+\int\limits_{\Omega'}\nabla\cdot(p\hat{\mathbf{f}})\,\mathrm{d}\mathbf{x}=\int\limits_{\partial\Omega}p(\mathbf{f}-\hat{\mathbf{f}})\cdot\boldsymbol{\nu}\,\mathrm{d}\sigma=0,$$

inequality (10.46) implies

$$\begin{aligned}|((\mathbf{f},\mathbf{g}))|&=\left|\int\limits_{\Omega}\mathbf{f}\cdot\mathbf{a}\,\mathrm{d}\mathbf{x}+\int\limits_{\Omega}\mathbf{f}\cdot\nabla p\,\mathrm{d}\mathbf{x}+\int\limits_{\Omega'}\hat{\mathbf{f}}\cdot\nabla p\,\mathrm{d}\mathbf{x}\right|=\left|\int\limits_{\Omega}\mathbf{f}\cdot\mathbf{a}\,\mathrm{d}\mathbf{x}\right|\\&\le|\mathbf{f}|_{-3/2}|\mathbf{a}|_{3/2}\le C_6|\mathbf{f}|_{-3/2}\|\mathbf{g}\|_{3/2}=C_6|\mathbf{f}|_{-3/2}\|\mathbf{h}\|_{1/2}.\end{aligned} \tag{10.47}$$

We substitute inequalities (10.43), (10.44) and (10.47) into identity (10.42), and apply Hölder's inequality and the interpolation inequality $\|\mathbf{h}\|_0^2 \le \|\mathbf{h}\|_{1/2}\|\mathbf{h}\|_{-1/2}$ to obtain

$$\eta\|\mathbf{h}\|_{1/2}^2 + \lambda\|\mathbf{h}\|_{-1/2}^2 \le \frac{\eta}{2}\|\mathbf{h}\|_{1/2}^2 + C_7\|\mathbf{h}\|_{-1/2}^2 + C_8|\mathbf{f}|_{-3/2}^2.$$

We have therefore demonstrated (10.40) for $C' = C_7, C = 2C_8/\eta$.

Furthermore, we have

$$|((\nabla \times (\alpha h^\varphi \mathbf{i}_\varphi), (-\nabla^2)^{-1/2}\mathbf{h}))| = \left| \int_\Omega \alpha h^\varphi \mathbf{i}_\varphi \cdot (\nabla \times (-\nabla^2)^{-1/2}\mathbf{h})\, \mathrm{d}\mathbf{x} \right|$$
$$\le \sup_\Omega |\alpha||\mathbf{h}|_0|\nabla \times (-\nabla^2)^{-1/2}\mathbf{h}|_0 \le C_9\|\mathbf{h}\|_0^2.$$

Consequently, the arguments that we have used to establish (10.40) demonstrate (10.41) for suitably chosen constants C and C'. □

Consider now $\mathscr{L}^n - \lambda$ and $\mathscr{L}^\infty - \lambda : \mathbb{H}^0(\Omega) \to \mathbb{H}^0(\Omega)$ as closed linear operators. The operator $(\eta\nabla^2 - \lambda)^{-1} : \mathbb{H}^q(\Omega) \to \mathbb{H}^{q+2}(\Omega)$ is well-defined for any $\lambda \ge 0$, and hence $\mathscr{R}^n \equiv (\eta\nabla^2 - \lambda)^{-1}\nabla \times (\mathbf{v}^n \times \cdot) : \mathbb{H}^0(\Omega) \to \mathbb{H}^0(\Omega)$ is a compact operator. One can choose a constant $\lambda^n \ge 0$ such that the norm of $\mathscr{R}^n$ is less than 1 for any $\lambda \ge \lambda^n$. Therefore, for any $\lambda \ge \lambda^n$ the operator $\mathscr{L}^n - \lambda$ has a bounded inverse

$$(\mathscr{L}^n - \lambda)^{-1} = \sum_{j \ge 0}(-\mathscr{R}^n)^j(\eta\nabla^2 - \lambda)^{-1}.$$

By inequality (10.40), $\|\mathbf{h}\|_0 \le C_0\|(\mathscr{L}^n - \lambda)\mathbf{h}\|_0$ for $\lambda \ge C'$. Analyticity of the resolvent $(\mathscr{L}^n - \lambda)^{-1}$ in λ together with this bound implies that even if $\lambda^n > C'$, the resolvents $(\mathscr{L}^n - \lambda)^{-1}$ are well-defined for all $\lambda \ge C'$. All these arguments remain applicable to $(\mathscr{L}^\infty - \lambda)^{-1}$.

10.3.2 Convergence of the Resolvents

Lemma 10.6 *If $\lambda > C'$ (here C' is the constant referred to in Lemma* 10.5), *then*

$$\|(\mathscr{L}^n - \lambda)^{-1} - (\mathscr{L}^\infty - \lambda)^{-1}\|_0 \le C_\lambda n^{-1/2}. \tag{10.48}$$

Proof For an arbitrary $\mathbf{f} \in \mathbb{H}(\Omega)$, we denote

$$\begin{aligned} \mathbf{h}^\infty &\equiv (\mathscr{L}^\infty - \lambda)^{-1}\mathbf{f}, \\ \mathbf{h}^n &\equiv (\mathscr{L}^n - \lambda)^{-1}\mathbf{f} \end{aligned} \tag{10.49}$$

and $\mathbf{g} \equiv (h^\infty)^\varphi \mathbf{W}'/\eta$. A straightforward calculation yields

$$\begin{aligned}(\mathscr{L}^n-\lambda)(\mathbf{h}^\infty+n^{-1/2}\mathbf{g}-\mathbf{h}^n) &= n^{1/2}(\mathcal{N}(\mathbf{Q}\times\mathbf{h}^\infty)+\nabla_\mathbf{x}\times(\mathcal{N}\mathbf{W}\times\mathbf{h}^\infty))\\ &+n^{-1/2}\left(\nabla_\mathbf{x}\times(\mathbf{Q}\times\mathbf{h}^\infty)+\eta\left(\nabla^2_\mathbf{x}\mathbf{g}+\frac{2n}{r^2}\frac{\partial}{\partial\Phi}\left(\frac{\partial g^r}{\partial\varphi}\mathbf{i}_r+g^r\mathbf{i}_\varphi+\frac{\partial g^z}{\partial\varphi}\mathbf{i}_z\right)\right)\right.\\ &+\nabla_\mathbf{x}\times(\mathbf{U}\times\mathbf{g})+n\mathcal{N}(\mathbf{U}\times\mathbf{g})+n^{-1/2}\nabla_\mathbf{x}\times(\mathbf{Q}\times\mathbf{g})\\ &\left.+n^{1/2}\mathcal{N}(\mathbf{Q}\times\mathbf{g})+n^{1/2}\nabla_\mathbf{x}\times((\mathcal{N}\mathbf{W}\times\mathbf{g})-(h^\infty)^\varphi\alpha\mathbf{i}_\varphi)+n^{3/2}\mathcal{N}(\mathcal{N}\mathbf{W}\times\mathbf{g})-\lambda\mathbf{g}\right).\end{aligned}$$

By the corollary to Lemma 10.4, the norm $|\cdot|_{-3/2}$ of every term in this sum is bounded by quantities $C^1n^{-1/2}|\mathbf{h}^\infty|_2$ (as before, C^j and $C^{j,k}$ denote various appropriate constants emerging in the course of calculations, that depend only on Ω and the data in the problem: $\eta, \mathbf{U}$ and $\mathbf{W}$). Consequently,

$$|(\mathscr{L}^n-\lambda)(\mathbf{h}^\infty+n^{-1/2}\mathbf{g}-\mathbf{h}^n)|_{-3/2}\le C^2n^{-1/2}|\mathbf{h}^\infty|_2. \tag{10.50}$$

A solution to the Dirichlet problem

$$\nabla^2p'=\nabla\cdot\mathbf{g}\ \text{in}\ \Omega,\quad p'|_{\partial\Omega}=0$$

satisfies the inequality

$$|p'|_{q+2}\le C^3_q|\nabla\cdot\mathbf{g}|_q \tag{10.51}$$

for any $q\ge 0$. We define

$$p|_\Omega\equiv h_{(\boldsymbol{\nu}\cdot\nabla)p'}-p',\quad p|_{\Omega'}\equiv h_{(\boldsymbol{\nu}\cdot\nabla)p'}$$

(here h_F is the single layer potential calculated by formula (10.6)); $p|_{\partial\Omega}$ can be defined by continuity. Evidently, for a field $\mathbf{g}$ infinitely continuous in $\overline{\Omega}$, this construction yields a solution $p\in\mathbb{C}^1(\mathbb{R}^3)$ (note that $\nabla p'|_{\partial\Omega}$ is normal to the boundary $\partial\Omega$ by virtue of the boundary condition for p'). Applying again a bound for solutions to the Dirichlet problem for the Laplace equation that the harmonic function $h_{(\boldsymbol{\nu}\cdot\nabla)p'}$ satisfies in Ω, inequality (10.7) for the single layer potential, the trace theorem and the bound (10.51), we establish (similarly to how we derived inequality (10.8)), that

$$\begin{aligned}|h_{(\boldsymbol{\nu}\cdot\nabla)p'}|_2 &\le C^{4,1}|h_{(\boldsymbol{\nu}\cdot\nabla)p'}|_{3/2,\partial\Omega}\le C^{4,2}|(\boldsymbol{\nu}\cdot\nabla)p'|_{1/2,\partial\Omega}\\ &\le C^{4,3}|\nabla p'|_1\le C^{4,4}|p'|_2\le C^{4,5}|\nabla\cdot\mathbf{g}|_0.\end{aligned}$$

Thus, we have constructed a function $p\in\mathbb{C}^1(\mathbb{R}^3)$ such that $\nabla^2p+\nabla\cdot\mathbf{g}=0$ in $\Omega, \nabla^2p=0$ in Ω', $p\to 0$ for $|\mathbf{x}|\to\infty$ and $|p|_2\le C^{5,1}|\nabla\cdot\mathbf{g}|_0\le C^{5,2}|\mathbf{h}^\infty|_1$. By the corollary to Lemma 10.4 and this bound,

$$\begin{aligned}|(\mathscr{L}^n-\lambda)\nabla p|_{-3/2} &\le C^{6,1}\Big(|\nabla p|_{1/2}+|\mathbf{U}\times\nabla p|_{-1/2}\\ &\qquad +n^{-1/2}\Big(|(\nabla_{\mathbf{x}}\times\mathbf{W})\times\nabla p|_{-1/2}+n|\mathscr{N}\mathbf{W}\times\nabla p|_{-1/2}\Big)\Big)\\ &\le C^{6,2}\Big(|\nabla p|_{1/2}+n^{-1/2}(|\nabla p|_0+n\cdot n^{-1/2}|\nabla p|_{1/2})\Big)\\ &\le C^{6,3}|\nabla p|_1\le C^{6,4}|\mathbf{h}^\infty|_1.\end{aligned}$$

Combining this inequality with the bound (10.50), we obtain

$$|(\mathscr{L}^n-\lambda)(\mathbf{h}^\infty+n^{-1/2}(\mathbf{g}+\nabla p)-\mathbf{h}^n)|_{-3/2}\le C^7 n^{-1/2}|\mathbf{h}^\infty|_2. \tag{10.52}$$

By construction, $\mathbf{h}^\infty+n^{-1/2}(\mathbf{g}+\nabla p)-\mathbf{h}^n\in\mathbb{H}^0(\Omega)$.

Scalar multiplying (10.49) by $\nabla^2\mathbf{h}^\infty$, we find $\|\mathbf{h}^\infty\|_2\le C^{7,1}|\mathbf{f}|_0+C^{7,2}|\mathbf{h}^\infty|_0$, and hence, by inequalities (10.3) from Lemma 10.2 and (10.41), $|\mathbf{h}^\infty|_2\le C^{7,3}|\mathbf{f}|_0$. By virtue of the latter bound, inequalities (10.52) and (10.40) imply

$$\|\mathbf{h}^\infty+n^{-1/2}(\mathbf{g}+\nabla p)-\mathbf{h}^n\|_{1/2}\le C^8 n^{-1/2}|\mathbf{f}|_0.$$

Furthermore,

$$\|\mathbf{g}+\nabla p\|_0\le C^{9,1}(|\nabla p|_0+|\mathbf{g}|_0)\le C^{9,2}|\mathbf{h}^\infty|_1\le C^{9,3}|\mathbf{f}|_0.$$

The last two inequalities establish the bound (10.48) for convergence of resolvents. □

The two following theorems were proved in [306] by application of the Kato [152] theory of generalised convergence of linear operators in Banach spaces. For a linear operator $\mathscr{L}$, we denote by $\sigma(\mathscr{L})$ and $\rho(\mathscr{L})$ the spectrum and the resolvent set of $\mathscr{L}$, respectively.

Theorem 10.3 *Let $\mathbb{X}$ be a complex Banach space and $\mathscr{L}:\mathbb{X}\to\mathbb{X}$ a closed linear operator. Suppose σ_1 is a bounded part of the spectrum $\sigma(\mathscr{L})$, separated from $\sigma_2=\sigma(\mathscr{L})\backslash\sigma_1$ by a piecewise smooth curve $\tau\subset\rho(\mathscr{L})$. Then the space $\mathbb{X}$ can be decomposed into a direct sum of two $\mathscr{L}$-invariant subspaces $\mathbb{X}_i$ such that the spectrum of $\mathscr{L}|_{\mathbb{X}_i(\mathscr{L})}$ is σ_i (= 1, 2); $\mathscr{L}|_{\mathbb{X}_1(\mathscr{L})}$ is a bounded linear operator.*

Theorem 10.4 *Let $\mathscr{L}^n,\mathscr{L}^\infty:\mathbb{X}\to\mathbb{X}$ be closed linear operators. Suppose, for some N, there exists $\lambda\in\rho(\mathscr{L}^n)\cap\rho(\mathscr{L}^\infty)$ for all $n\ge N$, and*

$$\|(\mathscr{L}^n-\lambda)^{-1}-(\mathscr{L}^\infty-\lambda)^{-1}\|_{\mathbb{X}}=\varepsilon_n\to 0.$$

Let $\Lambda\in\sigma(\mathscr{L}^\infty)$ be an m-fold eigenvalue of the operator $\mathscr{L}^\infty$, separated from $\sigma_2=\sigma(\mathscr{L}^\infty)\backslash\{\Lambda\}$ by a piecewise smooth curve $\tau\subset\rho(\mathscr{L}^\infty)$. Then for all sufficiently large n the following holds true:

(i) *$\tau\subset\rho(\mathscr{L}^n)$, and the part of $\sigma(\mathscr{L}^n)$ inside the curve τ consists of precisely m eigenvalues (counted with their multiplicities) λ_i^n. They converge to Λ:*

$$|\lambda_i^n-\Lambda|\le C_i\varepsilon_n^{1/m}.$$

(ii) *Let* $\mathscr{P}^\infty$ *and* $\mathscr{P}^n$ *be the projections onto the m-dimensional eigenspaces* $\mathbb{X}_1(\mathscr{L}^\infty)$ *and* $\mathbb{X}_1(\mathscr{L}^n)$, *respectively, constructed in Theorem* 10.3 *for the* τ*-separation of the spectra. Then*

$$\|\mathscr{P}^n - \mathscr{P}^\infty\|_X \leq C\varepsilon_n.$$

(iii) *The restriction* $(\mathscr{L}^n - \lambda)^{-1}$ *to the invariant complement to the image of* $\mathscr{P}^n$ *is bounded uniformly in all sufficiently large n and* λ *inside the curve* τ.

By these theorems, the bound (10.48) of Lemma 10.3.2 on convergence of resolvents implies a key result of this chapter:

Theorem 10.5 *Let* Λ^∞ *be an m-fold eigenvalue of the limit operator* $\mathscr{L}^\infty$. *Precisely m eigenvalues* Λ^n_{ik} *(counted with their multiplicities) of the magnetic induction operators* $\mathscr{L}^n$ *converge to* Λ^∞ *with the bound*

$$|\Lambda^n_{ik} - \Lambda^\infty_i| \leq C_i n^{-1/2m}.$$

The respective m-dimensional subspaces converge with the projections satisfying the inequality

$$\|\mathscr{P}^n - \mathscr{P}^\infty\|_0 \leq C_i n^{-1/2}. \tag{10.53}$$

10.4 Complete Asymptotic Expansion of Magnetic Modes with Internal Scaling

In this section we discuss, how to solve equations emerging from the expanded eigenvalue equation (10.26) at orders $j > 0$, and prove that the resultant power series expansions (10.24) and (10.25) are genuine asymptotic series for the magnetic modes and the associated eigenvalues of the magnetic induction operator. For the sake of simplicity, we assume here that the eigenvalue Λ^∞ of the limit operator $\mathscr{L}^\infty$ is of multiplicity $m = 1$; this case is generic.

10.4.1 Solution of the Hierarchy of Equations in all Orders

Lemma 10.7 *All terms in the series* (10.24) *and* (10.25) *can be found successively equating to zero terms in the series in the l.h.s. of Eq.* 10.26. *For any j, relations* (10.28) *for divergencies are satisfied.*

Proof We will solve successively equations

$$\sum_{m=-1}^{4} (\mathscr{A}_m\langle \mathbf{h}_{j+m}\rangle + \mathscr{B}_m\{\mathbf{h}_{j+m}\}) - \sum_{m=0}^{j} \lambda_{j-m}\mathbf{h}_m = 0 \tag{10.54}$$

obtained from Eq. 10.26 at order $n^{-j/2}$. Fluctuating parts of terms in the series (10.25) are sought in the form

$$\{\mathbf{h}_{j+4}\} = \eta^{-1}\Big(\langle h^{\varphi}_{j+3}\rangle\mathbf{W}' - r^2\mathscr{J}^2(\mathscr{A}_1\langle\mathbf{h}_{j+1}\rangle + \mathscr{B}_2(\langle h^{\varphi}_{j+1}\rangle\mathbf{W}'))\Big) + \mathbf{h}'_j, \qquad (10.55)$$

where $\mathscr{J}$ is the operator of zero-mean integration in Φ:

$$\langle\mathscr{J}\mathbf{g}\rangle = 0, \quad \frac{\partial}{\partial\Phi}\mathscr{J}\mathbf{g} = \mathbf{g}(\mathbf{x},\Phi),$$

and $\mathbf{h}'_j$ are fluctuating vector fields to be found. We will show by induction in k, that one can determine the fields $\mathbf{h}'_j$ and $\mathbf{h}_j \in \mathbb{C}^\infty(\overline{\Omega}\times[0,2\pi])$, and coefficients $\lambda_j, 1 \le j \le k$, such that the following equalities are satisfied:

- relations (10.55) for the fluctuating parts $\{\mathbf{h}_{j+4}\}$ for all $j \le k-4$;
- Eqs. 10.54 for all $j \le k$, upon substitution of $\mathbf{h}_j$ and λ_j for $j \le k$ and expressions (10.55) for $\{\mathbf{h}_{j+4}\}$ for $k-3 \le j \le k$;
- relations (10.28) for divergencies for all $j \le k+2$;
- all means $\langle\mathbf{h}_j\rangle$ satisfy the boundary conditions (10.1), the solenoidality conditions (10.22), and the normalisation condition

$$\mathscr{P}^\infty\langle\mathbf{h}_j\rangle = 0 \qquad (10.56)$$

  for $j > 0$, where $\mathscr{P}^\infty$ denotes the projection in $\mathbb{H}^0(\Omega)$ onto the invariant subspace spanned by the eigenfunction $\mathbf{h}_0 = \mathbf{h}^\infty$ associated with the eigenvalue Λ^∞. The fluctuating parts of the terms in the series comprising an eigenmode, $\{\mathbf{h}_j\}$, vanish on the boundary $\partial\Omega$, and hence each term $\mathbf{h}_j$ in the expansion of the magnetic mode satisfies the boundary conditions (10.1).

For $k = 0$ these statements are true by constructions of Sect. 10.2.5, and we must make the induction step. We start by examining the solvability condition for equation (10.54) for $j = k+1$. Averaging of (10.54) over Φ upon substitution of expression (10.55) for $j = k-2$ yields

$$(\mathscr{L}^\infty - \Lambda^\infty)\langle\mathbf{h}_{k+1}\rangle + \mathbf{h}''_{k+1} - \sum_{m=0}^{k}\lambda_{k+1-m}\langle\mathbf{h}_m\rangle = 0, \qquad (10.57)$$

where

$$\mathbf{h}''_{k+1} = \nabla_\mathbf{x}\times\big\langle\mathscr{N}\mathbf{W}\times\big(\mathbf{h}'_{k-2} - \eta^{-1}r^2\mathscr{J}^2(\mathscr{A}_1\langle\mathbf{h}_{k-1}\rangle + \mathscr{B}_2(\langle h^{\varphi}_{k-1}\rangle\mathbf{W}')) + \mathbf{Q}\times\{\mathbf{h}_k\}\big)\big\rangle.$$

Applying the projection $\mathscr{P}^\infty$ to Eq. 10.57 (note $\mathscr{P}^\infty\mathscr{L}^\infty = \mathscr{L}^\infty\mathscr{P}^\infty = \Lambda^\infty\mathscr{P}^\infty$) and taking into account the normalisation condition (10.56) we find

$$\lambda_{k+1}\langle\mathbf{h}_0\rangle = \mathscr{P}^\infty\mathbf{h}''_{k+1}. \qquad (10.58)$$

This equation uniquely determines λ_{k+1}. Subtracting (10.58) from (10.57), we find

$$\langle \mathbf{h}_{k+1} \rangle = (\mathscr{L}^{\infty} - \Lambda^{\infty})^{-1}|_{Im(\mathscr{I}-\mathscr{P}^{\infty})} \left(\sum_{m=1}^{k} \lambda_{k+1-m} \langle \mathbf{h}_m \rangle - (\mathscr{I} - \mathscr{P}^{\infty}) \mathbf{h}''_{k+1} \right)$$

(here $\mathscr{I}$ is the identity operator). A linear bounded operator $(\mathscr{L}^{\infty} - \Lambda^{\infty})^{-1}|_{Im(\mathscr{I}-\mathscr{P}^{\infty})}$ exists and is bounded by Theorem 10.3; smoothness of $\mathbf{h}_{k+1}$ is a consequence of smoothness of the r.h.s. of Eq. 10.57 and ellipticity of the problem. Now, $\{\mathbf{h}_m\}$ are completely determined for all $m \le k+2$ by expressions (10.55) for $j = m-4$.

We consider now the fluctuating part of Eq. 10.54:

$$\begin{aligned} \mathbf{Z}_k \equiv & \frac{1}{r^2} \frac{\partial^2}{\partial \Phi^2} (\eta \{\mathbf{h}_{k+5}\} - \langle h^{\varphi}_{k+4} \rangle \mathbf{W}') + \mathscr{A}_1 \langle \mathbf{h}_{k+2} \rangle + \mathscr{A}_{-1} \langle \mathbf{h}_k \rangle \\ & + \sum_{m=-1}^{3} \{\mathscr{B}_m \{\mathbf{h}_{k+1+m}\}\} - \sum_{m=1}^{k+1} \lambda_{k+1-m} \{\mathbf{h}_m\} = 0. \end{aligned} \tag{10.59}$$

Zero-mean integrating this equation twice in Φ and substituting expressions (10.55) for $k-3 \le j \le k+1$, we determine $\mathbf{h}'_{k+1}$ (after the substitution, all terms involving the unknown means $\langle \mathbf{h}_j \rangle$ for $j \ge k+2$ cancel out in Eq. 10.59). Carrying out straightforward algebraic transformations and applying the induction assumptions that expressions (10.28) for divergencies are valid for all $j \le k+2$, we obtain

$$0 = \frac{1}{r} \frac{\partial}{\partial \Phi} \mathbf{Z}^{\varphi}_k + \nabla_{\mathbf{x}} \cdot \mathbf{Z}_{k-2} = \frac{\eta}{r^2} \frac{\partial^2}{\partial \Phi^2} \left(\frac{1}{r} \frac{\partial}{\partial \Phi} \{h^{\varphi}_{k+5}\} + \nabla_{\mathbf{x}} \cdot \{\mathbf{h}_{k+3}\} \right),$$

which implies relation (10.28) for $j = k+3$. The induction step for $j = k+1$ is completed. □

10.4.2 Convergence of Asymptotic Series

We denote

$$\mathbf{f}^n_k = \sum_{j=0}^{k} n^{-j/2} \mathbf{h}_j(\mathbf{x}, \Phi)|_{\Phi = n\varphi}.$$

Let $p^n_k \in \mathbb{C}^1(\mathbb{R}^3)$ be a solution to the problem

$$\nabla^2 p^n_k = \nabla \cdot \mathbf{f}^n_k \text{ in } \Omega, \quad \nabla^2 p^n_k = 0 \text{ in } \Omega', \quad p^n_k \to 0 \text{ for } |\mathbf{x}| \to \infty \tag{10.60}$$

(to produce it, we use a single layer potential: $p^n_k|_{\Omega} \equiv p' - h_{(\mathbf{v}\cdot\nabla)p'}$, where p' is a solution to the Dirichlet problem $\nabla^2 p' = \nabla \cdot \mathbf{f}^n_k, p'|_{\partial\Omega} = 0$). By construction, $\mathbf{f}^n_k - \nabla p^n_k \in \mathbb{H}^0(\Omega)$, and hence $\mathbf{h}^n_k \equiv \mathscr{P}^n(\mathbf{f}^n_k - \nabla p^n_k)$ are well-defined (here $\mathscr{P}^n$ are the invariant projections of $\mathbb{H}^0(\Omega)$ onto one-dimensional subspaces spanned by eigenfunctions of the operators $\mathscr{L}^n$ associated with the eigenvalues $\Lambda^n \to \Lambda^\infty$). We denote

$$\mathbf{g}_k^n \equiv (\mathscr{I} - \mathscr{P}^n)(\mathbf{f}_k^n - \nabla p_k^n) = \mathbf{f}_k^n - \nabla p_k^n - \mathbf{h}_k^n.$$

Lemma 10.8 *For any $k \geq 0, (k+1)/2 \geq s \geq 0$ the following statements are true*:

$$|\mathbf{h}_k^n|_0 = 1 + \mathrm{O}(n^{-1/2}), \tag{10.61}$$

$$\left| \Lambda^n - \sum_{j=0}^{k} n^{-j/2} \lambda_j \right| = \mathrm{O}(n^{-(k+1)/2}), \tag{10.62}$$

$$|\nabla p_k^n|_s = \mathrm{O}(n^{s-(k+1)/2}), \tag{10.63}$$

$$|\mathbf{g}_k^n|_s = \mathrm{O}(n^{s-(k+1)/2}). \tag{10.64}$$

The bounds (10.61)–(10.64) characterise asymptotic convergence of the partial sums $\mathbf{f}_k^n$ to the magnetic eigenmode $\mathbf{h}_k^n$ in the Sobolev spaces $\mathbb{W}^s(\Omega)$. By the Sobolev embedding theorem this implies asymptotic convergence of the series (10.25) in the norms of $\mathbb{C}^m(\Omega)$ for derivatives of an arbitrary order m.

Proof For any integer $s \geq 0$ we have

$$|\langle \mathbf{h}_j \rangle|_s = \mathrm{O}(1), \quad |\mathbf{h}_j(\mathbf{x}, n\varphi)|_s = \mathrm{O}(n^s), \tag{10.65}$$

and hence

$$|\mathbf{f}_k^n - \langle \mathbf{h}_0 \rangle|_0 = \left| \sum_{j=1}^{k} n^{-j/2} \mathbf{h}_j \right|_0 = \mathrm{O}(n^{-1/2}). \tag{10.66}$$

Since all $\langle \mathbf{h}_j \rangle$ are solenoidal (10.27), and $\{\mathbf{h}_j\}$ satisfy relations (10.28) for divergencies for $j \geq -2$, we find

$$\nabla \cdot \mathbf{f}_k^n = n^{-(k-1)/2} \nabla_{\mathbf{x}} \cdot \mathbf{h}_{k-1} + n^{-k/2} \nabla_{\mathbf{x}} \cdot \mathbf{h}_k.$$

Solutions to the elliptic problem (10.60) satisfy the inequality $|p_k^n|_{q+2} \leq C_q |\nabla \cdot \mathbf{f}_k^n|_q$; in combination with the above identity, it implies the estimate (10.63) for any integer $s \geq 1$.

As $\mathbf{f}_k^n - \nabla p_k^n$ are solenoidal for all k, the inequality

$$\begin{aligned} &|\nabla p_k^n|^2_{0,\mathbb{R}^3} + |(\mathbf{f}_k^n - \nabla p_k^n) - (\mathbf{f}_{k+m}^n - \nabla p_{k+m}^n)|^2_{0,\mathbb{R}^3} \\ &\quad = |\mathbf{f}_k^n - (\mathbf{f}_{k+m}^n - \nabla p_{k+m}^n)|^2_{0,\mathbb{R}^3} \leq 2|\mathbf{f}_k^n - \mathbf{f}_{k+m}^n|^2_{0,\mathbb{R}^3} + 2|\nabla p_{k+m}^n|^2_{0,\mathbb{R}^3} \end{aligned} \tag{10.67}$$

holds for any $m > 0$. The estimate (10.63) for $s = 1$, and equivalence of norms $|\cdot|_0$ and $\|\cdot\|_0$ imply $|\nabla p_k^n|_{0,\mathbb{R}^3} = \mathrm{O}(n^{-(k-1)/2})$ for all $k \geq 1$. Due to this bound and the estimate (10.65) for $s = 0$, the r.h.s. of the inequality (10.67) for $m = 2$ is $O(n^{-k-1})$, this proving the estimate (10.63) for $s = 0$. Together with the bound (10.53) for convergence of invariant projections and bound (10.66), the estimate (10.63) for $s = 0$ implies estimate (10.61). By interpolation properties of the norms $|\cdot|_s$, the estimate (10.63) can be extended to all $s \geq 0$.

The identity

$$\left(\mathscr{L}^n - \sum_{j=0}^{k} n^{-j/2}\lambda_j\right)\mathbf{f}^n_{k+4} = \sum_{j=k+1}^{k+5} n^{-j/2} \sum_{m=-1}^{k+4-j} (\mathscr{A}_m\langle \mathbf{h}_{j+m}\rangle + \mathscr{B}_m\{\mathbf{h}_{j+m}\}) - \sum_{j=k+1}^{2k+4} n^{-j/2} \sum_{m=j-k}^{k+4} \lambda_{j-m}\mathbf{h}_m, \tag{10.68}$$

which can be derived by a straightforward algebra, implies

$$\left|\left(\mathscr{L}^n - \sum_{j=0}^{k} n^{-j/2}\lambda_j\right)\mathbf{f}^n_{k+4}\right|_0 = O(n^{-(k+1)/2}).$$

By estimate (10.63), we have

$$\left|\left(\mathscr{L}^n - \sum_{j=0}^{k} n^{-j/2}\lambda_j\right)\nabla p^n_{k+4}\right|_0 = O(n^{-(k+1)/2}).$$

Denoting

$$\xi^n_k = \left(\mathscr{L}^n - \sum_{j=0}^{k} n^{-j/2}\lambda_j\right)(\mathbf{f}^n_{k+4} - \nabla p^n_{k+4}),$$

we find from the last two estimates, in view of equivalence of the norms $|\cdot|_0$ and $\|\cdot\|_0$, that $\|\xi^n_k\|_0 = O(n^{-(k+1)/2})$. Applying now the projection $\mathscr{P}^n$ to ξ^n_k (note the identities $\mathscr{P}^n\mathscr{L}^n = \mathscr{L}^n\mathscr{P}^n = \Lambda^n\mathscr{P}^n$), we obtain

$$\left\|\left(\Lambda^n - \sum_{j=0}^{k} n^{-j/2}\lambda_j\right)\mathbf{h}^n_{k+4}\right\|_0 = O(n^{-(k+1)/2}),$$

which, in view of (10.61), establishes estimate (10.62). Applying to ξ^n_k the projection $\mathscr{I} - \mathscr{P}^n$, we find

$$\|(\mathscr{L}^n - \Lambda^n)\mathbf{g}^n_{k+4}\|_0 \leq \left\|\left(\Lambda^n - \sum_{j=0}^{k} n^{-j/2}\lambda_j\right)\mathbf{g}^n_{k+4}\right\|_0 + \|(\mathscr{I} - \mathscr{P}^n)\xi^n_k\|_0 = O(n^{-(k+1)/2}).$$

Employing the uniform boundedness of the restriction $(\mathscr{L}^n - \Lambda^n)^{-1}|_{Im(\mathscr{I}-\mathscr{P}^n)}$ for all sufficiently large n (see Theorem 10.4, part iii), we deduce from this estimate that $|\mathbf{g}^n_{k+4}|_0 = O(n^{-(k+1)/2})$. Applying the latter bound in the inequality

$$
\begin{aligned}
|\mathbf{g}_k^n|_0 &= |(\mathscr{I} - \mathscr{P}^n)(\mathbf{f}_k^n - \nabla p_k^n)|_{0,\mathbb{R}^3} \\
&\le |(\mathscr{I} - \mathscr{P}^n)(\mathbf{f}_{k+4}^n - \nabla p_{k+4}^n)|_{0,\mathbb{R}^3} + |(\mathbf{f}_k^n - \nabla p_k^n) - (\mathbf{f}_{k+4}^n - \nabla p_{k+4}^n)|_{0,\mathbb{R}^3} \\
&\le |\mathbf{g}_{k+4}^n|_0 + |\mathbf{f}_k^n - \mathbf{f}_{k+4}^n|_{0,\mathbb{R}^3},
\end{aligned}
$$

we prove the bound (10.64) for $s = 0$.

By virtue of identity (10.68),

$$
\begin{aligned}
\eta\nabla^2\mathbf{g}_{k+4}^n &= \mathscr{L}^n\mathbf{g}_{k+4}^n - \nabla\times\Big((\mathbf{U}(\mathbf{x}) + n^{-1/2}\nabla\times\mathbf{W})\times\mathbf{g}_{k+4}^n\Big) \\
&= \sum_{j=k+1}^{k+5} n^{-j/2}\sum_{m=-1}^{k+4-j}(\mathscr{A}_m\langle\mathbf{h}_{j+m}\rangle + \mathscr{B}_m\{\mathbf{h}_{j+m}\}) - \sum_{j=k+1}^{2k+4} n^{-j/2}\sum_{m=j-k}^{k+4}\lambda_{j-m}\mathbf{h}_m \\
&\quad + \Lambda^n\mathbf{g}_{k+4}^n - \left(\Lambda^n - \sum_{j=0}^{k} n^{-j/2}\lambda_j\right)\mathbf{f}_{k+4}^n - (\mathscr{L}^n - \Lambda^n)\nabla p_{k+4}^n \\
&\quad - \nabla\times\Big((\mathbf{U} + n^{-1/2}\nabla\times\mathbf{W})\times\mathbf{g}_{k+4}^n\Big).
\end{aligned}
\tag{10.69}
$$

For integer $s \ge 0$, estimates (10.63) and (10.65) give rise to the following ones:

$$
\begin{aligned}
&|(\mathscr{L}^n - \Lambda^n)\nabla p_{k+4}^n|_s \le |\nabla p_{k+4}^n|_{s+2} + C_{us}|\nabla p_{k+4}^n|_{s+1} \\
&+ n^{-1/2}C_s\sum_{j=0}^{s+1}\sup_{\Omega}|D^{j+1}\mathbf{W}||\nabla p_{k+4}^n|_{s+1-j} = \mathrm{O}(n^{s-(k+1)/2})
\end{aligned}
$$

(here D^{j+1} denotes partial derivatives of order $j+1$; note that differentiation in the term $n^{-1/2}\nabla\times((\nabla\times\mathbf{W})\times\cdot)$ can be performed by Leibniz' theorem);

$$
\begin{aligned}
&\left|\left(\Lambda^n - \sum_{j=0}^{k} n^{-j/2}\lambda_j\right)\mathbf{f}_{k+4}^n\right|_s = \mathrm{O}(n^{s-(k+1)/2}); \\
&\left|\sum_{j=k+1}^{k+5} n^{-j/2}\sum_{m=-1}^{k+4-j}(\mathscr{A}_m\langle\mathbf{h}_{j+m}\rangle + \mathscr{B}_m\{\mathbf{h}_{j+m}\}) - \sum_{j=k+1}^{2k+4} n^{-j/2}\sum_{m=j-k}^{k+4}\lambda_{j-m}\mathbf{h}_m\right|_s \\
&\quad = \mathrm{O}(n^{s-(k+1)/2}); \\
&|\Lambda^n\mathbf{g}_{k+4}^n - \nabla\times((\mathbf{U} + n^{-1/2}\nabla\times\mathbf{W})\times\mathbf{g}_{k+4}^n)|_s \\
&\quad \le |\Lambda^n||\mathbf{g}_{k+4}^n|_s + C\left(|\mathbf{g}_{k+4}^n|_{s+1} + \sum_{j=0}^{s+1} n^{j+1/2}|\mathbf{g}_{k+4}^n|_{s+1-j}\right).
\end{aligned}
$$

Substituting them into identity (10.69) we conclude that

$$
|\nabla^2\mathbf{g}_{k+4}^n|_s \le C_1|\nabla^2\mathbf{g}_{k+4}^n|_s \le C_2\left(|\mathbf{g}_{k+4}^n|_{s+1} + \sum_{j=0}^{s+1} n^{j+1/2}|\mathbf{g}_{k+4}^n|_{s+1-j}\right) + \mathrm{O}(n^{s-(k+1)/2})
\tag{10.70}
$$

for any integer $s \ge 0$.

Equivalence of norms $|\cdot|_q$ and $\|\cdot\|_q$ for integer $q \geq 0$ (Lemma 10.2) and inequality (10.5) imply

$$\begin{aligned}|\mathbf{g}^n_{k+4}|_1^2 &\leq C_3^1 \|\mathbf{g}^n_{k+4}\|_1^2 = C_3^1 ((-\nabla^2 \mathbf{g}^n_{k+4}, \mathbf{g}^n_{k+4}))\\ &\leq C_3^1 \|\mathbf{g}^n_{k+4}\|_2 \|\mathbf{g}^n_{k+4}\|_0 \leq C_3^2 |\mathbf{g}^n_{k+4}|_2 |\mathbf{g}^n_{k+4}|_0 \leq C_3^3 |\nabla^2 \mathbf{g}^n_{k+4}|_0 |\mathbf{g}^n_{k+4}|_0.\end{aligned}$$

Combining this inequality with the inequality (10.70) for $s = 0$, we infer that the estimate (10.64) for $s = 1$ is satisfied for all $k \geq 4$. Now, using again inequalities (10.5) for $q > 0$ and (10.70) for $s \geq 0$, we derive, incrementally in s, the rough bound $|\mathbf{g}^n_{k+4}|_s = \mathrm{O}(n^{s-(k+1)/2})$. We will infer from it the estimate (10.64) for all $s > 0$ and $k \geq 0$, if we prove that for all $k \geq 0$

$$|\mathbf{g}^n_{k+1} - \mathbf{g}^n_k|_s = \mathrm{O}(n^{s-(k+1)/2}). \tag{10.71}$$

To show this, we note that estimates (10.65) and (10.63) imply

$$|(\mathbf{f}^n_{k+1} - \nabla p^n_{k+1}) - (\mathbf{f}^n_k - \nabla p^n_k)|_s = \mathrm{O}(n^{s-(k+1)/2}). \tag{10.72}$$

Using the identity $\eta \nabla^2 \mathbf{h}^n = \Lambda^n \mathbf{h}^n - \nabla \times (\mathbf{v}^n \times \mathbf{h}^n)$, we obtain (for instance, by induction in s) a bound $|\mathbf{h}^n|_s = \mathrm{O}(n^{\max(s-1/2,0)})$. Since $\mathbf{h}^n_k = \gamma^n_k \mathbf{h}^n$,

$$|(\gamma^n_{k+1} - \gamma^n_k)\mathbf{h}^n|_0 \leq |(\mathbf{f}^n_{k+1} - \nabla p^n_{k+1}) - (\mathbf{f}^n_k - \nabla p^n_k)|_0 = \mathrm{O}(n^{-(k+1)/2});$$

recalling that $|\mathbf{h}^n|_0 = 1 + \mathrm{O}(n^{-1/2})$ (see (10.61)), we deduce

$$|\mathbf{h}^n_{k+1} - \mathbf{h}^n_k|_s = |\gamma^n_{k+1} - \gamma^n_k| |\mathbf{h}^n|_s = \mathrm{O}(n^{\max(s-1/2,0)-(k+1)/2}). \tag{10.73}$$

Combined, bounds (10.72) and (10.73) demonstrate the estimate (10.71) for all non-negative k and s. This concludes the proof of the lemma. □

10.5 Physical Properties of the Axisymmetric Dynamo

Elliptic operators $\mathscr{L}^n$ and $\mathscr{L}^\infty$ have discrete spectra; each eigenvalue is associated with a finite-dimensional subspace of smooth eigenfunctions (ordinary or generalised ones) belonging to $\mathbb{H}^q(\Omega)$ for all $q \geq 0$. We have constructed asymptotic expansions of magnetic modes $\mathbf{h}^n_i$ and the associated eigenvalues Λ^n_i of the magnetic induction operator $\mathscr{L}^n$ for integer positive $n \to \infty$, (10.25) and (10.24), respectively. The leading-order terms of these power series, $\langle \mathbf{h}_0 \rangle$ and λ_0, are eigenfunctions and the associated eigenvalues of the limit operator $\mathscr{L}^\infty$, (10.37):

$$\mathscr{L}^\infty \mathbf{h} \equiv \eta \nabla^2 \mathbf{h} + \nabla \times (\mathbf{U} \times \mathbf{h}) + \nabla \times (\alpha h^\varphi \mathbf{i}_\varphi), \quad \alpha(\mathbf{x}) \equiv \frac{2}{\eta r} \left\langle W^z \frac{\partial W^r}{\partial \Phi} \right\rangle.$$

The new term $\nabla \times (\alpha h^{\varphi} \mathbf{i}_{\varphi})$ describes the magnetic α-effect of the theory of hydromagnetic mean-field dynamo [156]. In this section we will consider the main physical properties of the dynamo[2] under consideration.

10.5.1 The Dynamo Mechanism and the Magnetic Reynolds Numbers

The flow velocity (10.21) is predominantly (up to lower-order terms) meridional:

$$\mathbf{v}^n = n^{1/2} \frac{1}{r} \frac{\partial}{\partial \Phi} (W^z \mathbf{i}_r - W^r \mathbf{i}_z)|_{\Phi = n\varphi} + \mathrm{O}(1).$$

Therefore, on finite intervals of the fast time $T = n^{-1/2} t$ each trajectory remains asymptotically close to the half-plane $\varphi =$ constant. In a well-known kinematic dynamo mechanism, toroidal magnetic field is created from poloidal one by differential rotation [186]. Similarly, in the present dynamo the azimuthal component of the mean magnetic field becomes a source of the meridional fluctuating magnetic field created by advection of the mean field by the fluid motion, which can be described as differential rotation of fluid particles in the meridional half-planes. The variation of this rotation when observed on nearby meridional planes is fast, because the predominant meridional flow depends on the fast azimuthal variable Φ. The leading-order part of the fluctuating component $\{\mathbf{h}^n\}$ of the mode $\mathbf{h}^n$ emerges as a result of a competition between the diffusive decay and advection by the predominant fluctuating meridional flow. Since diffusion is efficient on small spatial scales, the fluctuating part of magnetic field efficiently decays. Consequently, its amplitude sets at a low, $\mathrm{O}(n^{-1/2} \langle \mathbf{h}^n \rangle)$, level selected as a compromise between the vigour of diffusion and advection. The fluctuating component of the magnetic field is predominantly meridional, as it is in the flow:

$$\{\mathbf{h}^n\} = n^{-1/2} \frac{\langle \mathbf{h}^n \rangle^{\varphi}}{\eta} (-W^z \mathbf{i}_r + W^r \mathbf{i}_z)|_{\Phi = n\varphi} + \mathrm{O}(n^{-1}),$$

since no advective mechanism is made available by the flow configuration, which could sustain the fluctuating azimuthal component of the magnetic field at this level.

[2] We do not claim that the mean-field equations that we have derived always imply magnetic field generation, so it is not always appropriate to talk about a dynamo. However, since the α-effect is beneficial for magnetic field generation and many instances of dynamos governed by the equations that we have derived are known, in this section we permit this frivolity of speech to slip out.

The distribution of the current density, $\mathbf{J}^n = \nabla \times \mathbf{h}^n/\mu$, for the mode $\mathbf{h}^n$,

$$\mathbf{J}^n = n^{1/2}(\eta\mu)^{-1}\langle \mathbf{h}^n \rangle^{\varphi} \frac{1}{r}\frac{\partial}{\partial \Phi}(W^r \mathbf{i}_r + W^z \mathbf{i}_z)|_{\Phi = n\varphi} + \mathrm{O}(1)$$

(where μ is magnetic permeability of the fluid), is predominantly meridional as well, and orthogonal everywhere in the conducting fluid volume to the leading-order component of the flow velocity.

The following analogy with the classical mean-field dynamo theory [156] can be drawn: Averaging over the fast variable Φ the eigenvalue equation (10.23) for the magnetic induction operator (10.20), we obtain the magnetic mean-field equation

$$\eta \nabla^2 \langle \mathbf{h}^n \rangle + \nabla \times (\mathbf{U} \times \langle \mathbf{h}^n \rangle) + \nabla \times \mathscr{E}^n = \Lambda^n \langle \mathbf{h}^n \rangle,$$

where a new generating term $\nabla \times \mathscr{E}^n$ emerges. As in the mean-field theory, the mean electromotive force is the averaged vector product of the fluctuating components of the flow and magnetic field,

$$\mathscr{E}^n = n^{-1/2}\langle (\nabla \times \mathbf{W}) \times \{\mathbf{h}^n\} \rangle = \alpha \langle \mathbf{h}^n \rangle^{\varphi} \mathbf{i}_{\varphi} + \mathrm{O}(n^{-1/2}),$$

and it is a linear functional of the mean magnetic field $\langle \mathbf{h}^n \rangle$.

The magnetic Reynolds number can be estimated in the present problem by the usual formula $R_m = |\mathbf{v}^n||L|/|\eta|$, where L is the characteristic spatial scale L. For the whole fluid volume $L = \mathrm{O}(1)$ and hence $R_m = \mathrm{O}(n^{1/2})$. Interestingly, this estimate does not match the ratio of the characteristic values of the advective and diffusive terms, $|\nabla \times (\mathbf{v}^n \times \mathbf{h}^n)|/|\eta \nabla^2 \mathbf{h}^n| = \mathrm{O}(1)$, which it is supposed to represent. Since the flow velocity depends on the fast variable Φ, the flow has an almost periodic structure with the angular size of the periodicity cell (along $\mathbf{i}_{\varphi}$) equal to $2\pi/n$ (the periodicity of the structure is exact, if the vector fields $\mathbf{U}$ and $\mathbf{W}$ are independent of the slow azimuthal variable φ). Therefore, we can also define the local (per a periodicity cell) magnetic Reynolds number $R^l_m = \mathrm{O}(n^{-1/2})$. This implies that the dynamo under consideration is slow.

10.5.2 Comparison with Braginsky's Dynamo

In this section we consider more closely the particular case, in which the data in the problem is axially symmetric (i.e., independent of the slow azimuthal variable φ). Let $\mathbf{U} = \mathbf{U}_p(r,z) + U(r,z)\mathbf{i}_{\varphi}$ and $\mathbf{h}^{\infty} = \nabla \times (\chi(r,z)\mathbf{i}_{\varphi}) + H(r,z)\mathbf{i}_{\varphi}$ be decompositions of the mean flow and an axisymmetric eigenfunction of the limit operator $\mathscr{L}^{\infty}$, respectively, into poloidal and toroidal components. In this setup, the mean part of the flow, $\mathbf{U}$, is responsible for the motion of fluid particles along the trajectories, which lie on toroidal surfaces filling in the region Ω. We recast the eigenvalue problem for an axisymmetric eigenfunction in the terms of the azimuthal and meridional components:

$$\eta(\nabla^2 - r^{-2})\chi - r^{-1}(\mathbf{U}_p \cdot \nabla)(r\chi) + \alpha H = \lambda_0 \chi, \tag{10.74}$$

$$\eta(\nabla^2 - r^{-2})H - r(\mathbf{U}_p \cdot \nabla)(H/r) + (\nabla(U/r) \times \nabla(r\chi))^{\varphi} = \lambda_0 H; \tag{10.75}$$

outside $\Omega, H = 0, (\nabla^2 - r^{-2})\chi = 0; [H]_{\partial\Omega} = [\chi]_{\partial\Omega} = 0$. Equations 10.74 and 10.75 indicate that the $\alpha\omega$-effect (combination of the α-effect and differential rotation) is present in the dynamo under consideration.

Formally, these equations are identical to the equations governing Braginsky's [33] dynamo. Thus, the dynamo presented in this chapter can be regarded as a generalisation of Braginsky's model in the following sense. In both cases, the flow velocity is a sum of a mean vector field and a fluctuating component, playing the rôle of a perturbation of the flow. This enables one to apply the theory of perturbations to the problem, and gives rise to a new term in the mean-field equations, which describes the α-effect. In Braginsky's model, the mean components (up to asymptotically small terms) of the flow velocity and magnetic field are supposed to be predominantly toroidal and axially symmetric; this results in the fluid motion along the circles $r =$ constant, $z =$ constant. Our constructions do not impose such restrictions on the shape of the mean flow velocity or the morphology of the mean magnetic mode in the leading order, nor we require them to have a symmetry of any kind. There are two main further differences between the problem considered in Braginsky's theory [33] and the one discussed in this chapter:

(i) In Braginsky's dynamo, the fluctuating components of the flow velocity and magnetic field are infinitesimally small in amplitude relative their mean parts. In the dynamo under consideration, this is true for the fluctuating component of the magnetic field; by contrast, the fluctuating component of the flow is asymptotically large in comparison to the mean flow.

(ii) Opposite to the model considered here, in Braginsky's theory equations (10.74) and (10.75) establish relations between the so-called "effective" quantities comprised of different-order terms of expansions.

10.5.3 *Helicity of the Flow Potential and the Coefficient of Magnetic α-Effect*

Since the famous papers [212, 213] were published, the presence of magnetic α-effect is often attributed to non-vanishing of helicity of the fluid flow (although it is known for a long time that not only helicity contributes to magnetic field generation [157–159]: "helicity is unnecessary for α-effect dynamos, but it helps" [119]; see a discussion in Sect. 3.3.2). Recall that the integral of helicity of the flow $\mathbf{V}$ is

$$H(\mathbf{V}) \equiv \int_{\Omega} (\nabla \times \mathbf{V}) \cdot \mathbf{V}\, d\mathbf{x}.$$

A simple Lemma 10.9 which we prove below shows that in the dynamo studied here the α-effect is naturally linked to the integral of helicity of the vector potential of the flow, $\mathbf{A}$ (i.e., $\mathbf{V} = \nabla \times \mathbf{A}$):

$$I(\mathbf{V}) \equiv \int_\Omega \mathbf{A} \cdot \mathbf{V}\,\mathrm{d}\mathbf{x}.$$

Although the potential $\mathbf{A}$ is determined up to an arbitrary gradient, it is easy to show that for flows $\mathbf{V}$ satisfying the condition of non-leakage through the boundary, $\mathbf{V} \cdot \boldsymbol{\nu}|_{\partial\Omega} = 0$, the integral of helicity of the vector potential is invariant with respect to the choice of the gradient component of the vector potential.

Lemma 10.9 *Suppose* (10.21) *is an incompressible fluid flow, i.e.* $\nabla \cdot \mathbf{v}^n = \nabla \cdot \mathbf{U} = 0$, *and the fluid does not penetrate through the boundary* $\partial\Omega$:

$$\mathbf{v}^n \cdot \boldsymbol{\nu}|_{\partial\Omega} = \mathbf{U} \cdot \boldsymbol{\nu}|_{\partial\Omega} = 0, \quad \mathbf{W}|_{\partial\Omega} = (\boldsymbol{\nu} \cdot \nabla_\mathbf{x})\mathbf{W}|_{\partial\Omega}.$$

The mean and fluctuating parts of the flow $\mathbf{v}^n$ *make asymptotically independent contributions to the integral of helicity of the vector potential of* $\mathbf{v}^n$, *equal to the integral of helicity of the vector potential of the mean component* $\mathbf{U}$, *and* $-\eta \int_\Omega \alpha\,\mathrm{d}\mathbf{x}$, *respectively.*

Proof We expand $\mathbf{W}$ in the Fourier series in the fast and slow azimuthal variables:

$$\mathbf{W}(\mathbf{x}, \Phi) = \sum_{p,q} \mathbf{W}_{pq}(r, z)\mathrm{e}^{\mathrm{i}(p\varphi + q\Phi)}.$$

Since $\mathbf{W}$ is a smooth (10.19) vector field, the inequality

$$\sum_q \sum_{|p| \geq E} |q| \sup_\Omega |\mathbf{W}_{pq}| \leq C_\beta E^{-\beta} \tag{10.76}$$

holds true for any $E \geq 1$ and $\beta \geq 0$.

Suppose $\mathbf{U} = \nabla \times \mathbf{w}$. Then

$$I(\mathbf{v}^n) = \int_\Omega (\mathbf{w} + n^{-1/2}\mathbf{W}(\mathbf{x}, n\varphi)) \cdot (\mathbf{U} + n^{-1/2}\nabla \times \mathbf{W}(\mathbf{x}, n\varphi))\,\mathrm{d}\mathbf{x}$$

$$= I(\mathbf{U}) + \int_\Omega \mathbf{W}(\mathbf{x}, n\varphi) \cdot \mathcal{N}\mathbf{W}\,\mathrm{d}\mathbf{x} + n^{1/2}\int_\Omega \mathbf{w} \cdot \mathcal{N}\mathbf{W}\,\mathrm{d}\mathbf{x} + \mathrm{O}(n^{-1/2}).$$

Applying the bound (10.76) for $\beta = 1/2$, we find

$$\int_\Omega \mathbf{W}(\mathbf{x}, n\varphi) \cdot \mathcal{N}\mathbf{W}\,\mathrm{d}\mathbf{x} = 2\pi\mathrm{i} \sum_{\substack{p,q,p',q':\\ p+p'+n(q+q')=0}} (q' - q) \int_\omega W^r_{p,q} W^z_{p',q'}\,\mathrm{d}r\,\mathrm{d}z$$

$$= -4\pi\mathrm{i} \sum_{p,q} q \int_\omega W^r_{p,q} W^z_{-p,-q}\,\mathrm{d}r\,\mathrm{d}z + \mathrm{O}(n^{-1/2}) = -\eta \int_\Omega \alpha\,\mathrm{d}\mathbf{x} + \mathrm{O}(n^{-1/2}).$$

Similarly, using (10.76) for $\beta = 1$ and noting $\langle \mathcal{N}\mathbf{W} \rangle = 0$ one can easily establish that

$$\int_{\Omega} \mathbf{w} \cdot \mathcal{N}\mathbf{W}\, d\mathbf{x} = \mathrm{O}(n^{-1}).$$

Therefore,

$$I(\mathbf{v}^n) = I(\mathbf{U}) - \eta \int_{\Omega} \alpha\, d\mathbf{x} + \mathrm{O}(n^{-1/2}),$$

as required. □

10.6 Conclusions

1. We have considered the kinematic dynamo problem of magnetic field generation by a flow (10.21) of electrically conducting fluid with an internal scale, due to dependence on the fast azimuthal variable, in an axisymmetric volume. We have constructed a complete asymptotic expansion of magnetic modes and the associated eigenvalues of the magnetic induction operator in power series in the square root of the spatial scale ratio characterising the fluctuating part of the flow velocity. We have demonstrated asymptotic convergence of the formal asymptotic series and have established bounds for convergence. The leading term of the expansion of a magnetic mode is independent of the fast variable. Interaction of the fluctuating parts of the flow and magnetic field results in emergence of the $\alpha\omega$-effect. When the mean velocity profile is axisymmetric, the leading term in the expansion of a predominantly axisymmetric magnetic mode is described by Braginsky's system of equations.
2. We have shown that the α-effect in the present dynamo is not related to the flow helicity, but rather can be linked with the helicity of the flow vector potential (see Lemma 10.9).
3. Our study is confined to the case of the boundary conditions for magnetic field, corresponding to the presence of a dielectric medium outside the volume occupied by the conducting fluid. A similar expansion for the flow velocity (10.21) satisfying the bounds for convergence (10.61)–(10.64) can be constructed for boundary conditions different from (10.1)—for instance, for a fluid volume Ω of the shape of a spherical shell, separating a non-conducting outer space from the internal infinitely-conducting core (such a geometry is natural for the study of the geodynamo). We also note that rescaling of the data $\eta \to n^{-1/2}\eta$, $\mathbf{v}^n \to n^{-1/2}\mathbf{v}^n$, accompanied by a similar rescaling of the eigenvalues of the magnetic induction operator, does not change the mathematics of this linear problem, but the rescaled problem, with the flow uniformly bounded

in n and magnetic diffusivity asymptotically small, seems more realistic for applications in the theory of planetary dynamos.

4. It is straightforward to construct similar expansions and demonstrate that Theorem 10.5 and Lemmas 10.5–10.9 remain true for flows from a wider class:

$$\mathbf{v}^n = \mathbf{U}(\mathbf{x}) + \nabla \times \left(\mathbf{W}(\mathbf{x}, \Phi, n^{-1/2})|_{\Phi=n\varphi}\right),$$

if $\mathbf{W}$ is zero-mean (10.18) and smooth, vanishes together with derivatives at the axis of symmetry (10.19) of the fluid volume Ω and at the boundary $\partial\Omega$ (10.39), and is analytical in the last argument:

$$\mathbf{W}(\mathbf{x}, \Phi, n^{-1/2}) = \sum_{m \geq 0} \mathbf{W}_m(\mathbf{x}, \Phi) n^{-m/2}.$$

Then, the structure of the limit operator $\mathscr{L}^\infty$ remains unaltered, and the α-effect coefficient is rendered by a similar expression:

$$\alpha(\mathbf{x}) = \eta^{-1}\langle \mathscr{N}\mathbf{W}_0 \times (-W_0^z \mathbf{i}_r + W_0^r \mathbf{i}_z)\rangle = \frac{2}{\eta r}\left\langle W_0^z \frac{\partial W_0^r}{\partial \Phi}\right\rangle.$$

5. Geodynamo models must mimic the two following fundamental properties of the geodynamo [47]: the ohmic dissipation prevails at high harmonics, and the toroidal component of magnetic field in the core is apparently stronger than the poloidal one. The present model, evidently, possesses the first property (because of the presence of the molecular diffusion operator in the limit operator). The second property can be acquired by changing amplitudes of the flow components: By virtue of expression (10.37), a transformation $\mathbf{W} \to \delta^{-1}\mathbf{W}$ results in modification of the α-effect coefficient $\alpha \to \delta^{-2}\alpha$. For instance, in the axisymmetric case considered in Sect. 10.5.2, a simultaneous transformation $U \to \delta^2 U$ gives rise to modification of solutions to Eqs. 10.74 and 10.75 $(\chi, H, \lambda_0) \to (\chi, \delta^2 H, \lambda_0)$. Since $|\delta|$ can be chosen arbitrarily large, the energy of the toroidal component of the modified solutions can be larger than the energy of the poloidal component by an arbitrary factor.

Chapter 11
Magnetic Field Generation by a Two-Scale Flow in a Sphere

In this chapter we consider the problem of magnetic field generation by steady flows of conducting fluid, which have an internal spatial scale and reside in a spherical region Ω surrounded by a dielectric medium. (Without any loss of generality we assume that Ω is an open ball of a unit radius). This problem is analogous to the one examined in the previous chapter. We will use the general results of that chapter concerning the scale of Sobolev spaces of vector fields defined in a region of an arbitrary shape and obeying the above-mentioned boundary conditions.

Unlike in previous chapter, we assume here that the flow velocity has a fluctuating component depending on three fast spatial spherical variables. Following the approach of the previous chapter, we apply methods for homogenisation of elliptic operators to the magnetic induction operator in order to construct complete asymptotic expansions of magnetic modes and the associated eigenvalues in power series in the scale ratio and to prove their asymptotic convergence under the new conditions. Dependence of the flow velocity on three fast spherical variables gives rise to the anisotropic magnetic α-effect with a tensor, whose all entries are in general non-zero. The exposition of the present chapter follows [322].

11.1 Bounds in the Norms of Sobolev Spaces

In this chapter, ρ, θ and φ are the slow spherical variables, denoting the spherical coordinates of a point $\mathbf{x}$, and $\mathrm{P} = n\rho, \Theta = n\theta$ and $\Phi = n\varphi$ are the fast spherical variables; here $n \to \infty$ is integer.

Demonstration of the uniform boundedness and convergence of resolvents of the magnetic induction operator (Sects. 11.3.1 and 11.3.2) and convergence of the expansions which we will construct (Sect. 11.4.2) relies on bounds

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established in Lemma 11.2. The proof of the latter lemma is based on the following one:

Lemma 11.1 *Suppose* $\mathbf{f} \in \mathbb{W}_2^s(\Omega), 0 \le s \le 2,$

$$\psi(\rho)\rho \sin^{5/2}\theta\, \mathbf{f}(\mathbf{x}) = \sum_{k,p,q} \mathbf{f}_{k,p,q} \mathrm{e}^{\mathrm{i}(2\pi k\rho + 2p\theta + q\varphi)}, \tag{11.1}$$

where $\psi(\rho) = \rho^2(1-\rho^2)^2$. *Then the inequality*

$$\sum_{k,p,q} (|k|^{2s} + |p|^{2s} + |q|^{2s})|\mathbf{f}_{k,p,q}|^2 \le C_s |\mathbf{f}|_s^2 \tag{11.2}$$

holds true, where C_s *is a constant independent of* $\mathbf{f}$ *and* $|\cdot|_s$ *denotes, as before, the norm in the Sobolev space* $\mathbb{W}_2^s(\Omega)$.

Since this vector field vanishes together with its first-order derivatives at the endpoints of the intervals of variation of $\rho \in [0,1]$ and $\theta \in [0,\pi]$, the field obtained by continuation by 1-periodicity in ρ and π-periodicity in θ has sufficient smoothness at the endpoints to belong to the Sobolev space $\mathbb{W}_2^s([0,1] \times [0,\pi] \times [0,2\pi])$ as a function of ρ, θ and φ, and the series in the l.h.s. of the inequality (11.2) converges.

Proof Envisaging to apply the interpolation theorem 10.2., we set $\mathbb{X}_1 = \mathbb{L}_2(\Omega)$, $\mathbb{X}_0 = \mathbb{W}_2^2(\Omega)$. Consider Fourier series (11.1) and

$$\psi(\rho)\rho \sin^{5/2}\theta\, \mathbf{g}(\mathbf{x}) = \sum_{k,p,q} \mathbf{g}_{k,p,q} \mathrm{e}^{\mathrm{i}(2\pi k\rho + 2p\theta + q\varphi)}.$$

We define the scalar product

$$(\mathbf{f},\mathbf{g})_\vartheta \equiv 2\pi^2 \sum_{k,p,q} \left(1 + |2\pi k|^{4(1-\vartheta)} + |2p|^{4(1-\vartheta)} + |\mathrm{q}|^{4(1-\vartheta)}\right) \mathbf{f}_{k,p,q} \cdot \overline{\mathbf{g}}_{k,p,q},$$

where $0 \le \vartheta \le 1$. Let the space $\mathbb{Y}_\vartheta$ be the closure of $\mathbb{C}^\infty(\overline{\Omega})$ in the norm induced by this scalar product: $\mathbb{Y}_\vartheta = [\mathbb{Y}_0, \mathbb{Y}_1]_\vartheta$. For any $\mathbf{f} \in \mathbb{L}_2(\Omega)$, we set $\mathscr{L}\mathbf{f} = \mathbf{f}$.

The mapping $\mathscr{L} : \mathbb{X}_1 \to \mathbb{Y}_1$ is continuous, because

$$(\mathbf{f},\mathbf{f})_1 = 4 \int_\Omega |\mathbf{f}|^2 \psi^2(\rho) \sin^4\theta \, \mathrm{d}x \le 4|\mathbf{f}|_0^2.$$

Since

$$(\mathbf{f},\mathbf{f})_0 = |\psi\mathbf{f}\sin^2\theta|_0^2 + \left|\frac{\sin^2\theta}{\rho}\frac{\partial^2}{\partial\rho^2}(\rho\psi\mathbf{f})\right|_0^2 + \left|\frac{\psi}{\sqrt{\sin\theta}}\frac{\partial^2}{\partial\theta^2}(\sin^{5/2}\theta\,\mathbf{f})\right|_0^2 + \left|\psi\sin^2\theta\frac{\partial^2\mathbf{f}}{\partial\varphi^2}\right|_0^2,$$

clearly, the mapping $\mathscr{L}:\mathbb{X}_0\to\mathbb{Y}_0$ is also continuous. Therefore, by the interpolation theorem 10.2, inequality (11.2) holds true for any $s=2(1-\vartheta)$. □

In this chapter, $\langle\cdot\rangle$ denotes the mean over the fast spherical variables P,Θ,Φ:

$$\langle\mathbf{g}(\mathbf{x},P,\Theta,\Phi)\rangle \equiv (2\pi^2)^{-1}\int_0^1\int_0^\pi\int_0^{2\pi}\mathbf{g}(\mathbf{x},P,\Theta,\Phi)\mathrm{d}P\mathrm{d}\Theta\mathrm{d}\Phi.$$

Lemma 11.2 *Suppose* $0\le\beta\le 2$, *matrix*

$$\mathscr{W}(\mathbf{x},P,\Theta,\Phi)\in\mathbb{C}^\infty(\overline{\Omega}\times[0,1]\times[0,\pi]\times[0,2\pi])$$

is periodic in P, Θ *and* Φ *with periods 1,* π *and* 2π, *respectively,*

$$\mathscr{W}(\mathbf{x},P,\Theta,\Phi)\ \textit{and all its partial derivatives vanish for}\ \rho=0,\theta=0,\pi\ \textit{and at}\ \partial\Omega, \tag{11.3}$$

and

$$\langle\mathscr{W}\rangle = 0. \tag{11.4}$$

Then

$$\left|\int_\Omega\mathscr{W}(\mathbf{x},n\rho,n\theta,n\varphi)\mathbf{h}\cdot\mathbf{f}\mathrm{d}\mathbf{x}\right| \le C_{\beta,\mathscr{W}}n^{-\beta}(|\mathbf{h}|_0|\mathbf{f}|_\beta + |\mathbf{h}|_\beta|\mathbf{f}|_0),$$

where the constant $C_{\beta,\mathscr{W}}$ *does not depend on* $\mathbf{h}$ *or* $\mathbf{f}$.

Proof Let $\psi(\rho)$ be the same as in the previous lemma. Consider the Fourier series

$$\psi(\rho)\rho\sin^{5/2}\theta\;\mathbf{h}(\mathbf{x}) = \sum_{k,p,q}\mathbf{h}_{k,p,q}\mathrm{e}^{\mathrm{i}(2\pi k\rho+2p\theta+q\varphi)},$$

$$\psi(\rho)\rho\sin^{5/2}\theta\;\mathbf{f}(\mathbf{x}) = \sum_{k,p,q}\mathbf{f}_{k,p,q}\mathrm{e}^{\mathrm{i}(2\pi k\rho+2p\theta+q\varphi)},$$

$$\frac{2\pi^2\mathscr{W}(\mathbf{x},P,\Theta,\Phi)}{\psi^2(\rho)\sin^4\theta} = \sum_{k,k',p,p',q,q'}\mathscr{W}_{k,k',p,p',q,q'}\mathrm{e}^{\mathrm{i}(2\pi k\rho+2\pi k'P+2p\theta+2p'\Theta+q\varphi+q'\Phi)}. \tag{11.5}$$

By the Cauchy–Schwarz–Buniakowski inequality,

$$\left|\int\limits_\Omega \mathscr{W}(\mathbf{x}, n\rho, n\theta, n\varphi)\mathbf{h}\cdot\mathbf{f}\mathrm{d}\mathbf{x}\right|$$

$$= \left|\sum_{\substack{\mathrm{k,k',p,p',q,q'},\\ k'',p'',q''}} \mathscr{W}_{k,k',p,p',q,q'}\mathbf{h}_{k'',p'',q''}\cdot\mathbf{f}_{-k-nk'-k'',-p-np'-p'',-q-nq'-q''}\right|$$

$$\le \sum_{k,k',p,p',q,q'} |\mathscr{W}_{k,k',p,p',q,q'}| \Bigg(\Big(\sum\nolimits^{(1)} |\mathbf{h}_{k'',p'',q''}|^2 \sum\nolimits^{(1)} |\mathbf{f}_{-k-nk'-k'',-p-np'-p'',-q-nq'-q''}|^2$$

$$\times (3/n)^{2\beta}(|k+k''+nk'|+|p+p''+np'|+|q+q''+nq'|)^{2\beta}\Big)^{1/2}$$

$$+\Big(\sum\nolimits^{(2)} |\mathbf{h}_{k'',p'',q''}|^2 \left(\frac{|k''|+|p''|+|q''|}{n/3}\right)^{2\beta} \sum\nolimits^{(2)} |\mathbf{f}_{-k-nk'-k'',-p-np'-p'',-q-nq'-q''}|^2\Big)^{1/2}$$

$$+\left(\sum\nolimits^{(3)} |\mathbf{h}_{k'',p'',q''}|^2 \sum\nolimits^{(3)} |\mathbf{f}_{-k-nk'-k'',-p-np'-p'',-q-nq'-q''}|^2\right)^{1/2}\Bigg). \qquad (11.6)$$

Here summation in the sums $\sum^{(m)}$ is over such indices k'', p'', q'' that

$$|k+k''+nk'|+|p+p''+np'|+|q+q''+nq'| \ge n/3 \quad \text{in } \sum\nolimits^{(1)},$$
$$|k''|+|p''|+|q''| \ge n/3 \quad \text{in } \sum\nolimits^{(2)},$$

and none of the two inequalities is satisfied in $\sum^{(3)}$. Smoothness of matrix (11.5) in the fast and slow spherical variables and condition (11.3) imply that for any $\beta \ge 0$ and $E \ge 1$

$$\sum_{\substack{\mathrm{k,k',p,p',q,q'}\\ |k|+|p|+|q|\ge E}} |\mathscr{W}_{k,k',p,p',q,q'}| \le C_\beta E^{-\beta}$$

(see, e.g., [85]). By virtue of condition (11.4), $|k'|+|p'|+|q'| \ne 0$ in sum (11.6), and hence sums $\sum^{(3)}$ involve non-empty sets of terms and do not vanish only for $|k|+|p|+|q| \ge n/3$. We develop (11.6) further, applying Lemma 11.1:

$$\left|\int\limits_\Omega \mathscr{W}(\mathbf{x}, n\rho, n\theta, n\varphi)\mathbf{h}\cdot\mathbf{f}\mathrm{d}\mathbf{x}\right| \le C'_{\beta,\mathscr{W}}(3/n)^\beta(|\mathbf{h}|_0|\mathbf{f}|_\beta + |\mathbf{h}|_\beta|\mathbf{f}|_0 + |\mathbf{h}|_0|\mathbf{f}|_0)$$
$$\le C_{\beta,\mathscr{W}} n^{-\beta}(|\mathbf{h}|_0|\mathbf{f}|_\beta + |\mathbf{h}|_\beta|\mathbf{f}|_0),$$

where the constant $C_{\beta,\mathscr{W}}$ is independent of vector fields $\mathbf{h}$ or $\mathbf{f}$. □

Corollary *Suppose* $\delta \ge \beta \ge 0, \beta \le 2$, *and let* $\mathscr{D}$ *be a differential operator of order* $m \ge 0$ *with smooth coefficients in* $\overline{\Omega}$. *Under the conditions of Lemma 11.2, the inequality*

$$|\mathscr{W}(\mathbf{x}, n\rho, n\theta, n\varphi)\mathscr{D}\mathbf{h}|_{-\delta} \leq C_{\delta,\beta,\mathscr{W}} n^{-\beta} |\mathbf{h}|_{\beta+m} \tag{11.7}$$

holds with a constant $C_{\delta,\beta,\mathscr{W}}$ *independent of* $\mathbf{h}$.

11.2 The Spherical Kinematic Dynamo

11.2.1 Statement of the Problem

In this chapter we will construct an asymptotic expansion of magnetic modes $\mathbf{h}_i^n$ and the associated eigenvalues Λ_i^n of the magnetic induction operator

$$\mathscr{L}^n \mathbf{h} \equiv \eta \nabla^2 \mathbf{h} + \nabla \times (\mathbf{v}^n \times \mathbf{h})$$

in the kinematic dynamo problem for the flow

$$\mathbf{v}^n = \mathbf{U}(\mathbf{x}) + n^{-1/2} \nabla \times \left(\mathbf{W}(\mathbf{x}, P, \Theta, \Phi)|_{P=n\rho, \Theta=n\theta, \Phi=n\varphi} \right). \tag{11.8}$$

The modes are supposed to be solenoidal and satisfy the boundary conditions (10.1) pertaining to the case, when the volume Ω occupied by fluid is surrounded by a dielectric, and $n \to \infty$ are positive integers. We do not assume that the fluid is incompressible, i.e. solenoidality of the flow (11.8) is not required (except for Sect. 11.5.2, where helicity of the vector potential of the flow is calculated).

We assume henceforth that vector field $\mathbf{W}(\mathbf{x}, P, \Theta, \Phi)$ defined in Ω is periodic in the fast variables P, Θ and Φ, with periods 1, π and 2π, respectively, and

$$\mathbf{W} \text{ and all its partial derivatives vanish for } \rho = 0 \text{ or } \theta = 0, \pi, \tag{11.9}$$

$$\left. \frac{\partial^k \mathbf{W}}{\partial \rho^k} \right|_{\partial\Omega} = 0 \tag{11.10}$$

for any $k \geq 0$, and

$$\langle \mathbf{W} \rangle = 0. \tag{11.11}$$

Conditions (11.9) and (11.10) are assumed for the same reasons, as were conditions (10.19) and (10.39) in the previous chapter. In particular, by virtue of (11.9), $\rho^\delta \sin^\beta \theta \mathbf{W}(\mathbf{x}, n\rho, n\theta, n\varphi) \in \mathbb{C}^\infty(\overline{\Omega})$ for any $\delta, \beta \in \mathbb{R}^1$.

11.2.2 Derivation of the Limit Operator

We construct a solution to the eigenvalue equation

$$\mathscr{L}^n \mathbf{h}^n = \Lambda^n \mathbf{h}^n \tag{11.12}$$

for the magnetic induction operator as formal power series

$$\Lambda^n = \sum_{j\geq 0} n^{-j/2}\lambda_j, \tag{11.13}$$

$$\mathbf{h}^n = \sum_{j\geq 0} n^{-j/2}\mathbf{h}_j(\mathbf{x}, P, \Theta, \Phi)|_{P=n\rho,\Theta=n\theta,\Phi=n\varphi}. \tag{11.14}$$

Each averaged term $\langle\mathbf{h}_j\rangle$ in the expansion of a magnetic mode $\mathbf{h}^n$ is supposed to satisfy the boundary conditions (10.1) and have the same periodicities in the fast variables, as the field $\mathbf{W}$.

Let us define

$$\mathscr{D}_P = \frac{\partial}{\partial P}, \quad \mathscr{D}_\Theta = \frac{1}{\rho}\frac{\partial}{\partial\Theta}, \quad \mathscr{D}_\Phi = \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\Phi},$$

$$\mathscr{N}\mathbf{g} = (\mathscr{D}_\Theta g^{(\varphi)} - \mathscr{D}_\Phi g^{(\theta)})\mathbf{i}_\rho + (\mathscr{D}_\Phi g^{(\rho)} - \mathscr{D}_P g^{(\varphi)})\mathbf{i}_\theta + (\mathscr{D}_P g^{(\theta)} - \mathscr{D}_\Theta g^{(\rho)})\mathbf{i}_\varphi,$$

where $g^{(\rho)}$, $g^{(\theta)}$, $g^{(\varphi)}$ denote components of a vector $\mathbf{g}$ in the basis of unit vectors of the spherical coordinate system $\mathbf{i}_\rho$, $\mathbf{i}_\theta$ and $\mathbf{i}_\varphi$. By the chain rule,

$$\nabla \times \left(\mathbf{g}(\mathbf{x}, P, \Theta, \Phi)|_{P=n\rho,\Theta=n\theta,\Phi=n\varphi}\right) = \nabla_\mathbf{x} \times \mathbf{g} + n\mathscr{N}\mathbf{g}.$$

Clearly, the operators $\mathscr{D}_P, \mathscr{D}_\Theta, \mathscr{D}_\Phi$ and $\mathscr{N}$ commute.

We substitute the flow (11.8) and series (11.13) and (11.14) into the eigenvalue equation (11.12), and expand the result. This yields a power series form of the eigenvalue equation:

$$\sum_{j\geq -4} n^{-j/2}\left(\sum_{m=-1}^{4}(\mathscr{A}_m\langle\mathbf{h}_{j+m}\rangle + \mathscr{B}_m\{\mathbf{h}_{j+m}\}) - \sum_{m=0}^{j}\lambda_{j-m}\mathbf{h}_m\right) = 0. \tag{11.15}$$

We have introduced here the notation

$$\begin{aligned}
\mathbf{Q} \equiv \nabla_\mathbf{x} \times \mathbf{W}, &\quad \mathscr{D} \equiv \mathscr{D}_P^2 + \mathscr{D}_\Theta^2 + \mathscr{D}_\Phi^2,\\
\mathscr{A}_{-1}\mathbf{h} \equiv \nabla_\mathbf{x} \times (\mathbf{Q} \times \mathbf{h}), &\quad \mathscr{B}_{-1}\mathbf{g} \equiv \nabla_\mathbf{x} \times (\mathbf{Q} \times \mathbf{g}),\\
\mathscr{A}_0\mathbf{h} \equiv \eta\nabla^2\mathbf{h} + \nabla \times (\mathbf{U} \times \mathbf{h}), &\quad \mathscr{B}_0\mathbf{g} \equiv \eta\nabla_\mathbf{x}^2\mathbf{g} + \nabla_\mathbf{x} \times (\mathbf{U} \times \mathbf{g}),\\
\mathscr{A}_1\mathbf{h} \equiv \nabla_\mathbf{x} \times (\mathscr{N}\mathbf{W} \times \mathbf{h}) + \mathscr{N}(\mathbf{Q} \times \mathbf{h}), &\quad \mathscr{B}_1\mathbf{g} \equiv \nabla_\mathbf{x} \times (\mathscr{N}\mathbf{W} \times \mathbf{g}) + \mathscr{N}(\mathbf{Q} \times \mathbf{g}),\\
\mathscr{A}_3\mathbf{h} \equiv (h^{(\rho)}\mathscr{D}_P + h^{(\theta)}\mathscr{D}_\Theta + h^{(\varphi)}\mathscr{D}_\Phi)\mathscr{N}\mathbf{W}, &\quad \mathscr{B}_3\mathbf{g} \equiv \mathscr{N}(\mathscr{N}\mathbf{W} \times \mathbf{g}),\\
\mathscr{A}_2\mathbf{h} = \mathscr{A}_4\mathbf{h} \equiv 0, &\quad \mathscr{B}_4\mathbf{g} \equiv \eta\mathscr{D}\mathbf{g},
\end{aligned}$$

$$\mathscr{B}_2\mathbf{g} \equiv 2\eta\left(\mathscr{D}_P\frac{\partial\mathbf{g}}{\partial\rho} + \frac{1}{\rho}\left(\mathscr{D}_P\mathbf{g} + \frac{1}{\sin\theta}\mathscr{D}_\Phi\frac{\partial\mathbf{g}}{\partial\varphi} + \mathscr{D}_\Theta\frac{\partial\mathbf{g}}{\partial\theta} + \frac{\mathrm{ctg}\theta}{2}\mathscr{D}_\Theta\mathbf{g}\right)\right) + \mathscr{N}(\mathbf{U} \times \mathbf{g})$$

(for any $j<0, \mathbf{h}_j = 0$ in (11.15), and the sum $\sum_{m=0}^{j}$ is absent). Let us consider successively the equations arising from (11.15) at different orders $n^{-j/2}$.

$j = -4$. At order n^2, (11.15) yields

$$\eta\mathscr{D}\{\mathbf{h_0}\} = 0.$$

Operator $\mathscr{D}$ is elliptic in the fast variables (with coefficients depending on ρ, θ and φ), it has an empty kernel in the subspace of zero-mean vector fields satisfying the imposed periodicity conditions in the fast variables, and hence is invertible in this subspace. We conclude therefore that $\{\mathbf{h_0}\} = 0$.

$j = -3$. At the next order, $n^{3/2}$, we obtain from (11.15)

$$\eta\mathscr{D}\{\mathbf{h_1}\} = -(\langle h_0^{(\rho)}\rangle\mathscr{D}_P + \langle h_0^{(\theta)}\rangle\mathscr{D}_\Theta + \langle h_0^{(\varphi)}\rangle\mathscr{D}_\Phi)\mathscr{N}\mathbf{W}. \tag{11.16}$$

Suppose $\boldsymbol{\xi} \in \mathbb{C}^\infty(\overline{\Omega} \times [0,1] \times [0,\pi] \times [0,2\pi])$ is a zero-mean ($\langle\boldsymbol{\xi}\rangle = 0$) solution to the equation $\mathscr{D}\boldsymbol{\xi} = -\mathbf{W}$, which has the same periodicities in P, Θ and Φ as $\mathbf{W}$. The solution exists, since the r.h.s. of (11.16) is zero-mean. Expanding $\mathbf{W}$ in the Fourier series in the fast variables:

$$\mathbf{W}(\mathbf{x}, P, \Theta, \Phi) = \sum_{|p|+|q|+|s|\neq 0} \mathbf{W}_{p,q,s}(\mathbf{x})\mathrm{e}^{\mathrm{i}(2\pi pP + 2q\Theta + s\Phi)},$$

we find

$$\boldsymbol{\xi}(\mathbf{x}, P, \Theta, \Phi) = \sum_{|p|+|q|+|s|\neq 0} \frac{\mathbf{W}_{p,q,s}(\mathbf{x})\rho^2 \sin^2\theta}{(2\pi p\rho\sin\theta)^2 + (2q\sin\theta)^2 + s^2}\mathrm{e}^{\mathrm{i}(2\pi pP + 2q\Theta + s\Phi)}$$

(note that $\rho^\delta \sin^\beta \theta\boldsymbol{\xi}(\mathbf{x}, n\rho, n\theta, n\varphi) \in \mathbb{C}^\infty(\overline{\Omega})$ for any $\delta, \beta \in \mathbb{R}^1$). Consequently,

$$\{\mathbf{h_1}\} = (\langle h_0^{(\rho)}\rangle\mathscr{D}_P + \langle h_0^{(\theta)}\rangle\mathscr{D}_\Theta + \langle h_0^{(\varphi)}\rangle\mathscr{D}_\Phi)\mathscr{N}\boldsymbol{\xi}/\eta. \tag{11.17}$$

$j = -2$. At order, series (11.15) yields the equation

$$\eta\mathscr{D}\{\mathbf{h_2}\} = -(\langle h_1^{(\rho)}\rangle\mathscr{D}_P + \langle h_1^{(\theta)}\rangle\mathscr{D}_\Theta + \langle h_1^{(\varphi)}\rangle\mathscr{D}_\Phi)\mathscr{N}\mathbf{W} - \mathscr{N}(\mathscr{N}\mathbf{W} \times \{\mathbf{h}_1\}). \tag{11.18}$$

The r.h.s. of this equation is zero-mean and, hence, the equation is solvable. Upon substitution of the expression (11.17) for $\{\mathbf{h}_1\}$, we find

$$\{\mathbf{h_2}\} = (\langle h_1^{(\rho)}\rangle\mathscr{D}_P + \langle h_1^{(\theta)}\rangle\mathscr{D}_\Theta + \langle h_1^{(\varphi)}\rangle\mathscr{D}_\Phi)\mathscr{N}\boldsymbol{\xi}/\eta + \mathscr{I}_1\langle\mathbf{h}_0\rangle. \tag{11.19}$$

Here, linear operator $\mathscr{I}_1$ is defined by the relation

$$\mathscr{I}_1\mathbf{h}(\mathbf{x}) \equiv h^{(\rho)}\mathbf{a}^P + h^{(\theta)}\mathbf{a}^\Theta + h^{(\varphi)}\mathbf{a}^\Phi,$$

where $\mathbf{a}^P(\mathbf{x}, P, \Theta, \Phi), \mathbf{a}^\Theta(\mathbf{x}, P, \Theta, \Phi)$ and $\mathbf{a}^\Phi(\mathbf{x}, P, \Theta, \Phi)$ are zero-mean solutions to equations

$$\begin{aligned}\eta^2 \mathscr{D}\mathbf{a}^P &= -\mathscr{N}(\mathscr{N}\mathbf{W} \times \mathscr{D}_P(\mathscr{N}\boldsymbol{\xi})),\\ \eta^2 \mathscr{D}\mathbf{a}^\Theta &= -\mathscr{N}(\mathscr{N}\mathbf{W} \times \mathscr{D}_\Theta(\mathscr{N}\boldsymbol{\xi})),\\ \eta^2 \mathscr{D}\mathbf{a}^\Phi &= -\mathscr{N}(\mathscr{N}\mathbf{W} \times \mathscr{D}_\Phi(\mathscr{N}\boldsymbol{\xi})),\end{aligned}$$

with the required periodicity in the fast variables.

$\underline{j = -1}$. The equation arising from (11.15) at order $n^{1/2}$ is

$$\begin{aligned}&\eta\mathscr{D}\{\mathbf{h}_3\} = -(\langle h_2^{(\rho)}\rangle \mathscr{D}_P + \langle h_2^{(\theta)}\rangle \mathscr{D}_\Theta + \langle h_2^{(\varphi)}\rangle \mathscr{D}_\Phi)\mathscr{N}\mathbf{W}\\ &- 2\eta\left(\mathscr{D}_P \frac{\partial\{\mathbf{h}_1\}}{\partial\rho} + \frac{1}{\rho}\left(\mathscr{D}_P\{\mathbf{h}_1\} + \frac{1}{\sin\theta}\mathscr{D}_\Phi \frac{\partial\{\mathbf{h}_1\}}{\partial\varphi} + \mathscr{D}_\Theta \frac{\partial\{\mathbf{h}_1\}}{\partial\theta} + \frac{\mathrm{ctg}\theta}{2}\mathscr{D}_\Theta\{\mathbf{h}_1\}\right)\right)\\ &- \nabla_\mathbf{x} \times (\mathscr{N}\mathbf{W} \times \langle\mathbf{h}_0\rangle) - \mathscr{N}(\mathbf{Q} \times \langle\mathbf{h}_0\rangle + \mathbf{U} \times \{\mathbf{h}_1\} + \mathscr{N}\mathbf{W} \times \{\mathbf{h}_2\}).\end{aligned} \tag{11.20}$$

The solvability condition for this equation is satisfied. After substitution of the expressions for $\{\mathbf{h}_1\}$ and $\{\mathbf{h}_2\}$ found from equations at orders $n^{3/2}$ and n, respectively, we find

$$\{\mathbf{h}_3\} = (\langle h_2^{(\rho)}\rangle \mathscr{D}_P + \langle h_2^{(\theta)}\rangle \mathscr{D}_\Theta + \langle h_2^{(\varphi)}\rangle \mathscr{D}_\Phi)\mathscr{N}\boldsymbol{\xi}/\eta + \mathscr{I}_1\langle\mathbf{h}_1\rangle + \mathscr{I}_2\langle\mathbf{h}_0\rangle. \tag{11.21}$$

Here $\mathscr{I}_2$ is a linear operator defined as

$$\mathscr{I}_2\mathbf{h}(\mathbf{x}) \equiv h^{(\rho)}\mathbf{b}^P + h^{(\theta)}\mathbf{b}^\Theta + h^{(\varphi)}\mathbf{b}^\Phi - (\mathscr{N}\boldsymbol{\xi} \cdot \nabla_\mathbf{x})\mathbf{h}/\eta.$$

It is straightforward to deduce from Eq. 11.20 the auxiliary problems defining zero-mean vector fields $\mathbf{b}^P(\mathbf{x}, P, \Theta, \Phi), \mathbf{b}^\Theta(\mathbf{x}, P, \Theta, \Phi)$ and $\mathbf{b}^\Phi(\mathbf{x}, P, \Theta, \Phi)$, periodic in the fast variables; however, the equations are quite bulky and, since they will not be used in what follows, we do not present him.

Denote

$$\mathscr{F}\mathbf{g}(\mathbf{x}, P, \Theta, \Phi) \equiv \mathscr{D}_P g^{(\rho)} + \mathscr{D}_\Theta g^{(\theta)} + \mathscr{D}_\Phi g^{(\varphi)}.$$

It is simple to show that the following identities hold for any $\mathbf{g}(\mathbf{x}, P, \Theta, \Phi)$:

$$\nabla \cdot \mathbf{g} = \nabla_\mathbf{x} \cdot \mathbf{g} + n\mathscr{F}\mathbf{g}; \quad \mathscr{F}\mathscr{N}\mathbf{g} = 0; \quad \mathscr{F}(\nabla_\mathbf{x} \times \mathbf{g}) + \nabla_\mathbf{x} \cdot \mathscr{N}\mathbf{g} = 0. \tag{11.22}$$

Applying the operator of divergence in the slow variables to Eq. 11.16 and operator $\mathscr{F}$ to (11.20), we find

$$\eta\mathscr{D}(\mathscr{F}\{\mathbf{h}_3\} + \nabla_\mathbf{x} \cdot \{\mathbf{h}_1\}) = 0,$$

and hence

$$\mathscr{F}\{\mathbf{h}_3\} + \nabla_\mathbf{x} \cdot \{\mathbf{h}_1\} = 0.$$

$\underline{j=0}$. At order n^0 Eq. 11.15 yields

$$\begin{aligned}
&\eta\mathscr{D}\{\mathbf{h}_4\} + (\langle h_3^{(\rho)}\rangle\mathscr{D}_P + \langle h_3^{(\theta)}\rangle\mathscr{D}_\Theta + \langle h_3^{(\varphi)}\rangle\mathscr{D}_\Phi)\mathscr{N}\mathbf{W} + \nabla_\mathbf{x}\times(\mathscr{N}\mathbf{W}\times\mathbf{h}_1)\\
&+2\eta\left(\mathscr{D}_P\frac{\partial\{\mathbf{h}_2\}}{\partial\rho} + \frac{1}{\rho}\left(\mathscr{D}_P\{\mathbf{h}_2\} + \frac{1}{\sin\theta}\mathscr{D}_\Phi\frac{\partial\{\mathbf{h}_2\}}{\partial\varphi} + \mathscr{D}_\Theta\frac{\partial\{\mathbf{h}_2\}}{\partial\theta} + \frac{\mathrm{ctg}\theta}{2}\mathscr{D}_\Theta\{\mathbf{h}_2\}\right)\right)\\
&+\mathscr{N}(\mathbf{Q}\times\mathbf{h}_1 + \mathbf{U}\times\{\mathbf{h}_2\} + \mathscr{N}\mathbf{W}\times\{\mathbf{h}_3\}) + \eta\nabla^2\langle\mathbf{h}_0\rangle + \nabla_\mathbf{x}\times(\mathbf{U}\times\langle\mathbf{h}_0\rangle)\\
&=\lambda_0\langle\mathbf{h}_0\rangle.
\end{aligned}\tag{11.23}$$

Averaging (11.23) over the fast spherical variables, we obtain the magnetic mean-field equation

$$\mathscr{L}^\infty\langle\mathbf{h}_0\rangle = \lambda_0\langle\mathbf{h}_0\rangle,\tag{11.24}$$

where

$$\mathscr{L}^\infty\mathbf{h} \equiv \eta\nabla^2\mathbf{h} + \nabla\times(\mathbf{U}\times\mathbf{h} + \mathscr{A}\mathbf{h}).$$

The linear operator of magnetic α-effect $\mathscr{A}$ can be determined from relations

$$\nabla\times\mathscr{A}\langle\mathbf{h}_0\rangle \equiv \langle\nabla_\mathbf{x}\times(\mathscr{N}\mathbf{W}\times\{\mathbf{h}_1\})\rangle$$

and (11.17); in the basis of unit vectors of the spherical coordinate system $(\mathbf{i}_\rho, \mathbf{i}_\theta, \mathbf{i}_\varphi)$ it is represented by a symmetric matrix

$$\mathscr{A} = \begin{vmatrix} \alpha^{\rho\rho} & \alpha^{\rho\theta} & \alpha^{\rho\varphi}\\ \alpha^{\theta\rho} & \alpha^{\theta\theta} & \alpha^{\theta\varphi}\\ \alpha^{\varphi\rho} & \alpha^{\varphi\theta} & \alpha^{\varphi\varphi}\end{vmatrix}\tag{11.25}$$

with the entries

$$\begin{aligned}
\alpha^{\rho\rho}(\mathbf{x}) &\equiv \langle\mathscr{D}_P^2\mathbf{W}\cdot\mathscr{N}\xi\rangle/\eta, \quad \alpha^{\theta\theta}(\mathbf{x}) \equiv \langle\mathscr{D}_\Theta^2\mathbf{W}\cdot\mathscr{N}\xi\rangle/\eta,\\
\alpha^{\varphi\varphi}(\mathbf{x}) &\equiv \langle\mathscr{D}_\Phi^2\mathbf{W}\cdot\mathscr{N}\xi\rangle/\eta,\\
\alpha^{\rho\theta}(\mathbf{x}) &= \alpha^{\theta\rho}(\mathbf{x}) \equiv \langle\mathscr{D}_P\mathscr{D}_\Theta\mathbf{W}\cdot\mathscr{N}\xi\rangle/\eta,\\
\alpha^{\rho\varphi}(\mathbf{x}) &= \alpha^{\varphi\rho}(\mathbf{x}) \equiv \langle\mathscr{D}_P\mathscr{D}_\Phi\mathbf{W}\cdot\mathscr{N}\xi\rangle/\eta,\\
\alpha^{\theta\varphi}(\mathbf{x}) &= \alpha^{\varphi\theta}(\mathbf{x}) \equiv \langle\mathscr{D}_\Theta\mathscr{D}_\Phi\mathbf{W}\cdot\mathscr{N}\xi\rangle/\eta.
\end{aligned}\tag{11.26}$$

By virtue of Eq. 11.24, λ_0 is an eigenvalue and $\langle\mathbf{h}_0\rangle = \mathbf{h}^\infty$ the associated eigenfunction of the "limit" operator $\mathscr{L}^\infty$. The eigenfunction $\langle\mathbf{h}_0\rangle$ is solenoidal and satisfies the boundary conditions (10.1). We normalise it in $L_2(\Omega)$.

Subtracting from Eq. 11.24 its mean part (11.23), we find

$$\begin{aligned}
&\eta\mathscr{D}\{\mathbf{h}_4\} + (\langle h_3^{(\rho)}\rangle\mathscr{D}_P + \langle h_3^{(\theta)}\rangle\mathscr{D}_\Theta + \langle h_3^{(\varphi)}\rangle\mathscr{D}_\Phi)\mathscr{N}\mathbf{W}\\
&+2\eta\left(\mathscr{D}_P\frac{\partial\{\mathbf{h}_2\}}{\partial\rho} + \frac{1}{\rho}\left(\mathscr{D}_P\{\mathbf{h}_2\} + \frac{1}{\sin\theta}\mathscr{D}_\Phi\frac{\partial\{\mathbf{h}_2\}}{\partial\varphi} + \mathscr{D}_\Theta\frac{\partial\{\mathbf{h}_2\}}{\partial\theta} + \frac{\mathrm{ctg}\theta}{2}\mathscr{D}_\Theta\{\mathbf{h}_2\}\right)\right)\\
&+\nabla_\mathbf{x}\times(\mathscr{N}\mathbf{W}\times\mathbf{h}_1 - \mathscr{A}\langle\mathbf{h}_0\rangle) + \mathscr{N}(\mathbf{Q}\times\mathbf{h}_1 + \mathbf{U}\times\{\mathbf{h}_2\} + \mathscr{N}\mathbf{W}\times\{\mathbf{h}_3\}) = 0,
\end{aligned}\tag{11.27}$$

which yields

$$\{\mathbf{h}_4\} = (\langle h_3^{(\rho)}\rangle \mathscr{D}_P + \langle h_3^{(\theta)}\rangle \mathscr{D}_\Theta + \langle h_3^{(\varphi)}\rangle \mathscr{D}_\Phi)\mathscr{N}\boldsymbol{\xi}/\eta + \mathscr{I}_1\langle \mathbf{h}_2\rangle + \mathscr{I}_2\langle \mathbf{h}_1\rangle + \mathbf{h}'_0. \tag{11.28}$$

Here $\mathbf{h}'_0$ is a zero-mean vector, which can be completely determined at this stage from the equation obtained by substitution of expressions (11.19), (11.21) and (11.28) for $\{\mathbf{h}_k\}, k = 2, 3, 4$, into (11.27). In this equation for $\mathbf{h}'_0$ all terms involving $\langle \mathbf{h}_k\rangle$ for $k = 1, 2, 3$ cancel out, and $\{\mathbf{h}_1\}$ is already completely determined by expression (11.17). Using identities (11.22), we find from Eqs. 11.27 and 11.18

$$\mathscr{F}\{\mathbf{h}_4\} + \nabla_{\mathbf{x}} \cdot \{\mathbf{h}_2\} = 0.$$

Solution of equations produced by series (11.13) at orders $n^{-j/2}$ for $j > 0$ is discussed in Sect. 11.4.

11.3 Uniform Boundedness and Convergence of Resolvents

Uniform boundedness and convergence of the resolvents $(\mathscr{L}^n - \lambda)^{-1}$ can be established following the approach that we used in the previous chapter.

11.3.1 Uniform Boundedness of Resolvents

Lemma 11.3 *There exist such constants* $C > 0$ *and* C' *that for any vector field* $\mathbf{h} \in H^{1/2}(\Omega)$ *and* $n > 0$ *the inequalities*

$$\|\mathbf{h}\|_{1/2} \le C|(\mathscr{L}^n - \lambda)\mathbf{h}|_{-3/2} \tag{11.29}$$

and

$$\|\mathbf{h}\|_{1/2} \le C|(\mathscr{L}^\infty - \lambda)\mathbf{h}|_{-3/2} \tag{11.30}$$

hold as long as $\lambda > C'$.

(The norm $\|\cdot\|_q$ was defined in the previous chapter).

Proof The demonstration follows closely the proof of Lemma 10.5. The only difference is that while in the case of one fast azimuthal variable, considered in the previous chapter, the proof of Lemma 10.5 relies on the bound from the corollary to Lemma 10.5, in the present case of three fast spherical variables a similar inequality (11.7) from the corollary to Lemma 11.2 must be applied. □

The bound $|\mathbf{h}|_0 \le C_0 |(\mathscr{L}^n - \lambda)\mathbf{h}|_0$ following from (11.29) and analyticity of the resolvent $(\mathscr{L}^n - \lambda)^{-1}$ in λ imply, that the resolvent is well-defined for all $\lambda \ge C'$. By a similar argument, the same is true for $(\mathscr{L}^\infty - \lambda)^{-1}$.

11.3.2 Convergence of the Resolvents

Lemma 11.4 *Suppose $\lambda > C'$, C' being the constant referred to in Lemma 11.3. Then*

$$\|(\mathscr{L}^n - \lambda)^{-1} - (\mathscr{L}^\infty - \lambda)^{-1}\|_0 \le C_\lambda n^{-1/2}, \tag{11.31}$$

where C_λ is a constant independent of n.

Proof We follow the approach which was used for the proof of Lemma 10.6 in the previous chapter. Let

$$(\mathscr{L}^n - \lambda)\mathbf{h}^n = \mathbf{f}, \quad (\mathscr{L}^\infty - \lambda)\mathbf{h}^\infty = \mathbf{f},$$

$$\mathbf{g} = (\langle h^{\infty(\rho)}\rangle \mathscr{D}_P + \langle h^{\infty(\theta)}\rangle \mathscr{D}_\Theta + \langle h^{\infty(\varphi)}\rangle \mathscr{D}_\Phi)\mathscr{N}\boldsymbol{\xi}/\eta.$$

By the corollary to Lemma 11.2, the norm $|\cdot|_{-3/2}$ of each term in

$$\begin{aligned}
&(\mathscr{L}^n - \lambda)(\mathbf{h}^\infty + n^{-1/2}\mathbf{g} - \mathbf{h}^n)\\
&= \nabla_\mathbf{x} \times (n^{-1}\mathbf{Q} \times \mathbf{g} + n^{-1/2}(\mathbf{Q} \times \mathbf{h}^\infty + \mathbf{U} \times \mathbf{g}) + (\mathscr{N}\mathbf{W} \times \mathbf{g} - \mathscr{A}\mathbf{h}^\infty)\\
&\quad + n^{1/2}\mathscr{N}\mathbf{W} \times \mathbf{h}^\infty)\\
&\quad + \eta n^{-1/2}\left(\nabla^2_\mathbf{x}\mathbf{g} + 2n\left(\mathscr{D}_P\frac{\partial \mathbf{g}}{\partial \rho} + \frac{1}{\rho}\left(\mathscr{D}_P\mathbf{g} + \frac{1}{\sin\theta}\mathscr{D}_\Phi\frac{\partial \mathbf{g}}{\partial \varphi} + \mathscr{D}_\Theta\frac{\partial \mathbf{g}}{\partial \theta} + \frac{\mathrm{ctg}\theta}{2}\mathscr{D}_\Theta\mathbf{g}\right)\right)\right)\\
&\quad + \mathscr{N}(\mathbf{Q} \times \mathbf{g} + n^{1/2}(\mathbf{Q} \times \mathbf{h}^\infty + \mathbf{U} \times \mathbf{g}) + n\mathscr{N}\mathbf{W} \times \mathbf{g}) - n^{-1/2}\lambda\mathbf{g}
\end{aligned}$$

admits a bound of the form $C_1 n^{-1/2}|\mathbf{h}^\infty|_2$, whereby

$$|(\mathscr{L}^n - \lambda)(\mathbf{h}^\infty + n^{-1/2}\mathbf{g} - \mathbf{h}^n)|_{-3/2} \le C_2 n^{-1/2}|\mathbf{h}^\infty|_2.$$

Let us construct a function $p \in \mathbb{C}^1(\mathbb{R}^3)$ such that

$$\nabla^2 p = \nabla \cdot \mathbf{g} \text{ in } \Omega, \quad \nabla^2 p = 0 \text{ in } \Omega', \quad p \to 0 \text{ for } \rho \to \infty.$$

It was shown in Chap. 10 (see the proof of Lemma 10.6) that $|p|_2 \le C_3|\nabla \cdot \mathbf{g}|_0$. Applying this bound, the corollary to Lemma 11.2 and the first identity in (11.22), we find

$$\begin{aligned}|(\mathscr{L}^n-\lambda)\nabla p|_{-3/2}\leq & C_3^1(|\nabla p|_{1/2}+|(\mathbf{U}\times\nabla p)|_{-1/2}\\ & +n^{-1/2}(|(\nabla_{\mathbf{x}}\times\mathbf{W})\times\nabla p)|_{-1/2}+n|(\mathcal{N}\mathbf{W}\times\nabla p)|_{-1/2}))\\ \leq & C_3^2\left(|\nabla p|_{1/2}+n^{-1/2}(|\nabla p|_0+n\cdot n^{-1/2}|\nabla p|_{1/2})\right)\\ \leq & C_3^3|\nabla p|_1\leq C_3^4|\nabla\cdot\mathbf{g}|_0=C_3^4|\nabla_{\mathbf{x}}\cdot\mathbf{g}|_0\leq C_4|\mathbf{h}^\infty|_1.\end{aligned}$$

The relation $(\mathscr{L}^\infty-\lambda)\mathbf{h}^\infty=\mathbf{f}$ implies that $|\mathbf{h}^\infty|_2\leq C_5|\mathbf{f}|_0$. Summarising now the estimates obtained so far, we conclude that

$$|(\mathscr{L}^n-\lambda)(\mathbf{h}^\infty+n^{-1/2}(\mathbf{g}-\nabla p)-\mathbf{h}^n)|_{-3/2}\leq C_6 n^{-1/2}|\mathbf{h}^\infty|_2. \tag{11.32}$$

By construction, $\mathbf{h}^\infty+n^{-1/2}(\mathbf{g}-\nabla p)-\mathbf{h}^n\in H^0(\Omega)$, and hence the bound (11.29) for the resolvent together with inequality (11.32) implies

$$\|\mathbf{h}^\infty+n^{-1/2}(\mathbf{g}-\nabla p)-\mathbf{h}^n\|_{1/2}\leq C_7 n^{-1/2}|\mathbf{f}|_0.$$

Since we also have

$$\|\mathbf{g}-\nabla p\|_0\leq C_8(|\nabla p|_0+|\mathbf{g}|_0)\leq C_9|\mathbf{h}^\infty|_1\leq C_{10}|\mathbf{f}|_0,$$

the estimate (11.31) is established, as required. □

By Theorems 10.3 and 10.4 , Lemma 11.4 implies the following result:

Theorem 11.1 *Suppose an eigenvalue Λ_i^∞ of the operator $\mathscr{L}^\infty$ is m-fold. Then precisely m eigenvalues Λ_{ik}^n (counted with their multiplicities) of magnetic induction operators $\mathscr{L}^n$ converge to Λ_i^∞ with the bound*

$$|\Lambda_{ik}^n-\Lambda_i^\infty|\leq C_i n^{-1/2m}.$$

The associated m-dimensional invariant subspaces converge with the bound

$$\|\mathscr{P}^n-\mathscr{P}^\infty\|_0\leq C_i n^{-1/2},$$

where $\mathscr{P}^n$ and $\mathscr{P}^\infty$ are projections onto the respective invariant subspaces.

11.4 Complete Asymptotic Expansion of Magnetic Modes

For the sake of simplicity, in this section we consider the generic case, when Λ^∞ is a simple eigenvalue of the operator $\mathscr{L}^\infty$.

11.4.1 Solution of the Hierarchy of Equations in all Orders

Lemma 11.5 *One can solve equations arising from series* (11.15) *at successive orders* $n^{-j/2}$ *for* $j > 0$ *and find all terms of the power series* (11.13) *and* (11.14). *For any* $j \geq -2$, *relations*

$$\mathscr{F}\{\mathbf{h}_{j+2}\} + \nabla_{\mathbf{x}} \cdot \{\mathbf{h}_j\} = 0 \tag{11.33}$$

and

$$\nabla_{\mathbf{x}} \cdot \langle \mathbf{h}_j \rangle = 0$$

(equivalent together to the solenoidality condition for the large-scale magnetic mode) are satisfied.

Proof At order $n^{-j/2}$, the equation

$$\sum_{m=-1}^{4} (\mathscr{A}_m \langle \mathbf{h}_{j+m} \rangle + \mathscr{B}_m \{\mathbf{h}_{j+m}\}) - \sum_{m=0}^{j} \lambda_{j-m} \mathbf{h}_m = 0 \tag{11.34}$$

arises from the eigenvalue equation in the series form (11.15). We will solve these equations successively for increasing j, seeking fluctuating parts of terms of the series (11.14) in the form

$$\begin{aligned}\{\mathbf{h}_{j+4}\} = (\langle h^{(\rho)}_{j+3} \rangle \mathscr{D}_P + \langle h^{(\theta)}_{j+3} \rangle \mathscr{D}_\Theta + \langle h^{(\varphi)}_{j+3} \rangle \mathscr{D}_\Phi) \mathscr{N} \boldsymbol{\xi}/\eta \\ + \mathscr{I}_1 \langle \mathbf{h}_{j+2} \rangle + \mathscr{I}_2 \langle \mathbf{h}_{j+1} \rangle + \mathbf{h}'_j.\end{aligned} \tag{11.35}$$

We will now establish by induction in k the following statement: one can determine such vector fields $\mathbf{h}_m \in \mathbb{C}^\infty(\overline{\Omega} \times [0,1] \times [0,\pi] \times [0,2\pi])$, zero-mean fields $\mathbf{h}'_m \in \mathbb{C}^\infty(\overline{\Omega} \times [0,1] \times [0,\pi] \times [0,2\pi])$ and coefficients λ_m for $m = 0, \ldots, k$, that the following is satisfied:

- relations (11.35) for $j \leq k - 4$;
- equations (11.34) for $j \leq k$, upon substitution of $\mathbf{h}_m$ and λ_m for $m \leq k$, and $\{\mathbf{h}_{j+4}\}$ in the form (11.35) for $k - 3 \leq j \leq k$;
- relations (11.33) for $j \leq k + 2$;
- all mean fields $\langle \mathbf{h}_j \rangle$ are solenoidal and satisfy boundary conditions (10.1) for a dielectric outside the fluid volume, as well as the normalisation condition

$$\mathscr{P}^\infty \langle \mathbf{h}_j \rangle = 0 \tag{11.36}$$

for $j > 0$ (here $\mathscr{P}^\infty$ is the projection in $\mathbb{H}^0(\Omega)$ onto the subspace spanned by the eigenfunction $\mathbf{h}_0 = \mathbf{h}^\infty$ associated with the eigenvalue Λ^∞); the fluctuating parts $\{\mathbf{h}_j\}$ vanish at the boundary $\partial\Omega$, and hence each term of the series (11.13) satisfies the boundary conditions (10.1).

For $k = 0$ the statement is true by constructions of Sect. 11.2.2. Let us make the induction step, supposing that the statement is true for some k.

Averaging of Eq. 11.34 for $j = k+1$ upon substitution of the expression (11.35) for $j = k-2$ yields

$$(\mathscr{L}^\infty - \Lambda^\infty)\langle \mathbf{h}_{k+1}\rangle + \mathbf{h}''_{k+1} - \sum_{m=0}^{k} \lambda_{k+1-m}\langle \mathbf{h}_m\rangle = 0, \tag{11.37}$$

where

$$\mathbf{h}''_{k+1} = \nabla_{\mathbf{x}} \times \langle \mathscr{N}\mathbf{W} \times (\mathscr{I}_1\langle \mathbf{h}_k\rangle + \mathscr{I}_2\langle \mathbf{h}_{k-1}\rangle + \mathbf{h}'_{k-2}) + \mathbf{Q} \times \{\mathbf{h}_k\}\rangle$$

is known at this stage. Applying to Eq. 11.37 projection $\mathscr{P}^\infty$, in view of the normalisation condition (11.36) we find

$$\lambda_{k+1}\langle \mathbf{h}_0\rangle = \mathscr{P}^\infty \mathbf{h}''_{k+1},$$

wherefrom determine λ_{k+1}. Subtraction of this relation from (11.37) yields

$$\langle \mathbf{h}_{k+1}\rangle = (\mathscr{L}^\infty - \Lambda^\infty)^{-1}|_{Im(\mathscr{I}-\mathscr{P}^\infty)} \left(\sum_{m=0}^{k} \lambda_{k+1-m}\langle \mathbf{h}_m\rangle - (\mathscr{I} - \mathscr{P}^\infty)\mathbf{h}''_{k+1} \right)$$

(here $\mathscr{I}$ is the identity operator).

Consider now the fluctuating part of Eq. 11.34:

$$\begin{aligned}
\mathbf{Z}_k \equiv\ & \eta\mathscr{D}\{\mathbf{h}_{k+5}\} + (\langle h^{(\rho)}_{k+4}\rangle \mathscr{D}_P + \langle h^{(\theta)}_{k+4}\rangle \mathscr{D}_\Theta + \langle h^{(\varphi)}_{k+4}\rangle \mathscr{D}_\Phi)\mathscr{N}\mathbf{W} \\
& + \mathscr{A}_1\langle \mathbf{h}_{k+2}\rangle + \mathscr{A}_{-1}\langle \mathbf{h}_k\rangle \\
& + \sum_{m=-1}^{3} \mathscr{B}_m\{\mathbf{h}_{k+1+m}\} - \nabla_{\mathbf{x}} \times \langle \mathscr{N}\mathbf{W} \times \{\mathbf{h}_{k+2}\} + \mathbf{Q} \times \{\mathbf{h}_k\}\rangle \\
& - \sum_{m=1}^{k+1} \boldsymbol{\lambda}_{k+1-m}\{\mathbf{h}_\mathbf{m}\} = 0.
\end{aligned}$$

Substituting expressions (11.35) for $k-3 \le j \le k+1$ and inverting the operator $\mathscr{D}$, we determine a zero-mean field $\mathbf{h}'_{k+1}$ thus satisfying this equation (the solvability condition can be easily verified for it). A straightforward algebra (involving relations (11.22) and the induction assumption that (11.33) holds for all $j \le k+2$) yields

$$0 = \mathscr{F}\mathbf{Z}_k + \nabla_{\mathbf{x}} \cdot \mathbf{Z}_{k-2} = \eta\mathscr{D}(\mathscr{F}\{\mathbf{h}_{k+5}\} + \nabla_{\mathbf{x}} \cdot \{\mathbf{h}_{k+3}\}),$$

implying the relation (11.33) for divergencies for $j = k+3$. The induction step is now concluded. □

11.4.2 Convergence of Asymptotic Series

We denote $\mathbf{f}_k^n \equiv \sum_{j=0}^{k} n^{-j/2}\mathbf{h}_j(\mathbf{x}, n\rho, n\theta, n\varphi)$, $p_k^n \in \mathbb{C}^1(\mathbb{R}^3)$ solution to the problem

$$\nabla^2 p_k^n = \nabla \cdot \mathbf{f}_k^n \text{ in } \Omega, \qquad \nabla^2 p_k^n = 0 \text{ in } \Omega', \qquad p_k^n \to 0 \text{ for } \rho \to \infty$$

(hence $\mathbf{f}_k^n - \nabla p_k^n \in H^0(\Omega)$),

$$\mathbf{h}_k^n \equiv \mathscr{P}^n(\mathbf{f}_k^n - \nabla p_k^n), \qquad \mathbf{g}_k^n \equiv (\mathscr{I} - \mathscr{P}^n)(\mathbf{f}_k^n - \nabla p_k^n) = \mathbf{f}_k^n - \nabla p_k^n - \mathbf{h}_k^n,$$

where $\mathscr{P}^n$ are $\mathscr{L}^n$-invariant projections in $H^0(\Omega)$ onto the one-dimensional subspaces spanned by the eigenfunctions associated with the eigenvalues Λ^n converging to Λ^∞.

Lemma 11.6 *For any* $k \ge 0$ *and* s *from the interval* $(k+1)/2 \ge s \ge 0$ *the following estimates hold true:*

$$|\mathbf{h}_k^n|_0 = 1 + O(n^{-1/2}),$$

$$|\Lambda^n - \sum_{j=0}^{k} n^{-j/2}\lambda_j| = O(n^{-(k+1)/2}),$$

$$|\nabla p_k^n|_s = O(n^{s-(k+1)/2}),$$

$$|\mathbf{g}_k^n|_s = O(n^{s-(k+1)/2}).$$

By the Sobolev embedding theorem, this implies asymptotic convergence of series (11.14) in the norms of the functional spaces $\mathbb{C}^q(\overline{\Omega})$ (the maximum in $\overline{\Omega}$ of the absolute value of the field and all derivatives of order up to q).

Proof A demonstration basically repeats the proof of Lemma 10.8, which does not explicitly require that the fast azimuthal variable only is considered, and relies just on the facts that $\mathbf{h}_j$ satisfy equations (11.34) stemming from the eigenvalue Eq. 11.15; $|\{\mathbf{h}_j\}|_s = O(n^s)$; relations (11.33) are satisfied, guaranteeing asymptotic solenoidality of the series (11.14); and that for the considered boundary conditions operators $\mathscr{L}^n$ converge to $\mathscr{L}^\infty$ in the sense of Theorem 11.1. All these facts remain true in the present setup. □

11.5 Physical Properties of the Spherical Dynamo

The asymptotic expansions of magnetic modes and their growth rates take the form of the series

$$\mathbf{h}^n = \mathbf{h}_0(\mathbf{x}) + \sum_{j \ge 1} n^{-j/2}\mathbf{h}_j(\mathbf{x}, P, \Theta, \Phi)|_{P=n\rho, \Theta=n\theta, \Phi=n\varphi}.$$

and (11.13), respectively. The leading-order terms of these series, $\mathbf{h}_0$ and λ_0, are, respectively, an eigenfunction and the associated eigenvalue of the limit operator

$$\mathscr{L}^\infty \mathbf{h} \equiv \eta \nabla^2 \mathbf{h} + \nabla \times (\mathbf{U} \times \mathbf{h}) + \nabla \times \mathscr{A}\mathbf{h}.$$

Here the linear operator $\mathscr{A}$ is represented in the basis of unit vectors of the spherical coordinate system by a symmetric matrix (11.25) with the entries (11.26).

Like in the case of a flow with a single fast azimuthal variable, considered in the previous chapter, there is a direct analogy with the classical mean-field dynamo theory [156]. As in this theory, the mean electromotive force

$$\mathscr{E}^n = n^{-1/2} \langle (\nabla \times \mathbf{W}) \times \{\mathbf{h}^n\} \rangle = \mathscr{A} \langle \mathbf{h}^n \rangle + O(n^{-1/2})$$

is the mean vector product of the fluctuating components of the flow velocity and magnetic field; it depends linearly on the mean magnetic field $\langle \mathbf{h}^n \rangle$ via the magnetic α-effect tensor $\mathscr{A}$. The respective new term $\nabla \times \mathscr{A}\mathbf{h}$ in the limit operator $\mathscr{L}^\infty$ describes the magnetic α-effect of the kinematic dynamo theory. Leading terms of expansions of the physical fields have now a more complex structure than in the dynamo considered in Chap. 10—they possess both poloidal and toroidal components.

As the dynamo problem considered in the previous chapter, the present one is characterised by the global magnetic Reynolds number $R_m = O(n^{1/2})$ and the local magnetic Reynolds number $R_m^l = O(n^{-1/2})$, the ratio of the typical magnitudes of the advective and diffusive terms having an intermediate asymptotics $|\nabla \times (\mathbf{v}^n \times \mathbf{h}^n)| / |\eta \nabla^2 \mathbf{h}^n| = O(1)$.

11.5.1 The Axisymmetric Dynamo

In this subsection we consider the particular case, when all the data possesses axial symmetry about the axis $\theta = 0, \pi$. More precisely, we assume $\mathbf{W} = \mathbf{W}(r, z, P, \Theta, \Phi)$,

$$\mathbf{U} = \mathbf{U}_p(r,z) + U(r,z)\mathbf{i}_\varphi, \quad \mathbf{h}^\infty = \nabla \times \chi(r,z)\mathbf{i}_\varphi + H(r,z)\mathbf{i}_\varphi$$

are decompositions of the mean flow $\mathbf{U}$ and an axisymmetric eigenfunction of the limit operator $\mathscr{L}^\infty$ into poloidal and toroidal components; here cylindrical coordinates (r, φ, z) are employed. The azimuthal component and the potential of the poloidal component of the eigenvalue equation constitute a closed system of equations:

$$\eta(\nabla^2 - r^{-2})\chi - r^{-1}(\mathbf{U}_p, \nabla)(r\chi) + r^{-1}\alpha^{\varphi\rho}\mathscr{D}_1(r\chi) + \alpha^{\varphi\theta}\mathscr{D}_2\chi + \alpha^{\varphi\varphi}H = \lambda_0\chi,$$

$$\begin{aligned} &\eta(\nabla^2 - r^{-2})H - r(\mathbf{U}_p, \nabla)(H/r) + (\nabla(U/r) \times \nabla(r\chi))^{(\varphi)} \\ &- \mathscr{D}_1(r^{-1}\alpha^{\rho\rho}\mathscr{D}_1(r\chi) + \alpha^{\rho\theta}\mathscr{D}_2\chi + \alpha^{\rho\varphi}H) \\ &- \mathscr{D}_2(r^{-1}\alpha^{\theta\rho}\mathscr{D}_1(r\chi) + \alpha^{\theta\theta}\mathscr{D}_2\chi + \alpha^{\theta\varphi}H) = \lambda_0 H, \end{aligned}$$

where

$$\mathscr{D}_1 \equiv (r^2 + z^2)^{-1/2}(z\partial/\partial r - r\partial/\partial z),$$
$$\mathscr{D}_2 \equiv -(r^2 + z^2)^{-1/2}(1 + r\partial/\partial r + z\partial/\partial z);$$

outside $\Omega, H = 0, (\nabla^2 - r^{-2})\chi = 0$, and at the boundary the continuity conditions $H_{\partial\Omega} = [\chi]_{\partial\Omega} = 0$ must be satisfied. This system is a generalisation of equations describing Braginsky's almost axisymmetric dynamo and of equations for axisymmetric eigenfunctions of the limit operator considered in Sect. 10.5.2. The toroidal component of magnetic field is generated both by the differential rotation and magnetic α-effect. In other words, in the dynamo under consideration the $\alpha\omega$-effect and α^2-effect are present simultaneously.

11.5.2 Helicity of the Flow Potential and its Relation to the Magnetic α-Effect

Lemma 11.7 *Suppose the field* $\mathbf{v}^n$ (11.8) *is the flow velocity of an incompressible fluid, i.e.*

$$\nabla \cdot \mathbf{v}^n = \nabla \cdot \mathbf{U} = 0,$$

and the boundary is impenetrable for fluid,

$$\mathbf{v}^n \cdot \mathbf{i}_\rho|_{\partial\Omega} = \mathbf{U} \cdot \mathbf{i}_\rho|_{\partial\Omega} = 0.$$

Then the mean and fluctuating parts of $\mathbf{v}^n$ *furnish asymptotically independent contributions into the integral of helicity of the vector potential of the flow,* $I(\mathbf{v}^n)$*, equal, respectively, to the integral of helicity of the potential of the mean flow,* $I(\mathbf{U})$ *, and* $-\eta\int_\Omega \mathrm{tr}\mathscr{A}\,d\mathbf{x}$.

Proof Suppose $\mathbf{U} = \nabla \times \mathbf{w}$. A simple calculation yields

$$I(\mathbf{v}^n) \equiv \int_\Omega (\mathbf{w} + n^{-1/2}\mathbf{W}) \cdot (\mathbf{U} + n^{-1/2}\nabla \times \mathbf{W})\big|_{P=n\rho,\Theta=n\theta,\Phi=n\varphi} d\mathbf{x}$$
$$= I(\mathbf{U}) + \int_\Omega \mathbf{W} \cdot \mathscr{N}\mathbf{W} d\mathbf{x} + O(n^{-1/2}).$$

Consider the Fourier series

$$\mathbf{W}(\mathbf{x}, P, \Theta, \Phi) = \sum \mathbf{W}_{k,k',p,p',q,q'} e^{i(2\pi k\rho + 2\pi k'P + 2p\theta + 2p'\Theta + q\varphi + q'\Phi)},$$
$$\rho^2 \sin\theta\mathscr{N}\mathbf{W} = \sum \mathbf{Y}_{k,k',p,p',q,q'} e^{i(2\pi k\rho + 2\pi k'P + 2p\theta + 2p'\Theta + q\varphi + q'\Phi)}.$$

Relations (11.26) imply $\mathrm{tr}\mathscr{A} = -\langle \mathbf{W} \cdot \mathscr{N}\mathbf{W} \rangle / \eta$, and hence

$$\int_{\Omega} \mathbf{W} \cdot \mathscr{N}\mathbf{W} + \mathrm{tr}\,\mathscr{A}\,\mathrm{d}\mathbf{x} = 2\pi^2 \sum \mathbf{W}_{k,k',p,p',q,q'} \cdot \mathbf{Y}_{k_1,k_1',p_1,p_1',q_1,q_1'},$$

where summation is over such $k, k', p, p', q, q', k_1, k_1', p_1, p_1', q_1, q_1'$ that

$$k + k_1 + n(k' + k_1') = p + p_1 + n(p' + p_1') = q + q_1 + n(q' + q_1') = 0$$

and at least one of the sums $k' + k_1'$, $p' + p_1'$ or $q' + q_1'$ does not vanish. Consequently,

$$|k| + |k_1| + |p| + |p_1| + |q| + |q_1| \geq n.$$

Since the field $\mathbf{W}$ is smooth and by virtue of conditions (11.9) and (11.10),

$$\sum |\mathbf{W}_{k,k',p,p',q,q'}| \leq CE^{-1/2} \quad \text{and} \quad \sum |\mathbf{X}_{k,k',p,p',q,q'}| \leq CE^{-1/2}$$

for any $E \geq 1$, where the sums are over such k, k', p, p', q, q' that $|k| + |p| + |q| \geq E$. This results in the estimate

$$\int_{\Omega} \mathbf{W} \cdot \mathscr{N}\mathbf{W} + \mathrm{tr}\mathscr{A}\mathrm{d}\mathbf{x} = O(n^{-1/2})$$

proving the lemma. □

11.6 Conclusions

1. As in the case of one azimuthal fast variable, for a flow velocity (11.8) depending on three fast variables proportional to spherical coordinates we have constructed full asymptotic expansions of magnetic modes and their growth rates in power series in the square root of the scale ratio, (11.13) and (11.14), respectively, and demonstrated asymptotic convergence of these series. The leading term of the expansion of a magnetic mode is independent of the fast variables. The interaction of fluctuating parts of the flow and magnetic field gives rise to the magnetic α-effect, described by a symmetric tensor $\mathscr{A}$ (see formulae (11.25) and (11.26) featuring simultaneously the $\alpha\omega$-effect and α^2-effect.
2. Similar asymptotic expansions of magnetic modes and the associated eigenvalues, satisfying the bounds for convergence stated in Lemma 11.6, can be constructed for the geometry of the fluid volume and boundary conditions different from the ones considered in this chapter—for instance, for a flow (11.8) in a spherical shell separating a dielectric outer space from a perfectly electrically conducting inner core.

3. It is simple to show that the statements of Theorem 11.1 and Lemmas 11.3–11.7 remain true for flows from a more general class,

$$\mathbf{v}^n = \mathbf{U}(\mathbf{x}) + \nabla \times \left(\mathbf{W}(\mathbf{x}, P, \Theta, \Phi, n^{-1/2})|_{P=n\rho,\Theta=n\theta,\Phi=n\varphi}\right),$$

provided $\mathbf{W}$ satisfies conditions (11.9)—(11.11) and is analytic in the last argument:

$$\mathbf{W}(\mathbf{x}, P, \Theta, \Phi, n^{-1/2}) = \sum_{m\geq 0} \mathbf{W}_m(\mathbf{x}, P, \Theta, \Phi) n^{-m/2}.$$

Then the limit operator $\mathscr{L}^\infty$ has the same structure, and coefficients of tensor $\mathscr{A}$ describing the α-effect can be determined by formulae (11.26), where $\mathbf{W}_0(\mathbf{x}, P, \Theta, \Phi)$ is assumed in place of $\mathbf{W}$ and $\mathscr{D}\xi = -\mathbf{W}_0$.

Chapter 12
Concluding Remarks

We have considered three problems arising in the theory of magnetohydrodynamic stability: kinematic magnetic dynamos (Chaps. 3, 4, 5, 10 and 11), linear and weakly nonlinear stability of three-dimensional MHD regimes of fluid residing in the entire space (Chaps. 6 and 7), and linear and weakly nonlinear stability of hydromagnetic regimes of convection in a horizontal rotating plane layer (Chaps. 8 and 9) and we supposed that the perturbed states (and in the case of kinematic dynamos the flows—except for Chaps. 10 and 11) are small-scale, and perturbations involve both small and large spatial and temporal scales. The presence of a small parameter—the scale ratio, ε—gives an opportunity to apply asymptotic methods, and we have constructed solutions in the form of asymptotic power series in the small parameter.

The statements of the problems that we have addressed considerably vary in details. We have explored kinematic magnetic field generation by steady and time-periodic flows periodic in space, and by convective plan forms in a layer. We have investigated linear stability of three-dimensional space-periodic MHD states; in the subsequent study of their weakly nonlinear stability we have noted that the periodicity condition is too restrictive—its rôle consists of ensuring existence of solutions to auxiliary problems provided that no neutral small-scale modes are present in the kernel of the operator of linearisation—and hence the condition has been dropped. Weakly nonlinear stability of regimes of forced and free hydromagnetic convection in a layer has been examined allowing for rotation about the vertical axis.

In spite of differences in the statements of the problems that we considered, their solutions share some structural properties. The leading terms of expansions of perturbations are linear combinations of small-scale neutral modes of the operator of linearisation around the regime, whose stability is studied. The neutral modes are modulated by amplitudes (i.e., the respective coefficients in the linear combinations) depending solely on the slow variables. The behaviour of perturbations both in the linear and weakly nonlinear phases of evolution is governed by

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equations for the amplitudes (which may have the sense of mean-field equations for perturbations). From the algebraic point of view, these equations are solvability conditions for a system of linear partial differential equations in fast variables, from which the fluctuating part of a higher-order term of expansion can be determined; the amplitude equations are obtained by averaging over the fast variables (in general, by spatially weighted averaging) of equations composing this system.

For a generic MHD system, the temporal and spatial scale ratios are of the same order, ε, and the amplitude equations comprise a system of first-order linear equations. They involve a partial differential operator in the slow spatial variables; being of the first order, it can be regarded as describing an anisotropic combined α-effect (in which both the magnetic and kinematic α-effects are interwoven). We call it the operator of significant α-effect in the leading order. Operators describing any other physical processes such as diffusion are absent, as well as any nonlinearities even when weakly nonlinear stability is considered. The specific structure of the operators of significant α-effect in the leading order implies that generically the amplitude equations possess solutions superexponentially growing in the slow time (till their amplitude becomes sufficiently large to violate the assumptions of smallness under which the amplitude equations are derived); in non-generic cases, the behaviour of large-scale modes consists of harmonic oscillations.

However, all coefficients of the α-tensor in the amplitude equations can simultaneously vanish—in our terminology, the α-effect can be insignificant in the leading order. This is a typical of MHD regimes possessing symmetries. For MHD regimes in a fluid occupying the entire space, insignificance of the α-effect in the leading order is guaranteed by parity invariance, and in a plane layer also by the symmetry about a vertical axis; for time-periodic regimes, these symmetries can involve a time shift (see the definitions in Sect. 8.4). If the α-effect is insignificant in the leading order, the fast and slow times scale ratio of the order of ε^2 is assumed. The amplitude equations for perturbations of such MHD regimes involve terms describing a variety of eddy effects, which originate in the nonlinear interaction of small-scale fluctuating parts of fields constituting the perturbation. The interaction being averaged over the small scales, eddy effects are acting on the mean perturbation (or on amplitudes of neutral modes composing the leading term of a large-scale perturbation).

Combined anisotropic eddy correction of diffusion is one of such effects. As in the case of the combined α-effect, the interpretation is suggested by the structure of the respective operator, which is a second-order partial differential operator. However, the operator of eddy diffusion (the sum of the operators of molecular diffusion and eddy correction) is not necessarily negative definite. It may have eigenvalues with positive real parts, and then the associated instability modes exponentially grow in time (at least, at the stage of linear evolution). In this case one speaks of the effect of negative eddy diffusivity. We have derived formulae for determination of the coefficients of the operator of eddy correction of diffusion and presented a method for their computation, in which the number of auxiliary problems to be solved numerically is minimised. Sign-definiteness of operators of magnetic and combined MHD eddy diffusion exhibited by synthetic small-scale flows and MHD steady states modelling turbulence has been studied numerically (see Chaps. 3, 4, 5 and 6, respectively). We have found that the effect of negative eddy diffusivity is present in a significant part of such flows and MHD states (for molecular diffusivities below the thresholds for the onset of kinematic magnetic field generation or MHD instability). As it is natural to expect, perturbations of MHD steady states (in which the magnetic field and flow velocity are perturbed simultaneously) are excited more easily than magnetic field in kinematic dynamos (in which zero magnetic field is perturbed only).

Operators of a different nature, namely anisotropic non-local second-order pseudodifferential operators in the slow variables, can arise in amplitude equations for large-scale perturbations of unsteady MHD regimes with the combined α-effect insignificant in the leading order. Because of this structure, we interpret them as operators describing a non-standard non-local eddy diffusion. Such operators emerge, when a small-scale MHD regime subjected to a perturbation possesses neither parity invariance nor the symmetry about a vertical axis (but may have the spatio-temporal symmetries of this kind with a non-zero time shift). Emergence of the operator of non-local eddy diffusion can be traced down to a pseudodifferential operator appearing, when solenoidality (in the slow variables) of the order ε term $\langle \mathbf{v}_1 \rangle_h$ in the expansion of the mean flow perturbation is imposed. Consequently, from the point of view of physics, the respective phenomenon has its origin in the action of the large-scale component of pressure, averaged over the fast spatial variables and maintaining incompressibility of the perturbation of the mean flow. Alternatively, this phenomenon can be also regarded as the α-effect produced by the component of the mean flow perturbation, fluctuating in the fast time and described by the pseudodifferential operator, which enforces solenoidality of the flow.

When weakly nonlinear stability of MHD regimes is considered, new quadratic terms involving first derivatives in the slow spatial variables appear in the amplitude equations, provided the α-effect is insignificant in the leading order. They emerge as means of the terms describing advection in the original equations; thus the respective physical effect can be identified as anisotropic eddy advection. The operators of the α-effect are absent in the amplitude equations; algebraically, this is a consequence of insignificance of the α-effect in the leading order (despite the amplitude equations are the result of averaging of the equations in the hierarchy, governing the fast-time evolution of the terms of perturbations, which are not of the leading, but of the second or third order in ε).

Operators describing the α-effect and eddy diffusion coexist in the amplitude equations, when stability to large-scale perturbations of weakly non-symmetric MHD regimes depending on the small parameter ε is examined, the amplitude of the antisymmetric part of the regime being the order of ε. (Here the symmetries guaranteeing insignificance of the α-effect in the leading order are meant.) In this context, the parameter ε also retains its original senses of the spatial scale ratio, and the amplitude of a perturbation when weakly nonlinear stability is considered. MHD regimes are weakly non-symmetric, when the forcing has an antisymmetric part the order of ε, or if a branch of regimes emerges in a symmetry-breaking bifurcation. Bifurcations occur, when zero-mean short-scale neutral modes, steady or oscillatory, are present in the kernel of the operator of linearisation; the number of amplitude equations is equal to the dimension of the kernel. Equations for the amplitudes of zero-mean short-scale modes involve cubic nonlinearity.

Coexistence of operators of the magnetic α-effect and molecular diffusion in magnetic dynamos also occurs in the mean-field equations, when generation by flows possessing the scale separation with the scale ratio ε is considered, and the flow amplitude increases as $\varepsilon^{-1/2}$ (see Chaps. 10 and 11). The kinematic α-effect, molecular viscosity and the standard advection coexist [93] in the mean-field equations describing flows obeying an equivalent parameterisation, when the spatial scale ratio equals to the square of the kinematic Reynolds number tending to zero. Evidently, the two results can be generalised: the mean-field equations arising in a full MHD problem with the same scalings will feature the same ingredients—the combined kinematic and magnetic α-effect, molecular viscosity and magnetic diffusion, and the usual advection.

In all the problems mentioned above, the amplitude equations are evolutionary, i.e. can be used to determine the partial derivative of each amplitude in the slow time. When stability of regimes of free thermal hydromagnetic convection in a horizontal rotating layer of fluid is investigated (when no source terms are present in the equations governing the regime, i.e. no forces act on fluid, except for the buoyancy, Lorentz and Coriolis forces), the system of amplitude equations for large-scale perturbations looses this property (see Chap. 9): only equations governing the mean magnetic field perturbation remain evolutionary. If the regime is steady and has a symmetry guaranteeing insignificance of the α-effect in the leading order, then one of the amplitude equations is a third-order partial differential equation in the slow spatial variables, involving a cubic nonlinearity (despite the stability problem is considered for an individual regime, and not for a branch of regimes, as in Chap. 8). Due to the presence of cubic nonlinearity and other structural peculiarities of the amplitude equations, it seems natural to expect that a large variety of different patterns of behaviour can be exhibited by perturbations of steady symmetric regimes of free hydromagnetic convection, as well as of regimes emerging in a pitchfork bifurcation. We have found numerically a number of steady and time-periodic small-scale free hydromagnetic convective regimes, periodic in horizontal directions, which are stable to small-scale perturbations and possess parity invariance or the symmetry about a vertical axis [55, 333]; we plan to study numerically the system of amplitude equations for large-scale perturbations of these regimes.

Notably, none of the eddy operators (including the α-effect operators), that we have formally derived, involves the flow or magnetic helicity—the closest match is the helicity of the vector potential of the flow encountered in the α-effect operator in mean-field equations arising in the kinematic dynamo problem for flows with an external scaling (see Chaps. 10 and 11). Neither our expressions for the α-effect tensors are similar to heuristic expressions suggested in the literature, although it is not ruled out, that such formulae can arise in asymptotic analysis of solutions to auxiliary problems in the limit of high magnetic and/or kinematic Reynolds numbers, or for extreme values of other parameters.

Our analytical results can be applied in designing geo- and astrophysical numerical models. In particular, they imply that the choice of eddy effects included in a model (α-effects, eddy and non-local diffusion, eddy advection) must be consistent with the properties of the small-scale turbulence responsible for their presence. For instance, the often used "justification" for the use in computations of inflated values for molecular diffusivities by the argument that "turbulent diffusivity is used" can be accepted only with a simultaneous explanation, why the α-effect is insignificant in the leading order.

We hope that the theory of large-scale perturbations of MHD regimes, that we have presented here, will help the reader to gain a better insight into various phenomena, by means of which the nonlinear interaction of short-scale motions of electrically conducting fluid and fluctuations of magnetic field can give rise to emergence of large-scale MHD structures. We also hope that our exposition is helpful in equipping the reader with mathematical tools necessary to raise qualitative theories of such phenomena on a solid mathematical footing.

Appendix A
Mean-Field and Amplitude Equations for Bifurcations of Three-Dimensional MHD Regimes

We return here briefly to the problem of weakly nonlinear stability of three-dimensional MHD regimes considered in Chap. 7. Derivation of mean-field and amplitude equations in this context is very similar logically and algebraically to the one presented in Sect. 8.6. Due to this similarity, the mean-field and amplitude equations that we have derived remain applicable in the MHD stability problem, provided certain straightforward modifications are implemented in the auxiliary problems and in Eqs. 8.209, 8.210 and 8.222.

We consider MHD regimes that are governed by Eqs. 7.1–7.3 and have the form of series (8.179) and (8.180). The regimes and their perturbations are six-dimensional and are comprised of three-dimensional flow and magnetic components. Perturbations are sought as power series (7.7–7.8). The heat transfer equation is absent, and there is no buoyancy force (i.e., $\beta = \beta_2 = 0$). As before, the slow time has the scaling $T = \varepsilon t$ if the $\alpha-$effect is present in the leading order, and $T = \varepsilon^2 t$ otherwise. The problem is considered, as in Chap. 7, in the entire space, and the slow spatial variable $\mathbf{X} = \varepsilon \mathbf{x}$ is now three-dimensional. No rotation is assumed (i.e., $\tau = 0$). To be specific, we assume that the bifurcation occurs on variation of viscosity:

$$\nu = \nu_0 + \nu_2 \varepsilon^2$$

(variation of other parameters results in changes, which can be implemented in a similar way); the possibility $\nu_2 = 0$ is not ruled out.

The following modifications of equations presented in Sect.8.6 render them applicable for the problem of weakly nonlinear stability to large-scale perturbations near points of bifurcations.

i. The linearisation $\mathscr{L} = (\mathscr{L}^v, \mathscr{L}^h)$ (equivalent to the one used in Sect. 7.1.1 and considered at the point of bifurcation) is employed [in particular, in the statements of auxiliary problems (8.208)]:

V. Zheligovsky, *Large-Scale Perturbations of Magnetohydrodynamic Regimes*, Lecture Notes in Physics, 829, DOI: 10.1007/978-3-642-18170-2,

$$\mathscr{L}^{v}(\mathbf{v},\mathbf{h},p) \equiv -\frac{\partial \mathbf{v}}{\partial t} + \nu_0 \nabla_{\mathbf{x}}^2 \mathbf{v} - (\mathbf{V}_0 \cdot \nabla_{\mathbf{x}})\mathbf{v} - (\mathbf{v} \cdot \nabla_{\mathbf{x}})\mathbf{V}_0 + (\mathbf{H}_0 \cdot \nabla_{\mathbf{x}})\mathbf{h} + (\mathbf{h} \cdot \nabla_{\mathbf{x}})\mathbf{H}_0 - \nabla_{\mathbf{x}} p,$$

$$\mathscr{L}^{h}(\mathbf{v},\mathbf{h}) \equiv -\frac{\partial \mathbf{h}}{\partial t} + \eta \nabla_{\mathbf{x}}^2 \mathbf{h} + \nabla_{\mathbf{x}} \times (\mathbf{V}_0 \times \mathbf{h} + \mathbf{v} \times \mathbf{H}_0).$$

The adjoint operator $\mathscr{L}^* = ((\mathscr{L}^*)^v, (\mathscr{L}^*)^h)$ is now

$$(\mathscr{L}^*)^v(\mathbf{v},\mathbf{h},p) = \frac{\partial \mathbf{v}}{\partial t} + \nu_0 \nabla_{\mathbf{x}}^2 \mathbf{v} + \sum_{i=1}^{3} \left(\mathbf{V}_0 \cdot \left(\nabla v_i + \frac{\partial \mathbf{v}}{\partial x_i} \right) \right) \mathbf{e}_i + \mathbf{H}_0 \times (\nabla \times \mathbf{h}) - \nabla p,$$

$$(\mathscr{L}^*)^h(\mathbf{v},\mathbf{h},p') = \frac{\partial \mathbf{h}}{\partial t} + \eta \nabla^2 \mathbf{h} - \sum_{i=1}^{3} \left(\mathbf{H}_0 \cdot \left(\nabla v_i + \frac{\partial \mathbf{v}}{\partial x_i} \right) \right) \mathbf{e}_i - \mathbf{V}_0 \times (\nabla \times \mathbf{h}) - \nabla p'.$$

The operators act in the space of pairs of three-dimensional solenoidal vector fields depending on the fast variables (the resultant vector fields can be made solenoidal by tuning scalar fields p and p').

ii. Neutral short-scale MHD stability modes $\mathbf{S}_k$ belong to the kernel of the linearisation $\mathscr{L}$. They are enumerated as follows: $\mathbf{S}_k = (\mathbf{S}_k^{vv} + \mathbf{e}_k, \mathbf{S}_k^{vh}, S_k^{vp})$ for $k = 1,2,3,$ and $\mathbf{S}_{k+3} = (\mathbf{S}_k^{hv}, \mathbf{S}_k^{hh} + \mathbf{e}_k, S_k^{hp})$ for $k = 1,2,3$ are determined by solving auxiliary problems I.1 (7.25)–(7.27) and I.2 (7.28)–(7.31), respectively (where the linearisation $\mathscr{L}$ is assumed in place of $\mathscr{M}$). For $k > 6$, $\mathbf{S}_k$ are all remaining short-scale zero-mean eigenfunctions from $\ker \mathscr{L}$ (understood in the generalised sense as explained in introduction to Sect. 8.6). Generically, $K = 7$ when a saddle-node or pitchfork bifurcation occurs, and $K = 8$ when a Hopf bifurcation occurs ($K = 6$ if $\nu_2 = 0$). As before, we assume that no Jordan cells of size 2 or more are associated with the eigenvalue zero.

The biorthogonal basis in $\ker \mathscr{L}^*$ consists of K eigenfunctions $\mathbf{S}_k^*$, i.e. relations

$$\left\langle\!\!\left\langle (\mathbf{S}_k^v, \mathbf{S}_k^h) \cdot \mathbf{S}_j^* \right\rangle\!\!\right\rangle = \delta_j^k,$$

hold for all $1 \le k,j \le K$ (where $\cdot$ denotes the scalar product of six-dimensional vector fields). In particular, $\mathbf{S}_k^* = (\mathbf{e}_k, 0)$ for $k = 1,2,3$, $\mathbf{S}_k^* = (0, \mathbf{e}_{k-3})$ for $k = 4,5,6$, and $\langle\!\langle (\mathbf{S}_k^*)^v \rangle\!\rangle = \left\langle\!\!\left\langle (\mathbf{S}_k^*)^h \right\rangle\!\!\right\rangle = 0$ for $k = 7, ..., K$. We assume that the solvability condition for a problem

$$\mathscr{L}(\mathbf{v},\mathbf{h},p) = \mathbf{f}, \qquad \nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{h} = 0$$

consists of orthogonality of the r.h.s., $\mathbf{f}$, to vector fields $\mathbf{S}_k^*$ for all $k \le K$ (8.191).

iii. New terms involving the viscosity correction ν_2 emerge in the equations: $\nu_2(\nabla_{\mathbf{x}}^2 \mathbf{v}_{n-2} + 2(\nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{X}})\mathbf{v}_{n-3} + \nabla_{\mathbf{X}}^2 \mathbf{v}_{n-4})$ in the l.h.s. of (8.191), which upon averaging yields $\nu_2\nabla^2_{\mathbf{X}}\langle\mathbf{v}_{n-4}\rangle_h$ in the l.h.s. of (1.188). Evidently, this does not affect order ε^0 and ε^1 equations (including the mean-field and amplitude equations when the $\alpha-$effect is absent in the leading order) or solutions to the auxiliary problems. The term $\nu_2\nabla^2_{\mathbf{x}}\mathbf{v}_0$ appears at order ε^2 in the l.h.s. of equation (8.211); this does not affect the mean-field equations (8.209) and (8.210), but gives rise to an additional term $\nu_2\nabla^2_{\mathbf{x}}\mathbf{S}^v_k$ in the flow component of $\mathbf{B}^1_k$ causing the respective changes in the amplitude equations (8.222) for $k > 6$.

iv. Any spatial or spatio-temporal mean of a vector field is now three-dimensional, i.e. no extraction of particular components—horizontal or vertical—is performed before averaging. Accordingly, the subscripts h and v are omitted in all equations considered in this Appendix. In particular, the mean-field equations for the flow (8.209) and magnetic (8.210) perturbations are three-dimensional, as is the gradient in the slow variables $\nabla_{\mathbf{X}}$. Any subscript, previously taking the values 1 and 2, is now assumed to have the range from 1 to 3, and any sum over the index values 1 and 2 in any equation is now promoted to a sum over the index values 1, 2 and 3. As before, it is advisable to eliminate redundancies when the orthogonality conditions are verified (e.g., for the system of Eqs. 8.194–8.196) or coefficients of the mean-field and amplitude coefficients are computed. This involves summation of coefficients of similar products of amplitudes, and simplification of the terms involving the inverse Laplacian using solenoidality of the three-dimensional mean magnetic field and flow perturbations and arranging terms on which it acts into sums of second-order derivatives.

After these changes are implemented, the amplitude equations obtained at order ε are

$$-\frac{\partial c_{j0}}{\partial T}=\sum_{k=1}^{K}\left(\left\langle\!\left\langle \mathbf{A}^1_k\cdot\mathbf{S}^*_j\right\rangle\!\right\rangle c_{k0}+\sum_{m=1}^{K}\left\langle\!\left\langle \mathbf{A}^3_{km}\cdot\mathbf{S}^*_j\right\rangle\!\right\rangle c_{k0}c_{m0}\right)$$
$$+\sum_{k=1}^{K}\sum_{m=1}^{3}\left(\left\langle\!\left\langle \mathbf{A}^2_{km}\cdot\mathbf{S}^*_j\right\rangle\!\right\rangle\frac{\partial c_{k0}}{\partial X_m}+\sum_{n=1}^{3}\sum_{j=1}^{3}\left\langle\!\left\langle \mathbf{A}^4_{kmnj}\cdot\mathbf{S}^*_j\right\rangle\!\right\rangle(\nabla^2_{\mathbf{X}})^{-1}\frac{\partial^3 c_{k0}}{\partial X_m\partial X_n\partial X_j}\right). \tag{A.1}$$

(former (8.204)), and for $j = 1,\ldots,6$ they reduce to mean-field equations for the flow

$$\frac{\partial\langle\!\langle\mathbf{v}_0\rangle\!\rangle}{\partial T}=\sum_{k=1}^{K}\sum_{m=1}^{3}\left\langle\!\left\langle -\mathbf{S}^v_k(V_0)_m-\mathbf{V}_0(S^v_k)_m+\mathbf{S}^h_k(H_0)_m+\mathbf{H}_0(S^h_k)_m\right\rangle\!\right\rangle\frac{\partial c_{k0}}{\partial X_m}-\nabla_{\mathbf{X}}\langle\!\langle p_0\rangle\!\rangle$$

(former (8.205)) and magnetic field

$$\frac{\partial\langle\!\langle\mathbf{h}_0\rangle\!\rangle}{\partial T}=\nabla_{\mathbf{X}}c_{k0}\times\left\langle\!\left\langle \mathbf{V}_0\times\mathbf{S}^h_k+\mathbf{S}^v_k\times\mathbf{H}_0\right\rangle\!\right\rangle$$

(former (8.206)) perturbations (these mean-field equations follow from the analogues of (8.188) and (8.189)). Vector fields $\mathbf{A}^q$ in amplitude Eq. A.1 are now as follows:

$$\begin{aligned}
\mathbf{A}^1_k \equiv \Big(& (\mathbf{V}_1 \cdot \nabla_\mathbf{x})\mathbf{S}^v_k - (\mathbf{H}_1 \cdot \nabla_\mathbf{x})\mathbf{S}^h_k + (\mathbf{S}^v_k \cdot \nabla_\mathbf{x})\mathbf{V}_1 - (\mathbf{S}^h_k \cdot \nabla_\mathbf{x})\mathbf{H}_1, \\
& - \nabla_\mathbf{x} \times (\mathbf{S}^v_k \times \mathbf{H}_1 + \mathbf{V}_1 \times \mathbf{S}^h_k)\Big); \\
\mathbf{A}^2_{km} \equiv & - \mathscr{L}(\nabla_\mathbf{x} s^v_{km}, \nabla_\mathbf{x} s^h_{km}, 0) - \Big(2\nu_0 \partial \mathbf{S}^v_k / \partial x_m - (V_0)_m \mathbf{S}^v_k + (H_0)_m \mathbf{S}^h_k + S^p_k \mathbf{e}_m, \\
& 2\eta \partial \mathbf{S}^v_k / \partial x_m + \mathbf{e}_m \times (\mathbf{S}^v_k \times \mathbf{H}_0 + \mathbf{V}_0 \times \mathbf{S}^h_k)\Big); \\
\mathbf{A}^3_{km} \equiv & \Big((\mathbf{S}^v_k \cdot \nabla_\mathbf{x})\mathbf{S}^v_m - (\mathbf{S}^h_k \cdot \nabla_\mathbf{x})\mathbf{S}^h_m, \quad -\nabla_\mathbf{x} \times (\mathbf{S}^v_k \times \mathbf{S}^h_m)\Big) \\
\mathbf{A}^4_{kmnj} \equiv & \Big(2\langle -(V_0)_m (S^v_k)_n + (H_0)_m (S^h_k)_n \rangle \mathbf{e}_j, \quad 0\Big).
\end{aligned}$$

The mean-field equations must be solved together with the solenoidality conditions

$$\nabla_\mathbf{X} \cdot \langle\!\langle \mathbf{v}_0 \rangle\!\rangle = \nabla_\mathbf{X} \cdot \langle\!\langle \mathbf{h}_0 \rangle\!\rangle = 0. \tag{A.2}$$

If the MHD regime is parity-invariant (perhaps with a time shift), then the α-effect is absent in the leading order, and the amplitude equations obtained at order ε^2 are

$$\frac{\partial c_{k0}}{\partial T} = \left\langle\!\!\left\langle (\mathbf{C} - (\nabla_\mathbf{X}(\nabla^2_\mathbf{X})^{-1}(\nabla_\mathbf{X} \cdot \langle \mathbf{C}^v \rangle_h), 0)) \cdot \mathbf{S}^*_k \right\rangle\!\!\right\rangle \tag{A.3}$$

(former (8.222)), for $k \le 6$ reducing to the equation for the mean perturbation of the flow velocity

$$\begin{aligned}
& -\frac{\partial}{\partial T}\langle\!\langle \mathbf{v}_0 \rangle\!\rangle + \nu_0 \nabla^2_\mathbf{X} \langle\!\langle \mathbf{v}_0 \rangle\!\rangle + \sum_{k=1}^{K} \sum_{n=1}^{3} \frac{\partial}{\partial X_n} \left(\sum_{m=1}^{K} \mathbf{A}^v_{kmn} c_{k0} c_{m0} + \mathcal{A}^v_{nk} c_{k0} \right. \\
& \left. + \sum_{m=1}^{3} \frac{\partial}{\partial X_m} \left(\mathbf{D}^v_{kmn} c_{k0} + \sum_{j=1}^{3} \sum_{i=1}^{3} \mathbf{d}^v_{kmnji} \frac{\partial^2}{\partial X_j \partial X_i} \nabla^{-2}_\mathbf{X} c_{k0} \right) \right) = 0
\end{aligned}$$

(former (8.209)) and equation for the mean perturbation of the magnetic field

$$\begin{aligned}
& -\frac{\partial}{\partial T}\langle\!\langle \mathbf{h}_0 \rangle\!\rangle + \eta \nabla^2_\mathbf{X} \langle\!\langle \mathbf{h}_0 \rangle\!\rangle + \nabla_\mathbf{X} \times \sum_{k=1}^{K} \left(\sum_{m=1}^{K} \mathbf{A}^h_{km} c_{k0} c_{m0} + \mathcal{A}^h_k c_{k0} \right. \\
& \left. + \sum_{m=1}^{3} \frac{\partial}{\partial X_m} \left(\mathbf{D}^h_{km} c_{k0} + \sum_{n=1}^{3} \sum_{j=1}^{3} \mathbf{d}^h_{kmj} \frac{\partial^2}{\partial X_n \partial X_j} \nabla^{-2}_\mathbf{X} c_{k0} \right) \right) = 0
\end{aligned}$$

(former (8.210)). These mean-field equations are also solved together with the solenoidality conditions (A.2) for mean fields of perturbation. Expressions defining eddy coefficients $\mathbf{A}$, $\mathcal{A}$, $\mathbf{D}$ and $\mathbf{d}$ in the mean-field equations are now identical to the ones arising in the case of perturbation of CHM regimes, but involve the new vector fields $\mathbf{A}^q$ and new solutions to auxiliary problems:

$$\begin{aligned}
\mathbf{A}^v_{kmn} &\equiv \langle\langle -(V_0)_n(\mathbf{G}^3_{km})^v - \mathbf{V}_0(G^3_{km})^v_n + (H_0)_n(\mathbf{G}^3_{km})^h + \mathbf{H}_0(G^3_{km})^h_n \\
&\quad - (S^v_k)_n\mathbf{S}^v_m + (S^h_k)_n\mathbf{S}^h_m\rangle\rangle; \\
\mathcal{A}^v_{nk} &\equiv \langle\langle -(V_1)_n\mathbf{S}^v_k - \mathbf{V}_1(S^v_k)_n + (H_1)_n\mathbf{S}^h_k + \mathbf{H}_1(S^h_k)_n \\
&\quad - (V_0)_n(\mathbf{G}^1_k)^v - \mathbf{V}_0(G^1_k)^v_n + (H_0)_n(\mathbf{G}^1_k)^h + \mathbf{H}_0(G^1_k)^h_n\rangle\rangle; \\
\mathbf{D}^v_{kmn} &\equiv \langle\langle -(V_0)_n(\widetilde{\mathbf{G}}^2_{km})^v - \mathbf{V}_0(\widetilde{G}^2_{km})^v_n + (H_0)_n(\widetilde{\mathbf{G}}^2_{km})^h + \mathbf{H}_0(\widetilde{G}^2_{km})^h_n\rangle\rangle; \\
\mathbf{d}^v_{kmnji} &\equiv \langle\langle -(V_0)_i(\mathbf{G}^4_{kmnj})^v - \mathbf{V}_0(G^4_{kmnj})^v_i + (H_0)_i(\mathbf{G}^4_{kmnj})^h + \mathbf{H}_0(G^4_{kmnj})^h_i\rangle\rangle; \\
\mathbf{A}^h_{km} &\equiv \langle\langle \mathbf{V}_0 \times (\mathbf{G}^3_{km})^h + (\mathbf{G}^3_{km})^v \times \mathbf{H}_0 + \mathbf{S}^v_k \times \mathbf{S}^h_m\rangle\rangle; \\
\mathcal{A}^h_k &\equiv \langle\langle (\mathbf{G}^1_k)^v \times \mathbf{H}_0 + \mathbf{V}_0 \times (\mathbf{G}^1_k)^h + \mathbf{S}^v_k \times \mathbf{H}_1 + \mathbf{V}_1 \times \mathbf{S}^h_k\rangle\rangle; \\
\mathbf{D}^h_{km} &\equiv \langle\langle (\widetilde{\mathbf{G}}^2_k)^v \times \mathbf{H}_0 + \mathbf{V}_0 \times (\widetilde{\mathbf{G}}^2_k)^h\rangle\rangle; \\
\mathbf{d}^h_{kmnj} &\equiv \langle\langle (\mathbf{G}^4_{kmnj})^v \times \mathbf{H}_0 + \mathbf{V}_0 \times (\mathbf{G}^4_{kmnj})^h\rangle\rangle.
\end{aligned}$$

The coefficients $\mathbf{B}^q$ in the sum

$$\begin{aligned}
\mathbf{C} \equiv \sum_{k=1}^{K}\Bigg(\mathbf{B}^1_k c_{k0} + \sum_{m=1}^{3}\Bigg(\mathbf{B}^2_{km}\frac{\partial c_{k0}}{\partial X_m} + \sum_{n=1}^{3}\Bigg(\mathbf{B}^3_{kmn}\frac{\partial^2 c_{k0}}{\partial X_n \partial X_m} + \sum_{j=1}^{3}\Bigg(\mathbf{B}^4_{kmnj}\frac{\partial^3 \nabla^{-2}_{\mathbf{X}} c_{k0}}{\partial X_m \partial X_n \partial X_j} \\
+ \sum_{i=1}^{3}\mathbf{B}^5_{kmnji}\frac{\partial^4 \nabla^{-2}_{\mathbf{X}} c_{k0}}{\partial X_m \partial X_n \partial X_j \partial X_i}\Bigg)\Bigg)\Bigg) + \sum_{m=1}^{K}\Bigg(\mathbf{B}^6_{km} c_{k0} c_{m0} + \sum_{n=1}^{3}\Bigg(\mathbf{B}^7_{kmn} c_{k0}\frac{\partial c_{m0}}{\partial X_n} \\
+ \sum_{j=1}^{3}\sum_{i=1}^{3}\mathbf{B}^8_{kmnji} c_{k0}\frac{\partial^3 \nabla^{-2}_{\mathbf{X}} c_{m0}}{\partial X_n \partial X_j \partial X_i} + \mathbf{B}^9_{kmn} c_{k0} c_{m0} c_{n0}\Bigg)\Bigg)\Bigg)
\end{aligned}$$

(formerly defined by relation (8.221)), which enters into the amplitude equation (A.3), are now the following six-dimensional fields:

$$\begin{aligned}
\mathbf{B}^1_k = \Big(& -(\mathbf{V}_1\cdot\nabla)(\mathbf{G}^1_k)^v - ((\mathbf{G}^1_k)^v\cdot\nabla)\mathbf{V}_1 + (\mathbf{H}_1\cdot\nabla)(\mathbf{G}^1_k)^h + ((\mathbf{G}^1_k)^h\cdot\nabla)\mathbf{H}_1 \\
& -(\mathbf{V}_2\cdot\nabla)\mathbf{S}^v_k - (\mathbf{S}^v_k\cdot\nabla)\mathbf{V}_2 + (\mathbf{H}_2\cdot\nabla)\mathbf{S}^h_k + (\mathbf{S}^h_k\cdot\nabla)\mathbf{H}_2 + \nu_2\nabla^2_{\mathbf{x}}\mathbf{S}^v_k, \\
& \nabla\times((\mathbf{G}^1_k)^v\times\mathbf{H}_1 + \mathbf{V}_1\times(\mathbf{G}^1_k)^h + \mathbf{S}^v_k\times\mathbf{H}_2 + \mathbf{V}_2\times\mathbf{S}^h_k)\Big)
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}^2_{km} = \mathscr{L}(\nabla g^{1,v}_{km}, \nabla g^{1,h}_{km}, 0, 0) + \Bigg(& -(\mathbf{V}_1\cdot\nabla)(\widetilde{\mathbf{G}}^2_{km})^v - ((\widetilde{\mathbf{G}}^2_{km})^v\cdot\nabla)\mathbf{V}_1 \\
& + (\mathbf{H}_1\cdot\nabla)(\widetilde{\mathbf{G}}^2_{km})^h + ((\widetilde{\mathbf{G}}^2_{km})^h\cdot\nabla)\mathbf{H}_1 - (V_0)_m(\mathbf{G}^1_k)^v + (H_0)_m(\mathbf{G}^1_k)^h \\
& -(V_1)_m\mathbf{S}^v_k + (H_1)_m\mathbf{S}^h_k + 2\nu_0\frac{\partial(\mathbf{G}^1_k)^v}{\partial x_m} - (G^1_k)^p\mathbf{e}_m, \\
& +\nabla\times((\widetilde{\mathbf{G}}^2_{km})^v\times\mathbf{H}_1 + \mathbf{V}_1\times(\widetilde{\mathbf{G}}^2_{km})^h) + \mathbf{e}_m\times((\mathbf{G}^1_k)^v\times\mathbf{H}_0 + \mathbf{V}_0\times(\mathbf{G}^1_k)^h \\
& + \mathbf{S}^v_k\times\mathbf{H}_1 + \mathbf{V}_1\times\mathbf{S}^h_k) + 2\eta\frac{\partial(\mathbf{G}^1_k)^h}{\partial x_m}\Bigg);
\end{aligned}$$

$$
\begin{aligned}
\mathbf{B}^3_{kmn} &= \mathscr{L}(\nabla g^{2,v}_{kmn}, \nabla g^{2,h}_{kmn}, 0, 0)\\
&\quad + \Big(-(V_0)_n (\widetilde{\mathbf{G}}^2_{km})^v + (H_0)_n (\widetilde{\mathbf{G}}^2_{km})^h - (\widetilde{G}^2_{km})^p \mathbf{e}_n + \nu_0 \Big(\delta^m_n \mathbf{S}^v_k + 2 \frac{\partial (\widetilde{\mathbf{G}}^2_{km})^v}{\partial x_n} \Big),\\
&\quad \mathbf{e}_n \times ((\widetilde{\mathbf{G}}^2_{km})^v \times \mathbf{H}_0 + \mathbf{V}_0 \times (\widetilde{\mathbf{G}}^2_{km})^h) + \eta \Big(\delta^m_n \mathbf{S}^h_k + 2 \frac{\partial (\widetilde{\mathbf{G}}^2_{km})^h}{\partial x_n} \Big) \Big);\\
\mathbf{B}^4_{kmnj} &= \Big(-(\mathbf{V}_1 \cdot \nabla)(\mathbf{G}^4_{kmnj})^v - ((\mathbf{G}^4_{kmnj})^v \cdot \nabla)\mathbf{V}_1\\
&\quad + (\mathbf{H}_1 \cdot \nabla)(\mathbf{G}^4_{kmnj})^h + ((\mathbf{G}^4_{kmnj})^h \cdot \nabla)\mathbf{H}_1,\\
&\quad \nabla \times ((\mathbf{G}^4_{kmnj})^v \times \mathbf{H}_1 + \mathbf{V}_1 \times (\mathbf{G}^4_{kmnj})^h) \Big);
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{B}^5_{kmnji} &= \mathscr{L}(\nabla g^{4,v}_{kmnji}, \nabla g^{4,h}_{kmnji}, 0, 0)\\
&\quad + \Big(-(V_0)_i (\mathbf{G}^4_{kmnj})^v + (H_0)_i (\mathbf{G}^4_{kmnj})^h + 2\nu_0 \frac{\partial (\mathbf{G}^4_{kmnj})^v}{\partial x_i} - (G^4_{kmnj})^p \mathbf{e}_i,\\
&\quad \mathbf{e}_i \times ((\mathbf{G}^4_{kmnj})^v \times \mathbf{H}_0 + \mathbf{V}_0 \times (\mathbf{G}^4_{kmnj})^h) + 2\eta \frac{\partial (\mathbf{G}^4_{kmnj})^h}{\partial x_i} \Big);\\
\mathbf{B}^6_{km} &= (-(\mathbf{S}^v_k \cdot \nabla)(\mathbf{G}^1_m)^v - ((\mathbf{G}^1_m)^v \cdot \nabla)\mathbf{S}^v_k + (\mathbf{S}^h_k \cdot \nabla)(\mathbf{G}^1_m)^h + ((\mathbf{G}^1_m)^h \cdot \nabla)\mathbf{S}^h_k\\
&\quad - (\mathbf{V}_1 \cdot \nabla)(\mathbf{G}^3_{km})^v - ((\mathbf{G}^3_{km})^v \cdot \nabla)\mathbf{V}_1 + (\mathbf{H}_1 \cdot \nabla)(\mathbf{G}^3_{km})^h + ((\mathbf{G}^3_{km})^h \cdot \nabla)\mathbf{H}_1,\\
&\quad \nabla \times ((\mathbf{G}^1_m)^v \times \mathbf{S}^h_k + \mathbf{S}^v_k \times (\mathbf{G}^1_m)^h + (\mathbf{G}^3_{km})^v \times \mathbf{H}_1 + \mathbf{V}_1 \times (\mathbf{G}^3_{km})^h));\\
\mathbf{B}^7_{kmn} &= \mathscr{L}(\nabla(g^{3,v}_{kmn} + g^{3,v}_{mkn}), \nabla(g^{3,h}_{kmn} + g^{3,h}_{mkn}), 0, 0) + (-(S^v_k)_n \mathbf{S}^v_m + (S^h_k)_n \mathbf{S}^h_m\\
&\quad - (\mathbf{S}^v_k \cdot \nabla)(\widetilde{\mathbf{G}}^2_{mn})^v - ((\widetilde{\mathbf{G}}^2_{mn})^v \cdot \nabla)\mathbf{S}^v_k + (\mathbf{S}^h_k \cdot \nabla)(\widetilde{\mathbf{G}}^2_{mn})^h + ((\widetilde{\mathbf{G}}^2_{mn})^h \cdot \nabla)\mathbf{S}^h_k\\
&\quad - (V_0)_n ((\mathbf{G}^3_{km})^v + (\mathbf{G}^3_{mk})^v) + (H_0)_n ((\mathbf{G}^3_{km})^h + (\mathbf{G}^3_{mk})^h)\\
&\quad - ((G^3_{km})^p + (G^3_{mk})^p)\mathbf{e}_n + 2\nu_0 \frac{\partial}{\partial x_n} ((\mathbf{G}^3_{km})^v + (\mathbf{G}^3_{mk})^v),\\
&\quad \nabla \times ((\widetilde{\mathbf{G}}^2_{mn})^v \times \mathbf{S}^h_k + \mathbf{S}^v_k \times (\widetilde{\mathbf{G}}^2_{mn})^h) + \mathbf{e}_n \times (\mathbf{S}^v_k \times \mathbf{S}^h_m + \mathbf{S}^v_m \times \mathbf{S}^h_k\\
&\quad + ((\mathbf{G}^3_{km})^v + (\mathbf{G}^3_{mk})^v) \times \mathbf{H}_0 + \mathbf{V}_0 \times ((\mathbf{G}^3_{km})^h + (\mathbf{G}^3_{mk})^h))\\
&\quad + 2\eta \frac{\partial}{\partial x_n} ((\mathbf{G}^3_{km})^h + (\mathbf{G}^3_{mk})^h));\\
\mathbf{B}^8_{kmnji} &= (-(\mathbf{S}^v_k \cdot \nabla)(\mathbf{G}^4_{mnji})^v - ((\mathbf{G}^4_{mnji})^v \cdot \nabla)\mathbf{S}^v_k + (\mathbf{S}^h_k \cdot \nabla)(\mathbf{G}^4_{mnji})^h + ((\mathbf{G}^4_{mnji})^h \cdot \nabla)\mathbf{S}^h_k,\\
&\quad \nabla \times ((\mathbf{G}^4_{mnji})^v \times \mathbf{S}^h_k + \mathbf{S}^v_k \times (\mathbf{G}^4_{mnji})^h));\\
\mathbf{B}^9_{kmn} &= (-(\mathbf{S}^v_k \cdot \nabla)(\mathbf{G}^3_{mn})^v - ((\mathbf{G}^3_{mn})^v \cdot \nabla)\mathbf{S}^v_k + (\mathbf{S}^h_k \cdot \nabla)(\mathbf{G}^3_{mn})^h + ((\mathbf{G}^3_{mn})^h \cdot \nabla)\mathbf{S}^h_k,\\
&\quad \nabla \times ((\mathbf{G}^3_{mn})^v \times \mathbf{S}^h_k + \mathbf{S}^v_k \times (\mathbf{G}^3_{mn})^h)).
\end{aligned}
$$

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V. Zheligovsky, *Large-Scale Perturbations of Magnetohydrodynamic Regimes*, Lecture Notes in Physics, 829, DOI: 10.1007/978-3-642-18170-2,

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Index

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